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On the Growth of Solutions of Second Order Linear Complex Differential Equations whose Coefficients Satisfy Certain Conditions

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Abstract:

In this paper, we study the growth of solutions of the second order linear complex differential equations $f''+A(z)f'+B(z)f=0$ insuring that any nontrivial solutions are of infinite order. It is assumed that the coefficients satisfy the extremal condition for Yang's inequality and the extremal condition for Denjoy's conjecture. The other condition is that one of the coefficients itself is a solution of the differential equation $f''+P(z)f=0$.

Keywords: Denjoy's conjecture, Entire functions, Order of growth, Yang's extremal function.

Introduction:

Since Wittich's work in (1), the solution's growth of linear complex differential equations became one of the interesting topics in complex analysis. The Nevanlinna theory of meromorphic functions is used to study this topic. The reader must have a background on the basic results and standard notations in Nevanlinna theory, for more details we refer the reader to see, for example, (2). The order of growth is used to measure the growth of entire functions.

In this paper, we consider the second order linear complex differential equation (2ndLCDE)

$$f''+A(z)f'+B(z)f=0 \qquad (1)$$

where $A(z)$and $B(z)\neq 0$ are entire functions. It is well known that all solutions of Eq. (1) are entire functions provided that$A(z)$ and$B(z)$ are entire functions, and if at least one of the coefficients is transcendental and f_1,f_2 are two linearly independent solutions of Eq. (1), then at least one of f_1,f_2 is of infinite order. Hence, most solutions of Eq. (1) have infinite order. Besides, there are equations of the form Eq. (1) that has a nontrivial solution of finite order; for example, $f(z)=e^z$ is a solution of $f''+e^{-z}f'-(e^{-z}+1)f=0$.

We shall study the growth of solutions of Eq. (1) when its coefficients satisfy extremal and conjecture conditions and the coefficient $A(z)$ itself is a solution of the following differential equation

$$f''+P(z)f=0 \qquad (2)$$

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where $P(z)=a_nz^n+\cdots+a_0,\ a_n\neq 0$ is a polynomial.

Materials:

In what follows, we introduce some basic concepts in Nevanlinna theory of meromorphic functions. For an entire function f, the order of growth and lower order of growth are defined by (3)

$$\rho(f)=\lim_{r\to\infty}\sup\frac{log^+log^+M(r,f)}{logr}$$

and

$$\mu(f)=\lim_{r\to\infty}\inf\frac{log^+log^+M(r,f)}{logr}$$

respectively, where $M(r,f)=\max_{|z|=r}|f(z)|$ and $log^+\alpha:=\max\ \{0,log\alpha\}$ for $\alpha\geq 0$.
The next concept due to Yang depends on the following result:

Theorem 1 (4) Assume that f is entire function of lower finite order. Let q be the number of Borel directions with order $\geq\mu$ and p be the number of finite deficient values of f, then $p\leq q/2$.

Definition 2 (4) The entire function f is extremal for Yang's inequality if the assumptions of Theorem 1 is satisfied with $p=q/2$.

Definition 3 (5, 6) Let $E\subseteq[0,\infty)$. We define the Lebesgue linear measure of E by

$$m(E)=\int_E dt$$

Definition 4 (7) Let $E \subseteq [1, \infty)$. We define The logarithmic measure of E by

$$m_l\,(E) = \int_E \frac{dt}{t}$$

We define the upper and lower logarithmic densities of E by

$$\overline{logdens}E = \lim_{r\to\infty} \sup \frac{m_l\,(E \cap [1, r])}{logr}$$

and

$$\underline{logdens}E = \lim_{r\to\infty} \inf \frac{m_l\,(E \cap [1, r])}{logr}$$

respectively. E has logarithmic density if $\overline{logdens}E = \underline{logdens}E.$

Now we recall a conjecture due to Denjoy.

Definition 5 (8) (Denjoy's Conjecture) Let f be an entire function of finite order ρ. If f has$k$ distinct finite asymptotic values, then $k \leq 2\rho$.
We say that an entire function f is an extremal function for Denjoy's conjecture if it has a finite order ρ and has $k = 2\rho$ distinct finite asymptotic values.

