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## CL-duo modules

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### Abstract:

In this paper, we introduce and study a new concept (up to our knowledge) named CL-duo modules, which is bigger than that of duo modules, and smaller than weak duo module which is given by Ozcan and Harmanci. Several properties are investigated. Also we consider some characterizations of CL-duo modules. Moreover, many relationships are given for this class of modules with other related classes of modules such as weak duo modules, P-duo modules.

**Key words:** CL-duo modules, Duo modules, Weak duo modules, P-duo module, closed submodules, fully invariant submodules.

### Introduction:

Throughout this paper all rings are commutative with identity and all modules are unitary left R-modules. A submodule  $A$  of  $U$  is said to be fully invariant, if  $f(A) \subseteq A$  for every endomorphism  $f$  of  $U$  [1]. An R-module  $U$  is called duo, if every submodule of  $U$  is a fully invariant [2]. A ring  $R$  is called duo, if it is duo R-module. It is clear that every commutative ring is a duo ring. Ozcan and Harmanci in [2] introduced and studied weak duo modules, where a module  $U$  is called weak duo module if every direct summand of  $U$  is fully invariant. Inaam in [3] introduced purely duo modules (briefly P-duo module), which is a module in which every pure submodule is fully invariant. A submodule  $A$  of  $U$  is called essential (briefly  $A \leq_e U$ ), if the intersection between  $A$  and any nonzero submodule

of  $U$  is not equal to zero [4]. A submodule  $A$  of  $U$  is called closed (briefly  $A \leq_c U$ ), if  $N$  has no proper essential extension in  $U$ , i.e if  $A \leq_e L \leq U$ , then  $A=L$  [1].

In this paper, we introduce a new class of modules (up to our knowledge) named CL-duo module, where a module  $U$  is said to be CL-duo, if each closed submodule in  $U$  is fully invariant. This class of modules lies between duo modules and weak duo. In fact it is well-known that every direct summand of any module  $U$  is closed; hence, every CL-duo module is weak duo module. In section two, the main properties of CL-duo modules are investigated. In section three, we find other characterizations of CL-duo and in section four we study the direct sum of this class of modules. In section five, we give the hereditary

property between any ring  $R$  and CL-duo  $R$ -modules. The last section of this paper is devoted to study the relationships of CL-modules with some other related modules such as weak duo and P-duo modules. So we give some conditions under which CL-duo modules is equivalent to weak duo and P-duo modules.

**1. CL-duo modules**

This section is devoted to study the main properties of CL-duo modules.

**Definition (1.1):** An  $R$ -module  $U$  is said to be CL-duo module, if each closed submodule in  $U$  is fully invariant. A ring  $R$  is called CL-duo, if  $R$  is CL-duo  $R$ -module.

**Examples and Remarks (1.2):**

1. It is clear that every duo module is CL-duo, but the converse is not true in general. For example, the  $Z$ -module  $Q$  is CL-duo module, in fact  $Q$  has only two closed submodules,  $0$  and  $Q$  itself which are both fully invariant. On the other hand,  $Q$  is not duo module since the submodule  $Z$  is not fully invariant submodule of  $Q$ , in fact there exists a homomorphism  $f: Q \rightarrow Q$  which is defined by  $f(x) = \frac{1}{8}x \forall x \in Z$ , and clearly  $f(Z) \not\subseteq Z$ .
2. Since there is no direct implication between closed and pure submodule, so we think that CL-duo module and P-duo module are independent, but we don't have examples about that belief.
3. Any commutative ring is CL-duo ring.
4. Every uniform module is CL-duo module, where a non-zero module  $U$  is called uniform if every two non-zero submodule of  $U$  have non-zero intersection [1]. In fact, in a uniform module  $U$ , the only closed submodules are  $(0)$  and  $U$ , and both of them are fully invariant.
5. Since every direct summand of any module  $U$  is closed in  $U$ , then every

CL-duo modules weak duo module. We think that weak duo module is not necessary CL-duo module, but we don't have examples for that thing.

6.  $U=Z_4 \oplus Z_4$  as  $Z$ -module is not CL-duo, since there exists a homomorphism  $f: Z_4 \oplus Z_4 \rightarrow Z_4 \oplus Z_4$  defined by  $f(\bar{x}, \bar{y}) = (\bar{y}, \bar{x})$  for each  $(\bar{x}, \bar{y}) \in Z_4 \oplus Z_4$ . The submodule  $N = (0) \oplus Z_4$  of  $U$  is closed but it is not fully invariant, since  $f(N) = Z_4 \oplus (0) \not\subseteq N$ .
7. It is clear that every multiplication  $R$ -module is a CL-duo module. In particular, every cyclic module over commutative ring is a CL-duo module.
8. A module  $U$  is a CL-duo  $R$ -module if and only if  $U$  is a CL-duo  $\bar{R}$ -module, where  $\bar{R} = \frac{R}{\text{ann } U}$ .

