Open Access
Baghdad Science Journal Vol.13(2)2016

DOI: http://dx.doi.org/10.21123/bsj.2016.13.2.0388

The Act Of An Operator

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Received 22/3/ 2015
Accepted 19/8/ 2015

Abstract:

In this paper we will study some of the properties of an operator by looking at the associated S-act of this operator, and conversely. We look at some operators, like one to one operators, onto operators. On the other hand, we look at some act theoretic concepts, like faithful acts, finitely generated acts, singular acts, separated acts, torsion free acts and noetherian acts. We try to determine what properties of T make the associated S-act has any of these properties.

Key words: faithful act, finitely generated act, singular act, separated act, torsion free act, noetherian act.

Introduction:

The S-acts have been introduced and studied, for example, in[1, 2]. In this paper we study S-act of linear operator. Let V be a normed space over a field F, T be a bounded operator on V, and let $S = \{e^x : x \in R\}$ be the semigroup. Define $\mu : S \times V \rightarrow V$ by $\mu(e^x, v) = e^T(v)$, this function makes V a left S-act, denote by V_T and we call it the associated S-act of T. we will explain this definition by some examples, and give some basic facts about the associated V_T We introduce the form of every element in V_T, see proposition (2). and study if two operator T and S are similar then V_T is isomorphic to V_S, see proposition (6). We show that for any operator T then the S-act V_T is faithful act, separated act, and torsion free act. We prove if T is onto and V_T is finitely generated, then V is finite dimensional, see proposition (9). We prove for any operator T and V_T is singular S-act then V is generated by one element, see proposition (12). We show if V is a finite dimensional normed space and T is similar to any operator J from R to R then V_T is Noetherian S-act if and only if S is Noetherian.

Main results:

In this section we introduce the construction of associated S-act to each bounded operator T on a normed space V. We illustrate the construction by some examples and we prove some basic facts about the act V_T. We start by the following:

Definition 1. Let V be a normed space over a field F, let T be a bounded linear operator acting on the elements of V on the left.

Let S be a semigroup such that S = {e^x: x ∈ R}

Define μ: S × V ⟶V by μ (e^x, v) = e^T (v), v ∈ V.

Note e^T is defined since T is bounded. [3]

We note that μ is well defined since every vector is act over a field (**[1]** ,p.46).

It is easy to check that μ makes V a left S-act. We shall denote this act by V_T, and we call it the associated S-act of T.

In this proposition we introduce the form of each element of V_T:

Proposition 2. If K = {V_j , j ∈ Λ} is a basis for V, then each element of V_T can be written in the form

$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{T^i}{i\,!}\sum_{j\in\Lambda}a_j\,v_j = \lim_{n\to\infty} p_n(T).V$

The symbol $\sum_{j\in\wedge}$ means the sum is taken over a finite subset of Λ.

Proof: We define μ: S×V ⟶ V by μ (e^x. v) = e^T (V)

Then μ (e^x. v) = e^T(v) = $\sum_{i=0}^{\infty}\frac{T^i}{i\,!}$ (v),

Let w ∈ V_T then w = $\sum_{i=0}^{\infty}\frac{T^i}{i\,!}$(v).

Thus $w=\left(I+T+\frac{T^2}{2!}+\frac{T^3}{3!}+\cdots\right)$ (v), since K = { V_j , j ∈ ∧} is abasis for V this give W $=\left(I+T+\frac{T^2}{2!}+\frac{T^3}{3!}+\cdots\right)\left(\sum_{j\in\wedge}a_j\,v_j\right)$ $=I\left(\sum_{j\in\wedge}a_j\,v_j\right)+T\left(\sum_{j\in\wedge}a_j\,v_j\right)+\frac{T^2}{2!}\left(\sum_{j\in\wedge}a_j\,v_j\right)+\cdots$

$=\sum_{i=0}^{\infty}\frac{T^i}{i\,!}\left(\sum_{j\in\wedge}a_j\,v_j\right)$

But the series $\sum_{i=0}^{\infty}\frac{T^i}{i\,!}$ converges in B (H), then

$w=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{T^i}{i\,!}\left(\sum_{j\in\wedge}a_j\,v_j\right)$

$=\lim_{n\to\infty}p_n(T).V$ By [4]

Examples 3.