Definition 6 (9) Let $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ be an entire function. f is said to has Fejer gaps if

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty \qquad (3)$$

and f has Fabry gaps if the gap condition (3) is replaced with

$$\frac{\lambda_n}{n} \to \infty, as\ n \to \infty \qquad (4)$$

Remark: The Fabry condition (4) is a weaker than Fejer condition (3), and the entire function which has Fabry gaps has positive order, see (9 p. 651).

Definition 7 (5) Let $0 < \alpha < \beta < 2\pi$, we put

$$S(\alpha, \beta) = \{z | \alpha < arg(z) < \beta\},$$
$$S(\alpha, \beta, r) = \{z | \alpha < arg(z) < \beta, |z| < r\}$$

and let $\bar{S}(\alpha, \beta)$ be the closure of $S(\alpha, \beta)$. Let A be an entire function with $(0 < \rho(A) < \infty)$. Set $\rho = \rho(A)$and$S = S(\alpha, \beta)$. A is said to be blows up exponentially in S if for any $\theta \in (\alpha, \beta)$ the equation

$$\lim_{r\to\infty} \frac{loglog\,|A(re^{i\theta})|}{logr} = \rho$$

holds. Also A is said to be decays to zero exponentially in S if for any $\theta \in (\alpha, \beta)$ the equation

$$\lim_{r\to\infty} \frac{loglog\,|A(re^{i\theta})|^{-1}}{logr} = \rho$$

holds.

Before we give the next definition, we introduce the following result:

Lemma 8 (10) Let f be an entire function with $(0 < \rho(f) < \infty)$. Then an angular domain $S(\alpha, \beta)$ exists with $\beta - \alpha \geq \pi/\rho(f)$, where α and β are constants, such that

$$\lim_{r\to\infty} \frac{loglog|f(re^{i\theta})|}{logr} = \rho(f) \qquad (5)$$

for all $\theta \in (\alpha, \beta)$.

Definition 9 (11) A half straight line L_θ : $argz = \theta$ from the origin is called a radial line of order $\rho(f)$ of f if f satisfies (5), and an angular domain $S(\alpha, \beta)$ is called the radial angular domain of order $\rho(f)$ of f if for every $\theta \in (\alpha, \beta)$, L_θ is a radial line of order $\rho(f)$ of f .

The following result is obtained by Gundersen:

Theorem 10(12) Assume that$A(z)$ and $B(z)$ are entire functions satisfying one of the following conditions:
i) $A(z) < \rho(B)$;
ii) $A(z)$ is a polynomial and $B(z)$ is a transcendental entire function;
Then every nontrivial solution of Eq. (1) has infinite order.

The authors in (13) proved the following results:

Theorem 11 (13) Assume that$A(z)$is an entire function with finite order and with finite deficient value, $B(z)$ is a transcendental entire function with $\mu(B) < 1/2$ .Then any solution $f \neq 0$ of Eq.1 has infinite order.

Long, J. R. and Qiu, K.E. (14) proved the following result concerning both Eq. 1 and Eq. 2:

Theorem 12 (14) Suppose that $A(z)$ is a nontrivial solution of Eq. (2), and $B(z)$ is a transcendental entire function with $\mu(B) < 1/2$ and $\rho(A) \neq \rho(B)$. Then any solution $f \neq 0$ of Eq. (1) has $\rho(f) = \infty$.

J. Long in (15) proved the following result under the assumption that one of the coefficients of Eq.1 satisfy the extremal condition:

Theorem 13 (15) Suppose that$A(z)$is an entire function extremal for Yang's inequality, and $B(z)$ is an entire function with Fabry gaps. Then every nontrivial solution of Eq. (1) has infinite order.

Methods of Work:

In this section, we shall survey some results that we will be used to prove our results.

