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St-Polyform Modules and Related Concepts

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Abstract:

In this paper, we introduce a new concept named St-polyform modules, and show that the class of St-polyform modules is contained properly in the well-known classes; polyform, strongly essentially quasi-Dedekind and κ-nonsingular modules. Various properties of such modules are obtained. Another characterization of St-polyform module is given. An existence of St-polyform submodules in certain class of modules is considered. The relationships of St-polyform with some related concepts are investigated. Furthermore, we introduce other new classes which are; St-semisimple and κ-non St-singular modules, and we verify that the class of St-polyform modules lies between them.

Keywords: $\mathcal{K}$-nonsingular modules, Polyform modules, Semi-essential submodules, St-closed submodules, Strongly essentially quasi-Dedekind modules.

Introduction:

Throughout this paper, all rings are assumed to be commutative with a non-zero unity element, and all modules are unitary left R-modules. The notations V $\leq_e$U and V $\leq_{sem}$U mean that V is an essential and semi-essential submodule of U respectively. A submodule V of U is called essential if every non-zero submodule of U has a non-zero intersection with V (1, P.15). A submodule V of U is called semi-essential if every non-zero prime submodule of U has a non-zero intersection with V (2). A submodule V of U is called closed if V has no proper essential extensions inside U (1, P.18). Ahmed and Abbas introduced the concept of St-closed submodule, where a submodule V of U is said to be St-closed, if V has no proper semi-essential extensions inside U (3).

In this paper, we introduce and study a new class named St-polyform modules. This type of modules is contained properly in some classes of modules such as polyform, strongly essentially quasi-Dedekind and κ-non St-singular modules. An R-module U is called polyform if for every submodule V of U and for any homomorphism f: V→U, ker$f$ is closed submodule in U (4). A module U is called strongly quasi-Dedekind, if $\mathrm{Hom}_R(\frac{U}{V}, U)=0$ for all semi-essential submodule V of U (5). An R-module U is called κ-nonsingular, if for each homomorphism $f\in$End(U) such that ker$f$ is essential submodule of V, then $f=0$ (6, P.95).

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We define in this work a proper class of κ-nonsingular modules named κ-non St-singular. We define St-polyform as follows: an R-module U is called St-polyform, if for every submodule V of U and for every homomorphism f: V→U, ker$f$ is St-closed submodule in V. We verify that an St-polyform module is smaller than all of the classes: polyform, strongly quasi-Dedekind, κ-nonsingular and κ-non St-singular modules, see remark 2, proposition 30, proposition 40 and proposition 56. Beside that we give another generalization for St-polyform modules.

This work consists of three sections. In the first section we provide another characterization of St-polyform modules, we show that a module U is St-polyform if and only if for each non-zero submodule V of U and for each non-zero homomorphism f: V→U; ker$f$ is not semi-essential submodule of V, see theorem 4. Also we present the main properties of St-polyform module, for example we show in proposition 7 the existence of St-polyform in certain class of modules, also we prove in the proposition 11; if W$\leq_{sem}$V for every submodule V of U with $\mathrm{Hom}_R(\frac{V}{W}, U) = 0$, then U is a St-polyform module, and we show in the proposition 13 that a module U is an St-polyform if its quasi-injective hull is St-polyform. In section two we investigate the relationships of St-polyform with polyform module and small polyform, where a submodule V of U is called small if V+W≠ U for every proper submodule W of U (1, P.20). An R-module U is called small polyform if for each non-zero small submodule V of U, and for each $f \in Hom_R(V,U)$; ker $f \nleq_e V$ (4). Furthermore, we introduce another generalization for St-polyform module named essentially St-polyform module, and we show in theorem 26; the two concepts are equivalent under the class of uniform modules. The last section of this paper is devoted to study the relationships of St-polyform with other related concepts such as quasi-Dedekind and some of its generalizations as well as κ-nonsingular and Baer modules. We show that under certain condition an strongly essentially quasi-Dedekind module can be St-polyform, see theorem 31. Also, we give a partial equivalence between St-polyform and κ-nonsingular modules, see theorem 42. Moreover, other related concepts of St-polyform module are introduced which are St-semisimple, and κ-non St-singular modules.

St-polyform modules:

In this section, various properties and anther characterization for St-polyform modules are investigated. We start by the following definition.

Definition 1: An R-module U is called St-polyform, if for every submodule V of U and for any homomorphism f:V→U, ker$f$ is St-closed submodule in V. A ring R is called St-polyform, if R is St-polyform R-module.

Remark 2: The St-polyform module is a proper class of polyform module. In fact if U is St-polyform module, then for every submodule V of U and for any homomorphism f: V→U, ker$f$ is St-closed submodule in V. Since the class of closed submodule is greater than the class of St-closed submodule, thus ker$f$ is closed submodule in U; hence U is a polyform module. On the other hand, not every polyform module is St-polyform for example; Z_2 as Z-module is clearly polyform module, but not St-polyform, since the identity homomorphism I: $Z_2 \to Z_2$ has zero kernel which is not St-closed submodule in Z_2 (3).