The converse of remark (1.2)(1) is true if  $U$  is a semisimple module, as in the following.

9. A semisimple CL-duo module is a duo module.

**Proof (9):** Let  $A$  be any submodule of  $U$ . By assumption  $A$  is a direct summand of  $U$ ; hence, it is a closed submodule. But  $U$  is a CL-duo module, then  $A$  is fully invariant, and we obtained the result.

**Proposition (1.3):** A direct summand of CL-duo module is CL-duo.

**Proof:** Assume that  $U$  is a CL-duo  $R$ -modules, and let  $K$  be a direct summand of  $U$ . Let  $N$  be a closed submodule in  $K$  and consider the projection homomorphism  $\rho: U \rightarrow K$ , and the injection homomorphism  $J: K \rightarrow U$ . Let  $f: K \rightarrow K$ , then  $h \equiv jf\rho \in \text{End}(U)$ , where  $\text{End}(U)$  is the endomorphism of  $U$ . Since  $K$  is direct summand of  $U$ , then  $K \subseteq_c U$ . So by the transitivity of the closed submodules,  $N \subseteq_c U$  [1]. But  $U$  is a CL-duo module, thus  $h(N) \subseteq N$ , that is  $(jf\rho)(N) = f(N) \subseteq N$ . Hence  $K$  is a CL-duo module.

It is known that a submodule of duo module is not duo module, in the following example we see that also for the class of CL-duo modules.

**Example (1.4):** Consider the ring  $R=Q \oplus Q^2$ , where  $Q$  is the set of all rational numbers. Define  $(.)$  on  $R$  as follows:

$(a, b).(c, d)=(ac, ad+cb)$  for each  $a, c \in Q$  and  $b, d \in Q^2$ , where  $Q^2 \cong Q \times Q$ .

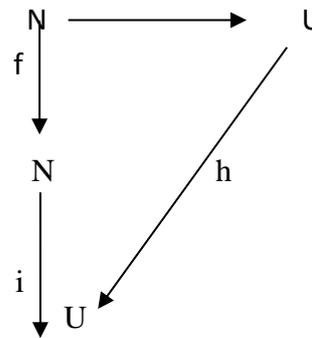
$R$  is commutative, so  $R$  is a CL-duo ring; hence it is CL-duo  $R$ -module. While the submodule  $N = Q \oplus Q$  is not CL-duo module, since the submodule  $K=\{(q,0) \mid q \in Q\}$  is a closed submodule in  $N$ , but it is not fully invariant, since there exists a homomorphism  $f: N \rightarrow N$  defined by  $f(x,y)=f(y,x) \forall x,y \in N$ , and  $f(K) = \{(0,q) \mid q \in Q\} \not\subseteq K$ .

Let  $V$  be any  $R$ -module. An  $R$ -module  $U$  is called  $V$ - $c$ -injective, provided that for every closed submodule  $K$  of  $U$  and for every homomorphism  $\phi: K \rightarrow V$  can be lifted to a homomorphism  $\theta: U \rightarrow V$  [5]. A module  $U$  is called self  $c$ -injective module if  $U$  is  $U$ - $c$ -injective module.  $U$  is called  $V$ -projective in case for each submodule  $K$  of  $U$ , every homomorphism  $f: V \rightarrow \frac{U}{K}$  can be lifted to a homomorphism  $g: U \rightarrow V$ , furthermore,  $U$  is self projective if  $U$  is  $U$ -projective [5].

We saw in example (1.4) that a submodule of a CL-duo module was not necessary CL-duo module. Also we think that a quotient of CL-duo module may not be CL-duo module, but we don't find an example about that thing. However, these are true under some conditions as the following two propositions show.

**Proposition (1.5):** Let  $U$  be a CL-duo module. If  $U$  is a  $c$ -injective module, then every closed submodule of  $U$  is a CL-duo module.