1. Let $\{v_j : j\in\wedge\}$ be a basis for a normed space V. Let O *be the* zero operator, recall $O^o=I$. If $w\in V_o$ then by proposition (2) we get $w=e^T(v)=e^o(v)$ *then* $w=I\left(\sum_{j\in\wedge}a_j\,v_j\right)$, then $w=\sum_{j\in\wedge}a_j$, *since* $e^o=I$ (**[3]**, p.26)
2. Let $I:V\to V$ be the identity operator on V, $\{v_j : j\in\wedge\}$ be a basis for a normed space V and let $w\in V_I$ then by proposition (2) we get
$w=e^T(v)=e^I(v)=$
$I.e\left(\sum_{j\in\wedge}a_j\,v_j\right)=$
$I.\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{j\in\wedge}a_j\,v_j\right)=$
$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{j\in\wedge}a_j\,v_j\right)$, since e^I= I. e (**[3]**, p.26)

$=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{i!}\left(\sum_{j\in\wedge}a_j\,v_j\right)$, *put* $a_n=\sum_{i=0}^{n}\frac{1}{i!}$,then
$w=\lim_{n\to\infty}a_n\left(\sum_{j\in\wedge}a_j\,v_j\right)$

3. Let $\{v_j : j\in\wedge\}$ be a basis for a normed space V, and T be a a nilpotent operator on V (i.e. $T^n=0$ *and* $T^{n-1}\neq 0$ for some positive integer n)

Let $w\in V_T$ then by proposition (2), we have

$w=e^T(v)=\left(1+T+\frac{T^2}{2!}+\cdots+\frac{T^n}{n!}+\cdots\right)v=\left(1+T+\frac{T^2}{2!}+\cdots+\frac{T^{n-1}}{(n-1)!}\right)\left(\sum_{j\in\wedge}a_j\,v_j\right)$
$=p_{n-1}(T)\left(\sum_{j\in\wedge}a_j\,v_j\right)=p_{n-1}(T).v$

Lemma 4. Let T be a nilpotent operator on V, then V_T is S- act if and only if V_T is R-module, where R be the ring of polynomials in one variable with coefficients in F.

Proof: Suppose V_T is S- act we want to prove V_T is R-module.

Let $w\in V_T$, then $w=p_{n-1}(T).v$, then V_T is R- module. [**5**]

Conversely, assume V_T is R-module then V_T is S-act. **([2], p.13)**

Remark 5. If M is a right R-module, then M is $(R,.)$-act. **([2], p.13)**

Note the converse is not true, to explain this we give example,

let S = {e^x: x ∈ R} then S is left S-act which denoted S_S ,to prove this

define $\mu: S \times S \to S$ by
$\mu(e^{x1}, e^{x2}) = e^{x1+x2} \quad \forall x_1, x_2 \in R, \forall x_1.x_2 = x_2.x_1$

Then $s_1.(s_2 s_3) = e^{x1}(e^{x2}e^{x3}) = e^{x1}(e^{x2+x3}) = e^{x1+x2+x3} = (e^{x1+x2}).e^{x3} = (s_1 s_2).s_3$, then S is act over S . But S_s is not module with (+, .), since $(S, +, .)$ *is* not Ring because it doesn't satisfy a binary operation with (+) ,this give S_s is act but not module.

Proposition 6. Let T and S be two operators on V. If S is similar to T then V_S is isomorphic to V_T.

Proof: Assume that T and S are similar, and then there exist an invertible operator h on V**.** Such that $h\,T\,h^{-1} = S$ ([6],p.156)**,** then $(h\,S\,h^{-1} = T).h$ this give $h\,S = T\,h$

since $h\,S = T\,h$, then

$h\,S^n = hSS^{n-1} = T\,hS^{n-1} = ThSS^{n-2} = TTh\,S^{n-2} = T^2hS^{n-2} = T^2hSS^{n-3} = T^2ThS^{n-3} = T^3hS^{n-3} = T^3hSS^{n-4} = T^3ThS^{n-4} = T^4hS^{n-4} = \cdots = T^n h$, Then $h\,e^s = e^T\,h$ ……..(1)

Define $h' : V_s \to V_T$ by $\left(e^s(v)\right)h' = e^T\left(h(v)\right)$ …….(2)

To prove h' is isomorphism we must prove:

h' is well defined, let $e^s(v_1) = e^s(v_2)$ then $h\left(e^s(v_1)\right) = h\left(e^s(v_2)\right)$ (since h is well defined). Then by equation (1), we get