Examples and Remarks 3:

- **i.** Simple module is not St-polyform module. The proof is similar as proving Z_2 is not St-polyform in remark 2.
- **ii.** Z_8 is not St-polyform module. In fact there exists f: $(\bar{2}) \to Z_8$ defined by $f(\bar{x}) = \bar{2}x$ $\forall \bar{x} \in (\bar{2})$. Note that ker $f = (\bar{4})$, and $(\bar{4})$ is not St-closed submodule in Z_8.
- **iii.** Epimomorphic image of St-polyform module may not be St-polyform; for example Z_{10} is St-polyform, while $\frac{Z_{10}}{(\bar{2})} \cong Z_5$. By i, Z_5 is not St-polyform.
- **iv.** Monoform module need not be St-polyform. For example, Z_2 is a monoform Z-module, but it is not St-polyform as we seen in remark 2.
- **v.** Uniform may not be St-polyform module, where a non-zero module U is called uniform if U every non-zero two submodules of U have non-zero intersection (1, P.85).
- **vi.** Q as Z is not St-polyform. In fact Q is uniform module, hence it is semi-uniform, and the result follows by v.
- **vii.** Z_6 is an St-polyform module, since every submodule of Z_6 is St-closed. So the kernel of any homomorphism from each submodule to Z_6 is St-closed. For the same argument Z_{10} is St-polyform.
- **viii.** Z_{12} is not St-polyform Z-module.
- **ix.** A submodule of St-polyform module may not be St-polyform, for example; by vii, Z_6 is an St-polyform module, but A = $(\bar{2})$ $\leq Z_6$ is not St-polyform, since A is simple module, which is not St-polyform as we showed in i.

The following theorem gives another characterization of St-polyform module.

Theorem 4: An R-module U is St-polyform, if and only if for each non-zero submodule V of U and for each non-zero homomorphism f: V→U; ker$f$ is not semi-essential submodule of V.

Proof: **⇒)** Assume that there exists a non-zero submodule V of U and a non-zero homomorphism f:V→U such that ker$f$ is semi-essential submodule of V. But ker$f \leq_{Stc} V$, therefore ker$f$=V, hence f=0 which is a contradiction. That is ker$f \nleq_{sem} V$. ■

⇐) Suppose that there exists a submodule V of U and a homomorphism f: V→U such that ker$f$ is not St-closed submodule in V. By definition of St-closed, there exists a submodule W of V such that ker$f \leq_{sem} W \leq V$. Consider the homomorphism $f \circ i$: W → U. It is clear that $f \circ i \neq 0$, and since ker$f \subseteq$W, then ker$(f \circ i) \leq_{sem} W$ (2). But this is contradict with our assumption, thus ker$f$ is St-closed submodule of V.

The following examples are checked by using theorem 4.

Examples 5:

- **i.** Any semi-uniform module is not St-polyform module, where a non-zero R-module U is called semi-uniform if every non-zero submodule has non-zero intersection with all prime submodules of U (2).

Proof i: Let V be a non-zero submodule of U, and f: V→U be a non-zero homomorphism. Assume that U is St-polyform module, so ker$f \nleq_{sem}$ V, hence ker$f \nleq_{sem}$ U (2). But this contradicts the definition of semi-uniform module, thus U is not St-polyform. ■

ii. Z is not St-polyform Z-module. In fact since Z is semi-uniform module, so the result follows by i.

iii. Z_4 is not St-polyform module. In fact if we take $V=Z_4$ in the theorem 4 as a submodule of itself, so there exists a homomorphism $f \in Hom_R(Z_4,Z_4)$ defined by $f(x)=2x\ \forall x\in Z_4$, note that $\ker f = (\bar{2})$ which is semi-essential submodule of Z_4. Thus Z_4 is not St-polyform module.

iv. $Z\oplus Z_2$ is not St-polyform Z-module. To show that; assume there exists a submodule $V=Z\oplus Z_2$ and a homomorphism f: $V\to U$ defined by $f(x, \bar{y}) = (0,\bar{x})$, where $x\in Z$, $\bar{y}\in Z_2$. Note that $f\neq 0$, and $\ker f=\{(x, \bar{y}) \in V|\ f(x, \bar{y}) = (0,0)\} = \{(x, \bar{y}) \in V| \bar{x}= \bar{0}\}= 2Z\oplus Z_2$, hence $\ker f \leq_{sem} V$. So $Z\oplus Z_2$ is not St-polyform module.

Proposition 6: A direct summand of St-polyform module is St-polyform.

Proof: Let $U=U_1\oplus U_2$ be a St-polyform module, where U_1 and U_2 are R-submodules of U. Let V_1 be a non-zero submodule of U_1, and f: $V_1 \to U_1$ be a non-zero homomorphism. Consider the following sequence:

$$V_1 \xrightarrow{f} U_1 \xrightarrow{j} U_1\oplus U_2$$

where j is an injection homomorphism. Now, $j\circ f$: $V_1 \to U$, and U is St-polyform, then $\ker(j\circ f) \nleq_{sem} V_1$. Since $\ker(j\circ f) = \{v_1\in V_1|\ (j\circ f)(v_1) = 0\} = \{v_1\in V_1|\ f(v_1) = 0\}= \ker f\oplus U_2$, then $\ker f\oplus U_2 \nleq_{sem} U$. But $U_2 \leq_{sem} U_2$, thus $\ker f \nleq_{sem} U_1$ (5, Lemma(1.18)). That is U_1 is St-polyform. ■

The converse of proposition 6 is not true in general; for example each of Z_{10} and Z_6 are St-polyform Z-modules; see 3vii, but $Z_{10} \oplus Z_6$ is not St-polyform Z-module.