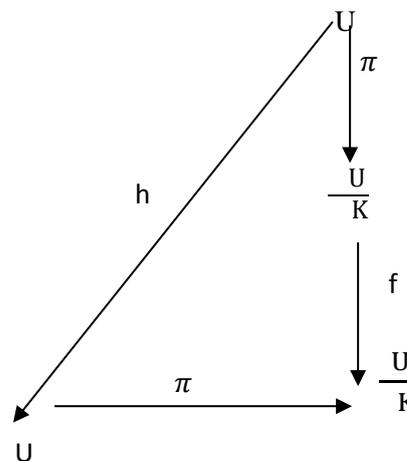
**Proof:** Let  $N$  be a closed submodule in  $U$ , and assume that  $L$  is a closed submodule in  $N$ . Let  $f \in \text{End}(N)$ . Consider the following diagram:



Where  $i$  is the inclusion homomorphism, since  $N \leq_c U$  and  $U$  is self  $c$ -injective, then there exists a homomorphism  $h: U \rightarrow U$  such that  $hi = if$ . Now,  $(hi)(L) = h(L)$ . Since  $L$  is a closed submodule in  $N$  and  $N$  is a closed submodule in  $U$ , then  $L$  is a closed submodule in  $U$  [1]. But  $U$  is a CL-duo module; therefore,  $h(L) \subseteq L$ . On the other hand,  $(hi)(L) = (if)(L) = f(L)$ , thus  $h(L) = f(L) \subseteq L$ . That is  $N$  is CL-duo module.

**Proposition (1.6):** Let  $U$  be a CL-duo module. If  $U$  is a self projective module, then  $\frac{U}{K}$  is a CL-duo module for each closed submodule  $K$  in  $U$ .

**Proof:** Let  $\frac{H}{K} \leq_c \frac{U}{K}$  and let  $f \in \text{End}(\frac{U}{K})$ . Let  $\pi: U \rightarrow \frac{U}{K}$  be the natural epimorphism. Consider the following diagram:



Where  $\pi$  is the natural epimorphism. Since  $\frac{U}{K}$  is a self projective module, so there exists  $h: U \rightarrow U$  such that  $\pi h = f \pi$ . Now,  $(\pi h)(u) =$

$h(u)+K = f(u+K) \forall u \in U$ . But  $\frac{H}{K}$  is a closed submodule in  $\frac{U}{K}$  and  $K$  is closed submodule in  $U$ ; thus,  $H$  is closed submodule in  $U$  [6, Prop.6.28, P.218]. Since  $U$  is CL-duo module, then  $h(H) \subseteq H$ , hence  $f(\frac{H}{K}) = h(H)+K \subseteq \frac{H}{K}$ . That is  $\frac{U}{K}$  is a CL-duo module.

It is well-known that the intersection of any two closed submodules is not necessary closed submodule. The following proposition deals with this fact. Before that, we need the following lemma which appeared in [2].

**Lemma (1.7):** let  $U$  be an  $R$ -module such that  $U = \bigoplus_{i \in I} U_i$ . If  $N$  is a fully invariant submodule of  $U$ , then  $N = \bigoplus_{i \in I} (N \cap U_i)$ .

**Proposition (1.8):** The intersection of any closed submodule in a CL-duo module  $U$  with any direct summand of  $U$  is closed in  $U$ .

**Proof:** Let  $N_1$  be a closed submodule in  $U$ , and  $N_2$  be a direct summand of  $U$ . So there exists a submodule  $L$  of  $U$  such that  $U = N_2 \oplus L$ . On the other hand,  $N_1$  is a closed submodule in  $U$ , and  $U$  is CL-duo module; thus,  $N_1$  is fully invariant. By lemma (1.5),  $N_1 = (N_1 \cap N_2) \oplus (N_1 \cap L)$ . That is  $(N_1 \cap N_2)$  is a direct summand of  $N_1$ . This implies that  $(N_1 \cap N_2)$  is closed in  $N_1$ . But  $N_1$  is closed in  $U$ ; thus,  $(N_1 \cap N_2)$  is a closed submodule in  $U$ .

**Proposition (1.9):** The sum of any closed submodule in a CL-duo module  $U$  with any direct summand of  $U$  is fully invariant.

**Proof:** Let  $N_1$  be a closed submodule in  $U$ , and  $N_2$  be a direct summand of  $U$ . So  $U = N_2 \oplus L$  for some submodule  $L$  of  $U$ . Since  $N_1$  is a closed submodule in  $U$ , then by assumption  $N_1$  is fully invariant, and by lemma (1.5),  $N_1 = (N_2 \cap N_1) \oplus (L \cap N_1)$ . Now,  $N_1 + N_2 = (N_2 \cap N_1) \oplus (L \cap N_1) + N_2$ . But  $(L \cap N_1)$  is a direct

summand of  $N_1$ , and since  $N_1$  is a closed submodule in  $U$ , so  $L \cap N_1$  is a closed submodule in  $U$ . It follows that  $L \cap N_1$  is fully invariant in  $U$ . As  $N_2$  is a direct summand of  $U$ , hence  $N_2$  is fully invariant in  $U$ . Thus  $(L \cap N_1) + N_2$  is a fully invariant submodule of  $U$ , i.e,  $N_1 + N_2$  is fully invariant in  $U$ .