Lemma 14 (16, 17) Let (f,Γ) be a pair contains a finite order transcendental meromorphic function f and

$$\Gamma = \{(k_1,j_1),(k_2,j_2),\dots,(k_q,j_q)\}$$

denote a set of distinct integers order pairs satisfying $k_i > j_i \geq 0, i = 1,2,\dots,q$. Let $\varepsilon > 0$ be a constant. Then the following hold:

i) There is $E_1 \subset [0,2\pi)$ with zero linear measure, such that, when $\psi_0 \in [0,2\pi)\backslash E_1$, then a real constant $R_0 = R_0(\psi_0) > 1$ exists such that, for z with $arg\, z = \psi_0$ $|z| \geq R_0$, and for each $(k,j) \in \Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)} \qquad (6)$$

ii) There is $E_2 \subseteq (1,\infty)$ with $m_l(E_2) < \infty$, such that, for each z with $|z| \notin E_2 \cup [0,1]$ and, for each $(k,j) \in \Gamma$, we have (6).

iii) There is $E_3 \subset [0,\infty)$ with linear measure is finite, such that

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho+\epsilon)} \qquad (7)$$

for each z with $|z| \notin E_3$ and $(k,j) \in \Gamma$.

Lemma 15 (15) Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is a finite order entire function with Fabry gaps, and g is an entire function with $(0 < \rho(g) < \infty)$. Then, for any given $\varepsilon \in (0,\varsigma)$, where $\varsigma = min\{1,\rho(g)\}$, there is a set $F \subseteq (1,\infty)$ with $\overline{logdens}(F) \geq \eta$, where $\eta \in (0,1)$ is a constant, such that for any z with $|z| = r \in F$,

$$logL(r,f) > (1-\varepsilon)logM(r;f); \quad logM(r,g) > r^{\rho(g)-\varepsilon}$$

where $logL(r,f) = \min_{|z|=r}|f(z)|$, $logM(r,f) = \max_{|z|=r}|f(z)|$.

Lemma 16 (18) Suppose that A is an entire function extremal for Yang's inequality. Assume that there is $argz = \theta$ with $\theta_j < \theta < \theta_{j+1}$, $1 \leq j \leq q$, such that

$$\lim_{r\to\infty} sup \frac{loglog\left|A(re^{i\theta})\right|}{logr} = \rho(A)$$

where $\mathrm{argz} = \theta_j$ $(j = 1,2,\dots,q)$ are Borel directions of A. Then $\theta_{j+1} - \theta_j = \frac{\pi}{\rho(A)}$.

Lemma 17 (8) Suppose that f be an extremal function for Denjoy's conjecture. Then, for any $\theta \in [0,2\pi)$, either $\Delta(\theta)$ is a Borel direction of f, or there is a constant $\sigma \in (0,\pi/4)$, satisfying

$$\lim_{\substack{|z|\to\infty \\ z\in(S(\theta-\sigma,\theta+\sigma)-E)}} sup \frac{loglog|f(z)|}{log|z|} = \rho(f),$$

where $E \subseteq S(\theta-\sigma,\theta+\sigma)$, and satisfies

$$\lim_{r\to\infty} m(S(\theta-\sigma,\theta+\sigma;r,\infty) \cap E) = 0$$

Lemma 18 (10) Suppose that f is an entire function with $(0 < \rho(f) < \infty)$. Then an angular domain $S(\alpha,\beta)$ exists with $\beta - \alpha \geq \pi/\rho(f)$, where α and β are constants, such that

$$\lim_{r\to\infty} \frac{loglog\left|f(re^{i\theta})\right|}{logr} = \rho(f) \qquad (8)$$

for all $\theta \in (\alpha,\beta)$.

Lemma 19 (3) Let $f \neq 0$ be a solution of Eq. 2. Put $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = S(\theta_j,\theta_{j+1}), j = 0,1,2,\dots,n+1$ and $\theta_{n+1} = \theta_0 + 2\pi$. Then f satisfies the following properties:

1) In each sector S_j, f either blows up or decays to zero exponentially.

2) If f decays to zero in S_j for some j, then it should blow up in S_{j-1} and S_{j+1}. However, it is probable for f to blow up in many adjacent sectors.

3) If f decays to zero in S_j, then f has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup \bar{S}_j \cup S_{j+1}$.