Recall that an R-module U is called Artinian if every descending chain of submodules in U is stationary (1,P.7). The following proposition indicates the existence of St-polyform submodules in certain class of modules.

Proposition 7: Every nonzero Artinian module has a submodule which is an St-polyform.

Proof: Let U be a non-zero Artinian module, and V be a submodule of U. If V is St-polyform, then we are done. Otherwise there exists a submodule V_1 of V and a homomorphism f_1: $V_1 \longrightarrow V$ with $\ker(f_1) \nleq_{St} V_1$ and $\ker(f_1) \leq_{St} V_2$ for some proper submodule V_2 of V_1. Now, if V_1 is St-polyform, then we are through, otherwise there exists a submodule V_3 of V_2 and a homomorphism f_2: $V_3 \longrightarrow V_2$ with $\ker(f_2) \nleq_{St} V_3$ and $\ker(f_2) \leq_{St} V_4$ for some proper submodule V_4 of V_3. We continue in this manner until we arrive in a finite number of steps at a submodule which is an St-polyform submodule. Otherwise, we have an infinite descending chain $V\supset V_1\supset V_2\supset \ldots\ldots$ of submodules of the module U. But this is a contradiction, since U is Artinian. Therefore U contains an St-polyform submodule. ■

Proposition 8: Let U be an R-module. If either V_1 or V_2 are St-polyform module, then $V_1\cap V_2$ is St-polyform module.

Proof: Assume that V_1 is St-polyform module. Let V be a non-zero submodule of $V_1\cap V_2$, and let f: $V\to V_1\cap V_2$ be a non-zero homomorphism. Consider the following sequence:

$$V \xrightarrow{f} V_1\cap V_2 \xrightarrow{i} V_1$$

Since V_1 is a St-polyform module, then $\ker(i\circ f) \nleq_{sem} V$. But $\ker f=\ker(i\circ f)$, then $\ker f \nleq_{sem} V$. That is $V_1\cap V_2$ is a St-polyform module. ■

Recall that an R-module U is called scalar if for any $f\in End_R(U)$, there exists $r\in R$ such that $f(x)=rx$ $\forall x\in U$, where $End_R(U)$ is the endomorphism ring of U (5).

Proposition 9: Let U be a faithful scalar R-module. Then R is an St-polyform ring if and only if $End_R(U)$ is an St-polyform ring.

Proof: Since U is a faithful scalar module, then $End_R(U) \cong R$ (7). So if R is an St-polyform module, then $End_R(U)$ is polyform, and vice versa. ■

An R-module U is called multiplication for every submodule V of U there exists an ideal I of R such that V= IU (8, P.200).

Corollary 10: Let U be a finitely generated faithful and multiplication R-module. Then R is St-polyform ring if and only if $End_R(U)$ is St-polyform module.

Proof: Since U is finitely generated and multiplication, then U is a scalar module (7), and the result follows by proposition 9. ■

Proposition 11: Let U be an R-module. If $W\leq_{sem}V$ for every submodule V of U, such that $Hom_R(\frac{V}{W}, U) = 0$, then U is a St-polyform module.

Proof: Assume U is not St-polyform module, so there exists a submodule V of U and a non-zero homomorphism α: $V\to U$ such that $\ker\alpha \leq_{sem} U$. Define φ: $\frac{V}{\ker\alpha} \to U$ by $\varphi(v+\ker\alpha) = \alpha(v)$ $\forall v+\ker\alpha\in\frac{V}{\ker\alpha}$. We can easily show that φ is well defined and homomorphism. Since α is a non-zero homomorphism, then φ is also non-zero, thus $Hom_R(\frac{V}{W}, U)\neq 0$. But this contradicts our assumption, therefore $\ker\alpha \nleq_{sem} U$. ■

Proposition 12: Let U be an R-module, and I be an ideal of R such that $I \subseteq ann_R(U)$, then U is St-polyform R-module if and only if U is St-polyform $\frac{R}{I}$ - module.

Proof: Assume that U is an St-polyform R-module. Since $I \subseteq ann_R(U)$, then it can be easily shown that $Hom_R(V, U) = Hom_{\frac{R}{I}}(V, U)$ for each submodule V of U, hence the result follows directly. ■

Recall that an R-module U is called injective if for every monomorphism f: A→B where A and B be any R-modules, and for every homomorphism g: A→U, there exists a homomorphism h: B →U such that $h \circ f = g$ (8, P.33). A module U is called quasi-injective if it is U-injective R-module (8, P.83). The injective hull (quasi- injective hull) of a module U is defined as an injective (quasi-injective) module with essential extension of U, it is denoted by E(U) (respectively $\overline{U}$) (8, P.39). Clark and Wisbauer in (9) proved that a module U is polyform if its quasi-injective hull is polyform. As analogue of that, we have the following result.