## 2. Characterizations of CL-duo modules

In this section we give some characterizations of CL-duo modules and other characterizations in certain types of modules. We start by the following proposition.

**Proposition (2.1):** An  $R$ -module  $U$  is CL-duo module if and only if for each  $f \in \text{End}(U)$  and for each cyclic closed submodule  $(u)$  of  $U$  there exists  $r \in R$  such that  $f(u) = ru$ .

**Proof:**  $\Rightarrow$ ) Let  $f \in \text{End}(U)$ . Let  $(u)$  be a cyclic closed submodule in  $U$ . Since  $U$  is a CL-duo module, so  $(u)$  is fully invariant that is  $f(u) \subseteq (u) = Ru$ . Hence, there exists  $t \in R$  such that  $f(u) = tu$ .

$\Leftarrow$ ) Let  $N \subseteq_c U$ , and let  $f \in \text{End}(U)$ . For each element  $n \in N$ ,  $f(n) \in U$ . By assumption, there exists  $t \in R$  such that  $f(n) = tn \in N$ . Hence  $f(N) \subseteq N$ , i.e  $U$  is a CL-duo module.

Recall that a ring  $R$  is called Bezout; if every finitely generated ideal of  $R$  is cyclic [6].

**Corollary (2.2):** Let  $R$  be a Bezout ring, then  $R$  is a CL-duo ring if and only if for each  $f \in \text{End}(R)$ , and for each finitely generated closed ideal  $A$  of  $R$ ,  $\exists r \in R$  such that  $f(A) = rA$ .

We can modify proposition (2.1) with extra conditions to characterize CL-duo module  $U$  such that the existence of the element  $r$  in  $R$  for all each cyclic closed submodule of  $U$ , as the following theorem shows. Before that an  $R$ -module  $U$  is said to be torsion free, if for each non zero element  $x \in U$  and  $\forall r \in R, 0 \neq r, 0 \neq r x$  [4].

**Theorem (2.3):** Let  $U$  be a torsion free module over an integral domain  $R$ . Then  $U$  is a CL-duo module if and only if for each  $f \in \text{End}(U) \exists d \in R$  such that  $f(k) = dk$  for every cyclic closed submodule  $(k)$  of  $U$ .

**Proof:**  $\Rightarrow$ ) Assume that  $U$  is a CL-duo module, and let  $f \in \text{End}(U)$ . Now, suppose that  $(x)$  and  $(y)$  be cyclic closed submodules of  $U$  with  $x \neq y$ . By proposition (2.1), there  $\exists r, s \in R$  such that  $f(x) = rx$  and  $f(y) = sy$ . We have two cases: either  $(x) \cap (y) = (0)$ , then  $f(x+y) = e(x+y)$ , where  $e \in R$ . On the other hand,  $f(x+y) = f(x) + f(y) = rx + sy$ . Thus  $(e-r)x = (s-e)y \in (x) \cap (y) = (0)$ . Thus  $(e-r)x = 0$ . But  $U$  is a torsion free module; therefore,  $e=r$ , and for the same reason  $s=e$ , this implies that  $r=s$ . The other case is  $(x) \cap (y) \neq (0)$ . Let  $0 \neq w \in (x) \cap (y)$ , and let  $f(w) = tw$ ,  $t \in R$ , then  $w = ex = vx$  for some  $e, v \in R$ . On the other hand,  $f(w) = tw = f(ex) = ef(x) = erx$ . This implies that  $tex = erx$ , hence  $(te-er)x = 0$ . But  $U$  is torsion free; therefore,  $te-er=0$ . Since  $R$  is a commutative ring, then  $e(t-r)=0$ . But  $R$  is an integral domain; therefore,  $t=r$ . That is in both cases we get the desired which is there exists one element  $d$  in  $R$  with  $f(k) = dk$  cyclic closed submodule  $(k)$  of  $U$ .

$\Leftarrow$ ) It follows by the proposition (2.1).

As a consequence of corollary (2.3), one can obtain the following corollary.

**Corollary (2.4):** Let  $U$  be a torsion free module over an integral domain  $R$ . Then  $U$  is a CL-duo module if and only if  $\text{End}(U) \cong R$ .