$e^T\left(h(v_1)\right) = h\left(e^s(v_1)\right) = h\left(e^s(v_2)\right) = e^T\left(h(v_2)\right)$, therefore

$e^T\left(h(v_1)\right) = e^T\left(h(v_2)\right)$ … (3), then by equations 2, 3 we get

$\left(e^s(v_1)\right)h' = \left(e^s(v_2)\right)h'$, then h' is well defined

To prove h' is one to one, let $\left(e^s(v_1)\right)h' = (e^s(v_2))h'$, then by equation (2) we get $e^T\left(h(v_1)\right) = e^T\left(h(v_2)\right)$, then by equation (1) we get $h\left(e^s(v_1)\right) = h\left(e^s(v_2)\right)$ but h is an invertable then $h^{-1}h\left(e^s(v_1)\right) = h^{-1}h\,(e^s(v_2))$, this gives $\left(e^s(v_1)\right) = (e^s(v_2))$ therefore h' is one to one.

To prove h' is onto, let $e^T(v) \in V_T$ then $h^{-1}(v) \in V$, for $v \in V$ and $e^s\left(h^{-1}(v)\right) \in V_s$, then by equation (2) we get $(e^s\left(h^{-1}(v)\right)h' = e^T(h\left(h^{-1}(v)\right)) = e^T(v)$ then h' is onto.

We illustrate in the following proposition the relation between the faithful S-act and the linear operator T. Now we need to give the definition of a faithful S-act.

Recall that A_s a faithful left S-act if for all $s, t \in S$ the equility $sa = ta$ for all $a \in A_s$, implies $s = t$. ([1], p.46)

Proposition 7. For any bounded operator T then V_T is a faithful S-act.

Proof: we want to show that V_T is a faithful S-act. Let $e^{x1}.e^T(v) = e^{x2}.e^T(v)$

Since e^T is linear transformation, this gives $e^{x1}.e^T(v) = e^{x2}.e^T(v)$, thus $e^T(e^{x1}.v) = e^T(e^{x2}.v)$ .Since e^T is one to one, then $e^{x1}.v = e^{x2}.v$, therefore $(e^{x1} - e^{x2})v = 0$, this give $e^{x1} = e^{x2}$, then V_T is a faithful S-act.

In ([1], p.63)**,** a subset $U \neq \emptyset$ of a right S-act A_s is said to be a set of generating elements or a generating set of A_s if every element $a \in A_s$ can be presented as $a = u.s$ for some $u \in U, s \in S$.In other words, U is a set of generating elements for A_s if $< U > = U\,uS = A_s$, $u \in U$ where $uS = \{us/s \in S\}$. we say that a right S-act A_s is finitely generated if $A_s = < U >$ *for some* $U, |U| < \infty$.

Remark 8. If V is a finite dimensional normed space, then V_T is finitely generated S-act.

The following proposition gives when the converse of remark (8) is true

Proposition 9. If T is onto and V_T is finitely generated, then V is finite dimensional.

Proof: we use the contradiction. Assume V is not finite dimensional. Let $k = k(T) = \{w \in V: Tw = 0\}$. It is clear that K is as invariant subspace of V ($since\ K \subseteq V\ and\ \forall\, w \in K, T(w) =$

0 *but* $0 \in$ *subspace* K *then* $T(K) \subseteq K$) and by the first isomorphism theorem of S-act *hen* $TV \simeq \frac{V}{K}$. ([1],P.53),since T is onto then TV=V, therefore $V \simeq \frac{V}{K}$.
By assuming that V is not finite dimensional then either K infinite dimensional or K finite dimensional. K is an *inv*ariant subspace of V then we can consider K_T.
If K is finite dimensional then K_T is finite generated by remark (10), the subact K_T is generated by the set $\{e^T(w_j) : j \in \wedge\}$ where $\{w_j : j \in \wedge\}$ is a basis for K. But $w_j \in K$ given that $Twj = 0$, since the restriction of T on K is the zero operator $\mathcal{O}$. Thus $K_T = K_{\mathcal{O}}$, therefore K_T can not be finitely generated (see example 1 from (3),this contradicition with suppose K is finite dimensional.
Then we must K is infinite dimensional, the subact K_T is generated by the set $\{e^T(w_j) : j \in \wedge\}$ where $\{w_j : j \in \wedge\}$ is a basis for K. But $w_j \in K$ given that $Twj = 0$, since the restriction of T on K is the zero operator $\mathcal{O}$. Thus $K_T = K_{\mathcal{O}}$, therefore K_T can not be finitely generated (see example 1 from (3),but K_T is a subact of V_T and V_T is finitely generated. This mean infinitely generated contain in finitely generated. This contradiction shows that V is finite dimensional.
[7] show that an S-act A separated if for each $a \neq b$ in A there exists $s \neq e$ such that $sa \neq sb$.