Proposition 13: Let U be an R-module. If the injective hull E(U) of U is St-polyform module, then U is St-polyform module.

Proof: Let V be a non-zero submodule of U, and f:V→U be a non-zero homomorphism. Suppose the converse is not true, that is $\ker f \leq_{sem} V$. Consider the following Fig. 1.

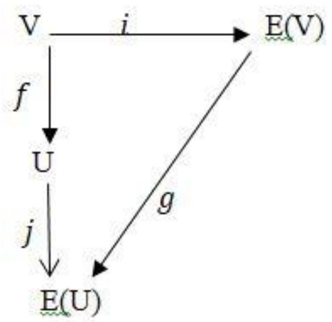

Figure 1. The diagram of injective the module E(U)

where i: V → E(V) and j: U → E(U) are the inclusion homomorphisms. Since E(U) is injective, then there exists a non-zero homomorphism g: V→ U such that $g \circ i = j \circ f$. It is clear that $\ker(g \circ i) \subseteq \ker g$ and $\ker f = \ker(j \circ f)$. Since E(U) is an St-polyform module, then $\ker(g) \not\leq_{sem} E(V)$. By definition of injective hull $V \leq_e E(V)$, hence $V \leq_{sem} E(V)$, and by our assumption $\ker f \leq_{sem} V$, then by transitivity of semi-essential submodules $\ker f \leq_{sem} E(V)$ (2). On the other hand, clearly $\ker f \subseteq \ker g$, therefore $\ker g \leq_{sem} E(V)$ (2), which is a contradiction. Therefore, $\ker f \not\leq_{sem} V$, i.e V is an St-polyform module. ■

In example 3ix, we verified that a submodule of St-polyform may not be St-polyform. In the following proposition, we satisfy that under certain condition.

Corollary 14: Let U be an injective and St-polyform module. If V is an essential submodule, then V is St-polyform module.

Proof: Since V is an essential submodule of U, then E(V) = E(U) (10, Prop(2.22), P.45). But U is injective module, so U = E(U). This implies that E(V) = U. Since U is St-polyform, then E(V) is St-polyform. The result follows by proposition 13. ■

Recall that a module over integral domain R is called divisible if U=rU ∀r∈R (10, P.32).

Corollary 15: Let R be a division ring, and U be an St-polyform R-module. If V is essential submodule of U, then V is an St-polyform module.

Proof: Since R is a division ring, then U is an injective module (10, P.30), and the result follows by corollary 14. ■

Corollary 16: If R is a division St-polyform ring, then each ideal of R is an St-polyform.

Proof: Let I be an ideal of R. Since R is a division ring, then clearly every ideal of R is essential. On the other hand, since every module over division ring is an injective module (10, P.30), therefore I is injective. But R is an St-polyform ring, so by corollary 14, I is a St-polyform ideal. ■

Corollary 17: Let U be a divisible St-polyform module over P.I.D. If V is an essential submodule of U, then V is St-polyform module.

Proof: Since U is divisible over P.I.D, then U is injective (10, Th(2.8), P.35). The result follows by corollary 14. ■

Recall that a commutative domain R is called Dedekind; if every non-zero ideal of R is invertible (10, P.36).

Corollary 18: Let U be a divisible module over Dedekind domain R, and $V \leq_e U$. If U is a St-polyform module, then V is St-polyform.

Proof: Since Every divisible module over a Dedekind domain is injective (10, P.36), then by corollary 14, we are done. ■

St-polyform and Polyform modules:

In this section, we investigate the relationships of St-polyform module with polyform and small polyform modules. Besides that, we introduce another generalization for St-polyform modules.

In the previous section, we verified that the class of St-polyform modules is a proper subclass of polyform modules. In the following theorems, we use certain conditions under which St-polyform module can be polyform module. Before that; an R-module U is called fully prime if every proper submodule of U is prime (2).

Theorem 19: Let U be a fully prime R-module, then U is St-polyform if and only if U is a polyform module.

Proof: ⇒) By remark 2.

⇐) Assume that U is polyform module, and let V be a submodule of U, and f: V→ U be a homomorphism. Since U is polyform, then ker$f$ is closed submodule in U. But U is fully prime, then ker$f$ is an St-closed in U (3), hence U is St-polyform. ■

Recall that an R-module U is called fully essential, if every semi-essential submodule of U is essential (2).

Theorem 20: Let U be a fully essential R-module, then U is St-polyform if and only if U is a polyform module.

Proof: ⇒) By remark 2.

⇐) Let V be a non-zero submodule of U, and f: V→ U be a non-zero homomorphism. Since U is polyform, then ker $f \not\leq_e$ V. But U is fully essential; therefore, ker $f \not\leq_{sem}$V (2), that is U is St-polyform module. ■

The following proposition shows that the class of St-polyform domain coincides with the class of polyform domain.

Theorem 21: An integral domain R is an St-polyform if and only if R is polyform domain.

Proof: ⇒) It is obvious.