We can rewrite the corollaries (2.3) and (2.4) as follows.

**Theorem (2.5):** Let  $U$  be a torsion free module over an integral domain  $R$ , then the following statements are equivalent.

1.  $U$  is a CL-duo module.
2. for each  $f \in \text{End}(U) \exists t \in R$  such that  $f(u) = tu$  for each cyclic closed submodule  $(u)$  of  $U$ .

3.  $\text{End}(U) \cong R$ .

In example (1.4), we see that a submodule of CL-duo is not necessary CL-duo module. However, this property is true under certain condition. Before that it is well-known that any module  $U$  is called countably generated, if  $U$  can be generated by a countable set.

**Proposition (2.6):** If every countably generated submodule of a module  $U$  is CL-duo, then  $U$  is CL-duo.

**Proof:** Assume that  $(u)$  is any cyclic closed submodule in  $U$ , and let  $f \in \text{End}(U)$ . Consider the sum of submodules of  $U$ :

$$Ru + R(f(u)) + R(f^2(u)) + \dots$$

This sum of submodules is a countably generated submodule of  $U$ . If we denote to that sum by  $K$ , and restrict  $f$  to  $K$ , then we have  $f|_K \in \text{End}(K)$ . By [6, P.215, Prop (6.24)], the cyclic submodule  $(u)$  is closed in  $N$ , and since  $K$  is a CL-duo module, so by Proposition (2.1),  $\exists t \in R$  such that  $f(u) = tu$ . Again Applying proposition (2.1) to obtain the result which is  $U$  is a CL-duo module.

### 3. The direct sum of CL-duo modules

The direct sum of CL-duo module is not necessary CL-duo module, for example,  $Z$  is CL-duo  $Z$ -module, but we will see later on, that the  $Z$ -module  $Z \oplus Z$  is not CL-duo. This section is devoted to study the cases in which the direct sum of CL-modules is CL-module.

**Proposition (3.1):** Assume that  $U$  is a CL-duo module. If  $U$  is a direct sum of  $U_1$  and  $U_2$ , then  $\text{Hom}(U_1, U_2) = 0$ .

**Proof:** Since  $U_1$  is a direct summand of  $U$ , then  $U_1$  is closed in  $U$ . but  $U$  is a CL-duo module, so  $U_1$  is fully invariant. This implies that  $\text{Hom}(U_1, U_2) = 0$  [2, Lemma1.9].

**Example (3.2):** The  $Z$ -module  $Z \oplus Z$  is not CL-duo module. In fact, if  $Z \oplus Z$  is CL-duo, then  $\text{Hom}(Z, Z)$  must be equal to  $(0)$  which is not true.

**Proposition (3.3):** Let  $U=U_1\oplus U_2$  be a direct sum of submodules  $U_1$  and  $U_2$  such that  $\text{ann } U_1+\text{ann } U_2=R$ . Then  $U$  is a CL-duo module if and only if  $U_1$  and  $U_2$  are CL-duo modules with  $\text{Hom}(U_i, U_j) = 0 \forall i,j=1,2$ , with  $i\neq j$ .

**Proof:**  $\Rightarrow$ ) It follows from proposition (1.3) and proposition (3.1).

$\Leftarrow$ ) Assume that  $N$  is a closed submodule in  $U$ . Since  $\text{ann } U_1+\text{ann } U_2 = R$ , so by [7],  $N=A_1\oplus A_2$  for some  $A_1\leq U_1$  and  $A_2\leq U_2$ . It can easily show that  $A_1$  is a closed submodule in  $U_1$  and  $A_2$  is closed in  $U_2$ . Now, let  $f\in\text{End}(U)$ , then  $\rho_j f i_j \in \text{End}(U_j)$ , where  $j=1,2$  and  $\rho_j$  is the projection homomorphism and  $i_j$  is the inclusion homomorphism. Since  $U_j$  a CL-duo module  $\forall j=1,2$ , then  $(\rho_j f i_j)(A_j) \subseteq A_j$ . So we obtain  $f(A_1)+f(A_2) \subseteq (\rho_1 f i_1)(A_1) + (\rho_2 f i_2)(A_2) = \sum_{j=1}^2 (\rho_j f i_j)(A_j)$ . But  $f(A_1)+f(A_2) = f(A_1\oplus A_2)=f(N)$ , therefore  $f(N) \subseteq \sum_{j=1}^2 (\rho_j f i_j)(A_j) \subseteq \sum_{j=1}^2 (A_j) = N$ . Thus  $f(N) \subseteq N$ , and the result is obtained.