4) If f blows up in S_{j-1} and S_j, then for each $\varepsilon > 0$, in each sector $\bar{S}(\theta_j-\varepsilon,\theta_j+\varepsilon)$, f has infinitely many zeros, furthermore, as $r \longrightarrow \infty$,

$$n(\bar{S}(\theta_j-\varepsilon,\theta_j+\varepsilon,r),0,f) = (1+o(1))\frac{2\sqrt{|a_n|}}{\pi(n+1)} r^{\frac{n+2}{2}}$$

where $n(\bar{S}(\theta_j-\varepsilon,\theta_j+\varepsilon,r),0,f)$ is the number of zeros of f in $\bar{S}(\theta_j-\varepsilon,\theta_j+\varepsilon,r)$.

Lemma 20 (5) Suppose that f is an entire function with $(1/2 \leq \mu(B) < \infty)$. Then a sector $S(\alpha,\beta) = \{z: \alpha < argz < \beta\}$ exists with $\beta - \alpha \geq \pi/\mu(B)$, such that

$$\lim_{r\to\infty} sup \frac{loglog\left|f(re^{i\theta})\right|}{logr} \geq \mu(f)$$

holds for each rays $arg\, z = \theta \in (\alpha,\beta)$, where $0 \leq \alpha < \beta \leq 2\pi$.

Lemma 21(18) Suppose that f is analytic function in $D = S(\alpha,\beta) \cap \{z: |z| > r_0\}$ and continuous in $\bar{D}$ and α,β,r_0 are constants with $0 < \beta - \alpha \leq 2\pi$ and $r_0 > 0$. Assume that there is a constant $M > 0$ such that $|f(z)| \leq M$ for $z \in \partial D$. If

$$\lim_{r\to\infty} inf \frac{loglogM(r,D,f)}{logr} < \frac{\pi}{\beta-\alpha}$$

where $M(r,D,f)=\max_{\substack{|z|=r\\ z\in D}}|f(z)|$, then $|f(z)|\le M$ for all $z\in D$.

Lemma 22 (12) Suppose that $A(z)$ and $B(z)$ are two entire functions such that for real constants $\alpha,\beta,\theta_1,\theta_2$, where $\alpha>0,\beta>0$ and $\theta_1<\theta_2$, we have

$$|A(z)|\ge \exp\{(1+o(1))\alpha\,|z|^{\beta}\}$$

and

$$|B(z)|\le \exp\{o(1)|z|^{\beta}\}$$

as $z\longrightarrow\infty$ in $\bar{S}(\theta_1,\theta_2)=\{z:\theta_1\le argz\le\theta_2\}$. Let $\varepsilon>0$ be a given small constant and let $\bar{S}(\theta_1+\varepsilon,\theta_2-\varepsilon)=\{z:\theta_1+\varepsilon\le argz\le\theta_2-\varepsilon\}$. If $f\ne 0$ is finite order solution of Eq. (1), then the following conclusions hold:

1) There is a constant $b(\ne 0)$ such that $f(z)\to b$ as $z\to\infty$ in $\bar{S}(\theta_1+\varepsilon,\theta_2-\varepsilon)$. Furthermore,

$$|f(z)-b|\le \exp\{-(1+o(1))\alpha\,|z|^{\beta}\}$$

as $z\longrightarrow\infty$ in $\bar{S}(\theta_1+\varepsilon,\theta_2-\varepsilon)$.

2) For each integer $k>1$,

$$\left|f^{(k)}(z)\right|\le \exp\{-(1+o(1))\alpha\,|z|^{\beta}\}$$

as $z\longrightarrow\infty$ in $\bar{S}(\theta_1+\varepsilon,\theta_2-\varepsilon)$.

Lemma 23 (19) Let (f,Γ) denote a pair that consists of a transcendental meromorphic function $f(z)$ and a set

$$\Gamma=\{(k_1,j_1),(k_2,j_2),\dots,(k_q,j_q)\}$$

of distinct pairs of integers satisfying $k_i>j_i\ge 0$ for $i=1,2,\dots,q$. Let $\alpha>0$ and $\varepsilon>0$ be given real constants. Then the following hold.

i) There is a set $E_1\subset[0,2\pi)$ that has linear measure zero, and there is a constant $c>0$ that depends only on α and Γ such that if $\varphi_0\in[0,2\pi)\backslash E_1$, then a constant $R_0=R_0(\varphi_0)>1$ exists such that for all z with $arg\,z=\varphi_0$ and $|z|=r\ge R_0$, and for all $(k,j)\in\Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le c\left(\frac{T(\alpha r,f)}{r}\,log^{\alpha}r\log T(\alpha r,f)\right)^{k-j} \quad (9)$$