⇐) Assume that R is a polyform domain. Let I be a non-zero ideal of R, and f: I →R be a non-zero homomorphism. Since R is integral domain, then ann(I)=0; that is $ann_R(I) \not\leq_{sem}$R. Thus R is St-polyform. ■

Hadi and Marhoon in (4) gave a generalization of polyform module as follows:

Definition 22: An R-module U is called small polyform module if for each non-zero small submodule V of U, and for each non-zero homomorphism f: V→U; ker $f \not\leq_e$ V.

Remark 23: Every St-polyform module is small polyform.

Proof: Since every St-polyform module is polyform, so the result follows directly. ■

Now, we need to introduce another class of polyform modules which is bigger than polyform modules.

Definition 24: An R module U is called essentially polyform module if for each non-zero proper essential submodule V of U, and for each non-zero homomorphism f: V→U; ker $f \not\leq_e$ U.

We can generalize St-polyform as follows:

Definition 25: An R module U is called essentially St-polyform module if for each non-zero proper essential submodule V of U, and for each non-zero homomorphism f: V→U; ker $f \not\leq_{sem}$ V.

It is clear that every St-polyform module is essentially St-polyform, and every essentially St-polyform module is essentially polyform module. Furthermore, it should be noted that the polyform module lies between St-polyform and essentially St-polyform module.

The following theorem gives a partial equivalence between St-polyform and essentially St-polyform module.

Theorem 26: Let U be a uniform module, then U is St-polyform if and only if U is essentially St-polyform.

Proof: ⇒) It is straightforward.

⇐) Assume that U is essentially St-polyform, and let V be a non-zero submodule of U, and f: V→ U be a non-zero homomorphism. Since U is a uniform module so V $\leq_e$U. But U is essentially St-polyform; therefore, ker $f \not\leq_{sem}$ V; that is U is an St-polyform module. ■

By replacing uniform module by hollow and essential submodule by small, we have the following; and the proof is in a similar way.

Proposition 27: Let U be a hollow module, then U is St-polyform if and only if U is small St-polyform.

We can summarize the main results of this section by the following implications of modules:

St-polyform ⇒ Polyform ⇒ Small polyform

St-polyform ⇒ Polyform ⇒ Essentially St-polyform
⇓
Essentially polyform

St-polyform and other related concepts:

This section is devoted to study the relationships of St-polyform with some related concepts such as quasi-Dedekind and some of its generalizations, κ-nonsingular, injective, extending, Baer and κ-non St-singular module.

Recall that an R-module U is called quasi-Dedekind, if for every non-zero homomorphism $f \in$End(U), ker$f$=0 (11).

Remark 28: It is worth mentioning that St-polyform modules and quasi-Dedekind modules are independent; for example the Z-module Z_6 is St-polyform module see example 3vii, but not quasi-Dedekind. On the other hand, Z is quasi-Dedekind (11), but not St-polyform, see example 5ii.

Proposition 29: Let U be a semi-uniform module. If U is St-polyform then U is a quasi-Dedekind module.

Proof: Assume that U is St-polyform module, and let $f \in$End(U). If V be a non-zero submodule of U, then we have the following sequence:

$$V \xrightarrow{i} U \xrightarrow{f} U$$

Where i is the inclusion homomorphism. Suppose that ker$f \neq 0$, since U is St-polyform. Note that $f \circ i \neq 0$. Since U is St-polyform, then ker$(i \circ f) \not\leq_{sem}$V, hence ker$(i \circ f) \not\leq_{sem}$U (2). But this is a contradiction since U is a semi-uniform module, thus ker$f$ = 0. ■

The converse of proposition 29 is not true in general, for example Z_2 is a quasi-Dedekind module, but not St-polyform.

Recall that an R-module U is called strongly essentially quasi-Dedekind if for each non-zero homomorphism $f \in End_R(U)$, $\ker f \not\leq_{sem}$ U (5).

Proposition 30: Every St-polyform module is strongly essentially quasi-Dedekind.

Proof: Let U be St-polyform module. Let V be a non-zero submodule of U, and f: V→U be a non-zero homomorphism. By assumption $\ker f$ is not semi-essential submodule in V. In particular, all non-zero endomorphisms of U have kernels which are not semi-essential in U, proving our assertion. ■

The converse of proposition 30 is not true in general, for example Z_2 is strongly essentially quasi-Dedekind module (5, Ex (1.11)) but not St-polyform as we saw in remark 2. In the following theorem we use a condition under which the converse is true.

Proposition 31: Let U be a quasi-injective R-module then U is U is St-polyform if and only if U is a strongly essentially quasi-Dedekind module.

Proof: ⇒) By proposition 30.

⇐) Let V be a non-zero submodule of U, and f:V→U be a non-zero homomorphism. Consider the following Fig. 2.

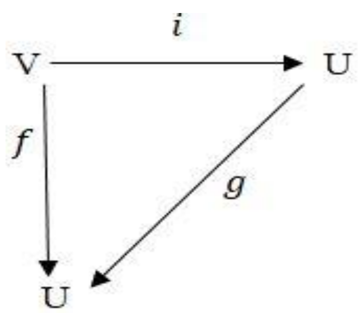

Figure 2. The diagram of injective module U

where i: V→ U is the inclusion homomorphism. Since U is quasi-injective, then there exists a homomorphism g: U→U such that $g \circ i = f$. Now, $g \in End(U)$ and U is essentially quasi-Dedekind; therefore, $\ker g \not\leq_{sem}$U. But $\ker f \subseteq \ker g$, then by transitivity of semi-essential submodule, $\ker f \not\leq_{sem}$ U (2), and we are done. ■

In (3) Ahmed and Abbas proved that if every submodule of U is St-closed, then every submodule of U is direct summand. This motivates us to introduce the following.