**Theorem (3.4):** Let  $U=\bigoplus_{i\in I} U_i$  with  $U_i$  submodule of  $U \forall i\in I$ , then  $U$  is a CL-duo module if and only if:

1.  $U_i$  is a CL-duo module  $\forall i\in I$ .
2.  $\text{Hom}(U_i, U_j)=0 \forall i,j\in I$ , with  $i\neq j$ .
3.  $N = \bigoplus_{i\in I} (N\cap U_i)$  for each closed submodule  $N$  of  $U$ .

**Proof:**  $\Rightarrow$ ) It follows from propositions (1.3), (3.1) and lemma (1.5).

$\Leftarrow$ ) Let  $N$  be a closed submodule in  $U$ . By (3),  $N = \bigoplus_{j\in I} (N\cap U_j)$ . Since  $N\cap U_j$  is a direct summand in  $N$ , then  $N\cap U_j$  is a closed in  $N$ . But  $N$  is a closed in  $U$ , therefore  $N\cap U_j$  is a closed in  $U$  [6], hence  $N\cap U_j$  is closed in  $U_j$  [2]. Let  $f\in\text{End}(U)$ . Consider the following sequence:

$$\begin{array}{ccccccc}
 & & f & & \rho_j & & \\
 i_j & & & & & & \\
 U_j & \longrightarrow & U & \longrightarrow & U & \longrightarrow & U_j
 \end{array}$$

Where  $i_j$  is the inclusion homomorphism and  $\rho_j$  is the projection homomorphism. So  $\rho_j f i_j \in \text{End}(U_j)$ . Since  $N\cap U_j$  is a closed in  $U_j$ ,  $(\rho_j f i_j)(N\cap U_j) \subseteq N\cap U_j \forall j\in I$ . By (2),  $\text{Hom}(U_i, U_j)=0 \forall i,j\in I$ , with  $i\neq j$ , this implies that  $(\rho_r f i_r)(N\cap U_i)=0 \forall r,j$  such that  $r\neq j$ . Hence  $f(N) = f(\bigoplus_{j\in I} (N\cap U_j)) \subseteq \bigoplus_{j\in I} ((\rho_j f i_j)(N\cap U_j)) \subseteq \bigoplus_{j\in I} (N\cap U_j) = N$ . So  $f(N) \subseteq N$ , that is  $U$  is a CL-duo module.

Recall that a module  $U$  satisfies the closed intersection property (briefly CIP), if for each direct summand  $L$  and  $N$ ,  $L\cap N$  is closed submodule in  $U$  [8]. In fact, there is no direct implication between CL-duo module and CIP, but we can prove the following.

**Proposition (3.5):** Let  $U = \bigoplus_{i\in I} U_i$ , where  $U_i$  is a submodule of  $U \forall i\in I$ . If the conditions hold:

1.  $\bigoplus_{i\in K} U_i$  is a CL-duo module for every finite subset  $K$  of  $I$ .
2.  $U$  satisfies CIP.

Then  $U$  is a CL-duo module.

**Proof:** We will satisfy the conditions of theorem (3.4), so let  $N$  be a closed submodule in  $U$ , and let  $x$  be any element in  $N$ , then  $x \in \bigoplus_{i\in K} U_i \equiv L$ . Put

$\bigoplus_{i\in K} U_i \equiv L$ , where  $K$  is a finite subset of  $I$ , so  $x \in N\cap L$ . Since  $\bigoplus_{i\in K} U_i$  is closed

submodule in  $U$ , then by condition (2),  $N\cap L$  is closed submodule in  $U$ . But  $N\cap L \subseteq L$ ; therefore,  $N\cap L$  is closed submodule in  $L$ . Since  $L$  is a CL-duo module, then  $N\cap L$  is a fully invariant submodule of  $U$ . By lemma (1.5),  $N\cap L = \bigoplus_{i\in K} [(N\cap L)\cap U_i] = \bigoplus_{i\in K} (N\cap U_i)$ . This

implies that  $x \in \bigoplus_{i\in K} (N\cap U_i)$ , and so  $x \in \bigoplus_{i\in I} (N\cap U_i)$ . On the other hand, it is clear that  $\bigoplus_{i\in I} (N\cap U_i) \subseteq N$ , thus  $N = \bigoplus_{i\in I} (N\cap U_i)$ . Moreover, by condition (1),

$U_i$  is a CL-duo module  $\forall i\in K$ . Also  $U_i\oplus U_j$  is a CL-duo module  $\forall i,j\in K, i\neq j$ .

By proposition (3.1),  $\text{Hom}(U_i, U_j) = 0$ , and by theorem (3.4),  $U$  is a CL-duo module.