Proposition 10. For any bounded operator T then V_T is separated S-act.

Proof: Let $a \neq b$ in V_T to prove that V_T is separated we have to show that there exist s in S, $s \neq e$ such that $sa \neq sb$.
Assume sa = sb, $e \neq s \in S$,a,b$\in V_T$,this give $e^x . e^T(v_1) = e^x . e^T(v_2)$ $v_1, v_2 \in V, x \in R$, Since e^T is operator then e^T is linear transformation, this gives $e^x . e^T(v_1) = e^x . e^T(v_2) \longrightarrow e^T(e^x . v_1) = e^T(e^x . v_2)$, bu e^T is one to one,then $e^x . v_1 = e^x . v_2 \longrightarrow (v_1 - v_2)e^x = 0$,but $e^x \neq 0$ then $v_1 = v_2$ this gives either $e^T(v_1) \neq e^T(v_2)$ or $e^T(v_1) = e^T(v_2)$ but if $e^T(v_1) \neq e^T(v_2)$ this gives $v_1 \neq v_2$ this contradction with assume $v_1 = v_2$, then $e^T(v_1) = e^T(v_2)$ this means a=b which is a contradction, *then* V_T is separated S-act.

In the following proposition we give the converse of the proposition (10)

Proposition 11. If V_T is separated then T is one to one**.**

Proof: Assume V_T is separated, we want to prove T is one to one, let $v_1 \neq v_2$ we must prove $T(v_1) \neq T(v_2)$,since $v_1 \neq v_2$ then either $e^T(v_1) \neq e^T(v_2)$ or $e^T(v_1) = e^T(v_2)$ if $e^T(v_1) = e^T(v_2)$ this contradction with $v_1 \neq v_2$, then
$e^T(v_1) \neq e^T(v_2)$, *but* V_T is separated S-act then $\exists$ $e \neq s \in S$ such that $e^x . e^T(v_1) \neq e^x . e^T(v_2)$, Since e^T is operator then e^T is linear transformation, this gives
$e^x . e^T(v_1) = e^x . e^T(v_2) \longrightarrow$
$e^T(e^x . v_1) \neq e^T(e^x . v_2)$, this mean
$(1 + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \cdots)(e^x . v_1) \neq \left(1 + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \cdots\right)(e^x . v_2)$, we get
$e^x . v_1 + T(e^x . v_1) + \frac{T^2}{2!}(e^x . v_1) + \frac{T^3}{3!}(e^x . v_1) + \cdots \neq e^x . v_2 + T(e^x . v_2) + \frac{T^2}{2!}(e^x . v_2) + \frac{T^3}{3!}(e^x . v_2) + \cdots$
$e^x . v_1 \neq e^x . v_2$, $T(e^x . v_1) \neq T(e^x . v_2)$, but T is operator, hence $e^x . T(v_1) \neq e^x . T(v_2)$,but $e^x \neq 0$ then $T(v_1) \neq T(v_2)$, *and* $\frac{T^2}{2!}(e^x . v_1) \neq \frac{T^2}{2!}(e^x . v_2)$
$\frac{T}{2!}\left(T(e^x . v_1)\right) \neq \frac{T}{2!}\left(T(e^x . v_2)\right)$, so $T(e^x . v_1) \neq T(e^x . v_2)$, by using the same way we get $T(v_1) \neq T(v_2)$,……,then we prove T is one to one.

In the following proposition we introduce another conditions to get V is cyclic which is every cyclic is finite dimensional.

Recall that M_s be an S-system and H a subset of S .then H is called reductive on M_s if and only if for each a,b ϵM_S , $ah = bh$ for all h$\in H$ implies a=b,an singular relation ψ_M on M_S by the set $\{(a,b)\epsilon M \times M \setminus ah = bh\ for\ some\ h\epsilon H for\ some\ reductive\ subset\ H of S\}$. [8]

Proposition 12. If V_T is singular S-act then V is generated by one element.