Definition 32: An R module U is called St-semisimple if every submodule of U is St-closed.

This concept is clearly a proper subclass of semisimple modules, and we can easily prove the following.

Remark 33: Every St-semisimple module is St-polyform module.

We think that the converse of the remark 33 is not true in general, but we cannot find example.

Definition 34: Let U be an R-module. We define St-singular submodule as follows:

$$\{u \in U \mid ann_R(u) \leq_{sem} R\}$$

It is denoted by St-sing(U). If St-sing(U) = U, then U is called St-singular module, and U is called non St-singular if St-sing(U) = 0. ■

Example 35: Q as Z-module is non St-singular, where Q is the set of all rational numbers, since St-sing(Q) = 0. For the same reason Z is non St-singular Z-module.

Proposition 36: Let U and V be R-modules. If $Hom_R(V,U)=0$ for each St-singular module V, then U is non St-singular module.

Proof: Consider the inclusion homomorphism i: St-sing(U) → U. It is clear that St-sing(U) is St-singular module, so by assumption i=0. But i(St-sing (U)) = St-sing(U), therefore sing U=0. That is U is non St-singular. ■

Remark 37: For any submodule V of an R-module U, St-sing(V)=St-sing(U)∩V.

Proof: It is clear that St-sing(V) ⊆ St-sing(U), so the result follows directly. ■

Remark 38: By using remark 37, we can easily show that the class of St-singular module is closed under submodules.

Recall that an R-module U is called κ-nonsingular, if for each $f \in End_R(U)$; $\ker f \leq_e$ U, then f=0 (6, P.95). In other words, for every non-zero homomorphism $f \in End_R(U)$; $\ker f \not\leq_e$ U. As example for this class of modules is Z-module Z_p, it is κ-nonsingular for every prime number P, since Z_p is a simple module; therefore, all non-zero endomorphisms are automorphisms.

Remark 39: The concept of κ-nonsingularity is strictly weaker than the concept of nonsingularity for modules (6, Ex(4.1.10), P.96), where an R-module U is called nonsingular if Z(U)=0, where $Z(U)= \{u \in U \mid ann_R(u) \leq_e R\}$ (1, P.30).

Proposition 40: Every St-polyform module is κ-nonsingular.

Proof: Let U be St-polyform module. Let V be a non-zero submodule of U, and f: V→ U be a non-zero homomorphism. By assumption, $\ker f \not\leq_{sem}$V. As we take V=U, then we obtain f: U → U, and $\ker f \not\leq_{sem}$ U. Since every essential submodule is semi-essential (2), then $\ker f \not\leq_e$ U, hence U is κ-nonsingular. ■

The converse of proposition 40 is not true in general as the following examples show:

Examples 41:

1. Every simple module is κ-nonsingular (12), but not St-polyform, see example 3i.
2. The Z-module Q is nonsingular module, hence it is κ-nonsingular (12). But Q is not St-polyform module, see example 3vi.
3. The Z-module U = $Q \oplus Z_2$ is not St-polyform module. In fact if V= $Z\oplus(0)$ be a non-zero submodule of U. Let f: V $\to$ U be a map defined by f(x,0) =(0,$\bar{x}$), where x$\in$Z. It is clear that f is a non-zero homomorphism, then ker$f$={(x,0)$\in$V| f(x,0)=(0,$\bar{0}$)}=2Z$\oplus$(0). We can easily verify that 2Z$\oplus$(0) $\leq_{\text{sem}}$V, hence U is not St-polyform module. On the other hand, U is κ-nonsingular Z-module (12).

The following proposition gives a partial equivalence between St-polyform and κ-nonsingular modules.

Theorem 42: Let U be a fully essential quasi-injective module, then U is St-polyform if and only if U is κ-nonsingular provided that Hom_R(V,U) $\neq$ 0.

Proof: $\Rightarrow$) By proposition 40.

$\Leftarrow$ **)** Suppose that U is a κ-nonsingular, and let V be a non-zero proper submodule of U. Let f: V$\to$ U, Since Hom_R(V,U)$\neq$0, so we can take $f\neq$0. Consider the following Fig. 3.

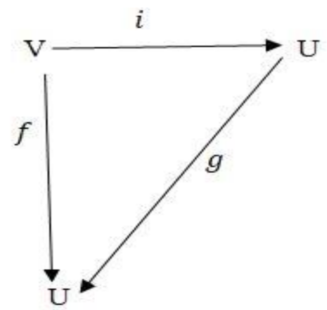

Figure 3. The diagram of injective module U

where i: V$\to$ U is the inclusion homomorphism. Since U is quasi-injective, then there exists a homomorphism g: U$\to$U such that $g \circ i = f$. Now, $g\in$ End_R(U) and U is κ-nonsingular, thus ker $g \not\leq_e$ U. But ker$f \subseteq$ ker$g$, thus ker $f \not\leq_e$ U. Since U is fully prime, then ker $f \not\leq_{\text{sem}}$ U. ■

Corollary 43: Let U be a fully prime injective module. Then U is an St-polyform module if and only if U is κ-nonsingular.