We can rewrite proposition (1.1.40) in [8] as follows:

**Proposition (3.6):** If  $U = \bigoplus_{i \in I} U_i$  be a CL-duo module, then  $U$  satisfies CIP if and only if  $U_i$  satisfies CIP  $\forall i \in I$ .

#### 4. The hereditary of CL-duo property

This section is devoted to study the hereditary of CL-duo property between the ring  $R$  and  $R$ -modules. We start by the following proposition. Before that, an  $R$ -module  $U$  is called multiplication, if for every submodule  $N$  of  $U$ , there exists an ideal  $I$  of  $R$  such that  $N = IU$  [9].

**Proposition (4.1):** If  $U$  is a finitely generated, faithful and multiplication module, then the localization  $U_P$  is a CL-duo module for each prime ideal  $P$  of  $R$ .

**Proof:** Since  $U$  is a finitely generated multiplication module, then  $U_P$  is a multiplication  $R_P$ -module [9]. But by remark (1.2)(7), every multiplication module is CL-duo module, thus  $U_P$  is a CL-duo module.

**Proposition (4.2):** If there exists a finitely generated and multiplication module over a ring  $R$ , then  $R_P$  is a CL-duo ring.

**Proof:** Assume that there exists a finitely generated, faithful and multiplication module  $U$  over a ring  $R$ . By proposition (4.1),  $U_P$  is a CL-duo module. Moreover,  $U_P$  is cyclic [9], thus  $U_P \cong \frac{R_P}{\text{ann}U_P}$  [10, P.35]. Clearly  $U_P$  is faithful, so  $U_P \cong R_P$ . Thus  $R_P$  is a CL-duo  $R_P$ -module.

**Proposition (4.3):** Every projective  $R$ -module is CL-duo module if and only if  $\bigoplus_{i \in I} R$  is CL-duo ring for every index set.

**Proof:**  $\Rightarrow$ ) Assume that  $\bigoplus_{i \in I} R$  is a CL-duo module, and let  $U$  be projective over the ring  $R$ . So there exists a free  $R$ -

module  $T$  and an epimorphism  $\varphi: T \rightarrow U$ . Since  $T$  is free, thus  $T \cong \bigoplus_{i \in I} R$  for some an index set  $I$  [4, lemma (4.4.1), P.88]. Consider the following sequence:

$$0 \longrightarrow \ker \varphi \xrightarrow{i} \bigoplus_{i \in I} R \xrightarrow{\varphi} U \longrightarrow 0$$

where  $i$  is the inclusion homomorphism. But  $U$  is a projective module; therefore, the sequence splits [4, Th.(4.4.1), P.90]. This implies that  $\bigoplus_{i \in I} R \cong \ker \varphi \oplus U$ . On the other hand,  $\bigoplus_{i \in I} R$  is a CL-duo module, so by proposition (1.1),  $U$  is a CL-duo module.

$\Leftarrow$ ) It is clear that  $R$  is an  $R$ -projective module, and by [9],  $\bigoplus_{i \in I} R$  is a projective  $R$ -module. By assumption,  $\bigoplus_{i \in I} R$  is a CL-duo module.

#### 5. CL-duo modules and related concepts

This section includes the study of the relationship of CL-duo modules with other related concepts. We start by the following remark. Firstly, an  $R$ -module  $U$  is called fully stable module, if every submodule of  $U$  is stable, where a submodule  $A$  of  $U$  is said to be stable, if  $\varphi(A) \subseteq A$  for each homomorphism  $\varphi$  of  $A$  into  $U$  [11].

**Theorem (5.1):** For a semisimple  $R$ -module, the following implications hold:  
 CL-duo module  $\Leftrightarrow$  Duo module  $\Leftrightarrow$   
 Weak duo module  $\Leftrightarrow$  P-duo module  $\Leftrightarrow$  Fully stable module

**Proof:** It is clear.

An  $R$ -module  $U$  is called extending, if each submodule of  $U$  is an essential in a direct summand [12]. In the following theorem, we put condition under which weak duo module be CL-duo module.

**Theorem (5.2):** In the class of extending modules, CL-duo module is equivalent to a weak duo module.

**Proof:** Assume that  $U$  is a CL-duo module, and let  $A$  be a direct summand

of  $U$ . So  $A$  is a closed submodule in  $U$  [1, P.18]. Since  $U$  is a CL-duo module, then  $A$  is fully invariant, that is  $U$  is weak duo. Conversely, let  $A$  be a closed submodule in  $U$ . Since  $U$  is an extending module, then  $A$  is a direct summand. But  $U$  is a weak module, therefore  $A$  is a fully invariant submodule, and we are done.