Proof: since V_T is singular S-act then $\psi_{V=}\{(e^T(v_1), e^T(v_2))\epsilon V_T \times V_T \setminus e^x.e^T(v_1) = e^x.e^T(v_2)\ for\ some\ e^x\epsilon H\ for\ some\ reductive\ subset\ H of\ S\}$ Then $e^x.e^T(v_1) = e^x.e^T(v_2)$… (1), hence $e^T(e^x.v_1) = e^T(e^x.v_2)$, since $e^T$is one to one, therefore $e^x.v_1 = e^x.v_2$, then

$e^x.v_1 + (-1)e^x.v_{2=}0$… (2), but H is reductive subset of S, then by (1). We find $e^T(v_1) = e^T(v_2)$, but $e^T$is one to one, then $v_1= v_2$, we replies $v_1= v_2$ on (2),then $e^x.v_1 + (-1)e^x.v_{1=}0$, then V is generated by one element, this give V is finite dimension.

Recall that an act A_S is called torsion free if for any x, y $\in A_S$,and for any right cancellable element c$\in S$, the equality xc = yc this implies x=y. [1]

Proposition 13. For any bounded operator T then V_T is torsion free S-act.

Proof: Assume $e^x.e^T(v_1) = e^x.e^T(v_2)$,∀ e^x is cancellable element in S, this gives $e^T(e^x.v_1) = e^T(e^x.v_2)$, since e^T is one to one, this gives $(e^x.v_1) = (e^x.v_2)$, thus $v_1= v_2$, then either $e^T(v_1) = e^T(v_2)$ or $e^T(v_1) \neq e^T(v_2)$, if $e^T(v_1) \neq e^T(v_2)$, we get a contradiction with $v_1= v_2$, then $e^T(v_1) = e^T(v_2)$,thus V_T is torsion free.

In the following proposition we introduce sufficient and necessary conditions on V in order that V_T is a Noetherian S-act, we need some definitions to clear this proposition.

Recall that a monid S is right Noetherian if and only if it satisfies the asending chain condition for right ideals, this mean for every a sending chain $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_n \subseteq K_{n+1} \subseteq \ldots$ Of its right ideals, there exists n ϵN such that K_n=K_{n+1}=…. [1]

In [9] show that an act A_S is called Noetherian if it satisfies the asending chain condition for right sub acts, this mean for every asending chain $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_n \subseteq K_{n+1} \subseteq \ldots$ Of its right sub acts, there exists n ϵN such that K_n=K_{n+1}=…

Theorem 14. If S is Noetherian and A is finitely generated S-act then A is Noetherain S-act. **[9]**

Proposition 15. Let V is a finite dimensional normed space and T is similar to any operator J from R to R then V_T is Noetherian S-act if and only if S is Noetherian.

Proof: Since V is finite dimensional then V_T is finitely generated S-act by remark (8), therefore V_T is Noetherian S-act, by theorem (14).

Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_n \subseteq K_{n+1} \subseteq \ldots$be any ascending sequence ideals of S, then it is a sequence of subacts of S_S denoted S_J , where J any operator from R to R, since T is similar to J, then by **Proposition (6)**, V_T is isomorphic to S_J , thus S_J is Noetherian S- act, therefore this sequence is finite, then S is Noetherian.

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الاثر التابع للمؤثر

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الخلاصة:

في هذا البحث تم دراسة خصائص المؤثرات الخطية من خلال الفضاءات S – المؤثر المرافقة لتلك المؤثرات وبالعكس حيث نظرنا الى بعض المؤثرات الخطية مثل المؤثرات الاحادية و المؤثرات الشاملة. من جهة اخرى نظرنا في بعض المفاهيم النظرية للفضاءات المرافقة للمؤثرات الخطية واثرها على طبيعة هذه المؤثرات مثل الاثر المخلص، الاثر منتهي التولد، الاثر الفريد، الاثر الفاصل، الاثر الملتوي الحر و الاثر النوتيري كما حاولنا معرفة صفات المؤثر T التي تجعل فضاءات الاثر V_T تمتلك احدى او بعض تلك الصفات .

الكلمات المفتاحية: الاثر المخلص، الاثر المنتهي التولد، الاثر الفريد، الاثر الفاصل، الاثر الملتوي الحر و الاثر النوتيري.