Proof: Since every fully prime module is fully essential (2), and End_R(U) $\neq$ 0, then the result follows by theorem 42. ■

Lemma 44: **(11)** If U is an injective module, then J(End_R(U)) = {$f\in$ End_R(U)| ker$f \leq_e$U}.

Corollary 45: Let U be a fully essential module. If U is injective and J(End_R(U)) = 0, then U is St-polyform.

Proof: Since J(End_R(U)) = 0, then It is clear that U is κ-nonsingular. Since End_R(U) $\neq$ 0, then by theorem 42 we are done ■

The following theorem gives some useful relationships of St-polyform ring with some related concepts. Before that, we need the following characterization of essential submodules.

Lemma 46: **(10, P.40)** Let U be an R-module. A submodule V of U is essential, if $\forall$0$\neq$u$\in$U, there exists r$\in$R such that 0 $\neq$ ru $\in$V.

Theorem 47: Let R be a fully essential quasi-injective ring. Consider the following statements:

1. R is an St-semisimple ring
2. R is an St-polyform ring.
3. R is a κ-nonsingular ring.
4. R is a polyform ring.
5. R is a semiprime ring.
6. R is a nonsingular ring.

Then: **(1) $\Rightarrow$(2) $\Leftrightarrow$(3)$\Rightarrow$(4)**,**(5) $\Rightarrow$(4), (5) $\Leftrightarrow$ (6) $\Rightarrow$(3), (6) $\Rightarrow$(4)** and **(5) $\Rightarrow$(2).**

Proof: **(1) $\Rightarrow$(2)** By remark 33.

(2) $\Leftrightarrow$(3) Since U is fully essential quasi-injective, then by theorem 42 we are done.

(4) $\Rightarrow$(3) (6, Prop(4.1.5), P.95).

(5) $\Rightarrow$(3) Assume that R is not κ-nonsingular ring, so there exists a non-zero homomorphism $\varphi\in\text{End}_R$(R) with ker$\varphi \leq_{\text{sem}}$R. If $\varphi\neq$0 then there exists 0$\neq$x$\in$R such that φ(x) = tx $\forall$t$\in$R. By lemma 46 there exists 0$\neq$k$\in$R such that 0$\neq$xk$\in$kerφ.This implies that 0=φ(xk) =x^2k, hence $(xk)^2$ = 0. But R is semiprime, therefore xk = 0 which is a contradiction. Thus φ = 0.

(5) $\Leftrightarrow$ (6) (1, Prop(1.27), P.35).

(6) $\Rightarrow$(3) By remark 39.

(6) $\Rightarrow$(4) (6, P.95).

(5) $\Rightarrow$(2) Assume that R is not St-polyform ring, so for each non-zero ideal I of R, there exists a homomorphism f:I$\to$R such that ker$f \leq_{\text{sem}}$ R. Since R is fully essential ring, then ker$f \leq_e$R. The remain steps of the proof are similar of the direction **(5) $\Rightarrow$(3).** ■

An R-module U is called extending, if every closed submodule of U is direct summand of U (8, P.118).

Proposition 48: Let U be a fully essential module. If U is an extending module, then U is St-polyform module.

Proof:. Let 0$\neq$V$\leq$ U and f: V $\to$ U be a non-zero homomorphism. Since U is an extending module, then ker$f \leq_c$U, hence ker$f \leq_c$V (1, Prop(1.5), P.18). But U is fully essential, thus ker$f \leq_{\text{St}}$U, so we are done. ■

We need to give the following definition.

Definition 49 (6, P.94): An R-module U is called Baer, if for every submodule V of U, $ann_S(V) = (f)$, where $f^2=f \in End_R(U)$.

In order to verify the relation of St-polyform with Baer module, we need to introduce the following proposition.

Proposition 50: Every Baer quasi-injective module is polyform.

Proof: Let V be a non-zero submodule of U, and f: V→U be a non-zero homomorphism. Suppose the converse; that is $\ker f \leq_e V$. Consider the following Fig. 4:

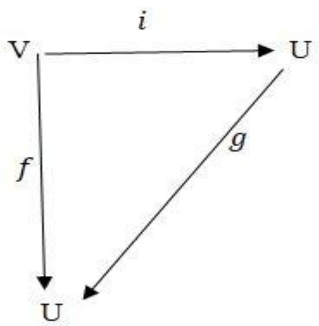

Figure 4. The diagram of injective module U

where i: V→ U is the inclusion homomorphism. Since U is quasi-injective, then there exists a homomorphism g: U→U such that $g \circ i = f$. Now, $g \in End_R(U)$ and U is Baer, so $\ker g$ = $ann_S\, g$ = e, e^2=e, and S= $End_R(U)$. This implies that $\ker g$ is direct summand of U (12). Since $\ker(i \circ g) \subseteq \ker g$, then clearly $\ker(i \circ g)$ is a direct summand of V. But $g \circ i = f$, thus $\ker f$ is direct summand of V. On the other hand, $\ker f \leq_e V$, therefore $\ker f = V$, hence $f = 0$ which is a contradiction with assumption, thus $\ker f \not\leq_e V$. ■

Corollary 51: For a fully prime (or fully essential) module, every Baer quasi-injective module is St-polyform.