It is well-known that the direct sum of extending module is not necessary extending module. However, under the class of CL-duo module that is true. In fact this result is obtained from [12, Prop. (3.7), P.27], and we can rewrite it as follows.

**Theorem (5.3):** Let  $U = \bigoplus_{i \in I} U_i$ , where  $U_i$  be a submodules of  $U \forall i \in I$ . If  $U$  is an extending module, then  $U_i$  is extending  $\forall i \in I$ . The converse is true whenever  $U$  is a CL-duo module.

It is known that a ring  $R$  is called principal ideal ring (briefly PIR), if  $R$  is commutative with identity and every ideal of  $R$  is principal. It is worth mentioning that we can't find a direct implication between CL-duo and P-duo module. However under certain conditions, we obtain the following propositions, before that recall that an  $R$ -module  $U$  is called purely extending module, if every submodule of  $U$  is essential in pure submodule of  $U$  [12].

**Proposition (5.4):** Every CL-duo module over PIR is a P-duo module.

**Proof:** Let  $U$  be CL-duo module, and  $A$  be a pure submodule of  $U$ . Since  $R$  is a PIR, then  $A$  is closed in  $U$  [6, exc.15, P.242]. But  $U$  is CL-duo, so  $A$  is fully invariant. That is  $U$  is a P-duo module

**Proposition (5.5):** Every P-duo module which is purely extending module is CL-duo.

**Proof:** Assume that  $U$  is a P-duo module, and Let  $A$  be a closed submodule of  $U$ . Since  $U$  is a purely extending module, then  $A$  is a pure submodule of  $U$  [12, Th.2.2, P.39]. But

$U$  is a P-duo module; therefore,  $A$  is fully invariant. That is  $U$  is a CL-duo module.

From proposition (5.4) and proposition (5.5) we obtain the following theorem.

**Theorem (5.6):** Let  $U$  be purely extending module over PIR. Then  $U$  is a CL-duo module if and only if  $U$  is P-duo.

A module  $U$  is called F-regular, if every submodule of  $U$  is pure [13]. In the following proposition we use this class of modules.

**Proposition (5.7):** Suppose that  $U$  is an F-regular module over PIR. If  $U$  is CL-duo, then  $U$  is a duo module.

**Proof:** Let  $A$  be a submodule of  $U$ . Since  $U$  is an F-regular module, then  $A$  is a pure submodule of  $U$ . By [6, exc.15, P.242],  $A$  is closed. But  $U$  is CL-duo; therefore,  $A$  is fully invariant. That is  $U$  is a duo module.

From proposition (5.7) and proposition 19 in [3], we obtain the following theorem.

**Theorem (5.8):** Let  $U$  be an F-regular module over PIR. Consider the following statements:

1.  $U$  is a CL-duo module.
2.  $U$  is a P-duo module.
3.  $U$  is a duo module.
4.  $U$  is weak duo.

Then: (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4), and if  $U$  is an extending module, then (4)  $\Rightarrow$  (1).

**Proof:** (1)  $\Rightarrow$  (2) proposition (5.7).

(2)  $\Rightarrow$  (3)  $\Rightarrow$  (4): Proposition (16) in [3].

(4)  $\Rightarrow$  (1): Let  $A$  be a closed submodule in  $U$ . Since  $U$  is extending, then  $A$  is direct summand. But  $U$  is a weak duo module, then  $A$  is fully invariant, and we are done.

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**المقاس الثنائي من النمط -CL**

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**الخلاصة:**

في هذا البحث قدمنا مفهوم جديد (على حد علمنا) أطلقنا عليه اسم المقاس الثنائي من النمط -CL وهو أكبر من صنف المقاس الثنائي، وأصغر من صنف المقاس الثنائي الضعيف المعطى من قبل Ozcan و Harmanci. يقال للمقاس U بأنه مقاس ثنائي من النمط -CL اذا كان كل مقاس جزئي مغلق فيه ثابت تماماً. تم التحقق من مجموعة من الخواص والتشخيصات الأخرى لهذا المفهوم. كذلك درسنا علاقته ببعض المقاسات الأخرى مثل، المقاس الثنائي الضعيف، المقاس الثنائي من النمط -P.

**الكلمات المفتاحية:** المقاس الثنائي من النمط -CL، المقاس الثنائي، المقاس الثنائي الضعيف، المقاس الثنائي من النمط P، المقاس الجزئي المغلق، المقاس الجزئي الثابت تماماً.