Proof: Since in the class of fully prime (or fully essential) modules the concept of essential submodules coincides with the concept of semi-essential, so the proof is in similar of the proposition 50. ■

Proposition 52: Let U be an extending module. If U is St-polyform, then U is a Baer module.

Proof: Since U is St-polyform, then by proposition 40, U is κ-nonsingular. On the other hand, U is extending, so U is Baer (6, Lemma(4.1.17), P.97). ■

Theorem 53: Let U be an quasi-injective module. Consider the following statements:

1. U is an St-polyform module.
2. U is a κ-nonsingular module.
3. U is a Baer module.
4. U is a polyform module.

Then: **(1)** ⇒ **(2)** ⇒ (3) ⇔ **(4),** and if U is fully prime then **(4)** ⇒ **(1).**

Proof: **(1)** ⇒ **(2)** By proposition 40.

(2) ⇒ **(3)** Since U is quasi-injective, so clearly U is extending. But U is St-polyform, thus U is a Baer module (6, Lemma(4.1.17), P.97).

(3) ⇔**(4)** Since U is Baer and quasi-injective, then by proposition 50, U is polyform. Conversely; Since U is polyform, then U is κ-nonsingular (6, Prop(4.1.5), P.95). But U is quasi-injective; therefore, U is extending. So U is κ-nonsingular and extending, this implies that U is a Baer module (6, Lemma(4.1.17), P.97).

(4) ⇒ **(1)** Since U is fully prime, then by theorem 20, we are done. ■

Now we introduce a subclass of κ-nonsingular module.

Definition 54: An R-module U is called κ-non St-singular, if for any non-zero homomorphism $f \in End_R(U)$ $\ker f \leq_{sem} U$, then f=0. In other words, for every non-zero homomorphism $f \in End_R(U)$; $\ker f \not\leq_{sem} U$.

Remark 55: Every κ-non St-singular R-module is κ-nonsingular.

Proof: Let $f \in End_R(U)$ be a non-zero homomorphism. Since U is a κ-non St-singular module, then $\ker f \not\leq_{sem} U$, hence $\ker f \not\leq_e U$ (2). Thus U is κ-non St-singular module. ■

The converse of remark 55 is true under certain condition as the following proposition shows.

Proposition 56: Let U be a fully essential module, then U is κ-non St-singular module if and only if U is κ-nonsingular.

Proof: ⇒) By remark 55.

⇐) Assume that U is a κ-nonsingular module. Let V be a non-zero submodule of U, and $f \in End_R(U)$ be a non-zero homomorphism, so $\ker f \not\leq_{sem} V$. Since U is a fully essential module, then $\ker f \not\leq_e V$ and we are done. ■

Proposition 57: Every St-polyform module is κ-non St-singular module.

Proof: It is similar of the proof of the proposition (40), but in this proposition we use the transitive property of semi-essential submodules (2), instead of the generalized property of semi-essential submodules. ■

We end this work by the following.

Remark 58: We can summarize the main results which were introduced in last section about the relationships of the St-polyform module with related concepts as follows:

St-polyform ⇒ strongly essentially quasi-Dedekind

St-polyform ⇒ polyform ⇒ κ-nonsingular

St-semisimple ⇒ St-polyform ⇒ κ-non St-singular
⇓
κ-nonsingular

Conflicts of Interest: None

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مقاس بوليفورم من النمط -St والمفاهيم ذات العلاقة

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الخلاصة:

في هذا البحث قدمنا نوع جديد من المفاهيم أطلقنا عليه إسم مقاس بوليفورم من النمط -St والذي برهنا أنه محتوى فعلياً في بعض أصناف المقاسات المعروفة، مثل مقاس بوليفورم، مقاس كواسي ديدكند واسع بقوة والمقاس غير الشاذ من النمط -κ. قمنا بالتحقق في هذا البحث من مجموعة من الخواص الأساسية لمقاس بوليفورم من النمط -St، وأعطينا تشخيصاً آخر له. كما تم البرهنة على وجود مقاس بوليفورم من النمط -St كمقاس جزئي في اصناف معينة من المقاسات. كذلك درسنا علاقة المقاس بوليفورم من النمط -St ببعض المقاسات الاخرى. إضافة الى ذلك تم اعطاء مفهومين جديدة لهما علاقة بالمقاس بوليفورم من النمط -St، هما المقاس شبه البسيط من النمط -St والمقاس الغير شاذ من النمط -κSt، وبرهنا أن المقاس بوليفورم من النمط – St يقع بينهما.

الكلمات المفتاحية: المقاسات غير الشاذة من النمط -κ، مقاسات بوليفورم، المقاسات الجزئية شبه الواسعة، المقاسات الجزئية المغلقة من النمط-St، المقاسات شبه الديديكاندية الواسعة بقوة.