In particular, if f has a finite order $\rho(f)<\infty$, then (9) is replaced with:

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\le c|z|^{(k-j)(\rho(f)-1+\epsilon)} \quad (10)$$

ii) There is a set $E_2\subset[1,\infty)$ that has finite logarithmic measure, and there is a constant $c>0$ that depends only on α and Γ such that for all z with $|z|=r\notin E_2\cup[0,1]$ and for all $(k,j)\in\Gamma$, the inequality (9) holds. In particular, if $\rho(f)<\infty$, then the inequality (10) holds.

iii) There is a set $E_3\subset[0,\infty)$ that has finite linear measure, and there exists a constant $c>0$ that depends only on α and Γ such that for all z with $|z|=r\notin E_3$ and for all $(k,j)\in\Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\le c(T(\alpha r,f)r^{\varepsilon}\,log\,T(\alpha r,f))^{k-j} \quad (11)$$

In particular, if $\rho(f)<\infty$, then (11) is replaced with

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\le c|z|^{(k-j)(\rho(f)+\epsilon)} \quad (12)$$

Results and Discussion:

In what follows, we generalize Theorem 13 in which the condition on $B(z)$ is replaced with the condition that $B(z)$ is an extremal function for Denjoy's conjecture:

Theorem 24 Let $A(z)$ be an entire function extremal for Yang's inequality, and let $B(z)$ be an extremal function for Denjoy's conjecture. Then every nontrivial solution of Eq. (1) is of infinite order.

Proof: Suppose that there exists a nontrivial solution f of Eq. (1) with $\rho(f)<\infty$, we hope getting a contradiction. Because $B(z)$ is entire function, then by Lemma 15, for any given $\varepsilon\in(0,\rho(B)/4)$, there is a set $E_1\subseteq(1,\infty)$ with $\overline{logdens}(E_1)>0$, such that for all z with $z=r\in E_1$, the following

$$|B(z)|>\exp(r^{\rho(B)-\varepsilon}) \quad (13)$$

hold. By Lemma 14 (ii), there is a set $E_2\subseteq(1,\infty)$ with $m_l(E_2)<\infty$, such that for all z satisfying $|z|\notin E_2\cup[0,1]$, the following

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\le|z|^{2\rho(f)},\qquad k=1,2 \quad (14)$$

hold.

Let $a_i<\infty,\ 1\le i\le p$ be all the deficient values of $A(z)$. Thus we have $2p$ sectors $S_j=\{z:\ \theta_j<argz<\theta_{j+1}\},\ j=1,2,\dots,2p$ such that $A(z)$ has the following property: in each sector S_j, either there is some a_i such that

$$log\frac{1}{|A(z)-a_i|} > C\left(\theta_j,\theta_{j+1},\varepsilon,\delta(a_i,A)\right)T(|z|,A) \quad (15)$$

holds for $z\in S(\theta_j+\epsilon,\theta_{j+1}-\varepsilon;r,\infty)$, where $C\left(\theta_j,\theta_{j+1},\varepsilon,\delta(a_i,A)\right)=C$ is a positive constant, or there is $\theta_j<argz<\theta_{j+1}$ such that

$$\lim_{r\to\infty}sup\frac{loglog\left|A(re^{i\theta})\right|}{logr}=\rho(A) \quad (16)$$

hold.

Observe that if there exists some a_i such that (15) holds in S_j, then there exists $argz=\theta$ such that (16) holds in S_{j-1} and S_{j+1}. If there exists

$\theta \in (\theta_j, \theta_{j+1})$ such that (16) holds, then there are a_i $(a_{i'})$ such that (15) holds in S_{j-1} and S_{j+1}, respectively.

We do not loss the generality if we assume that there is a ray $argz = \theta$ in S_1 such that (16) holds. Therefore, there exists a ray in each sector $S_3, S_5, \dots, S_{2p-1}$, such that (16) holds. By using Lemma 16, all the sectors have the same magnitude $\pi/\rho(A)$. Thus there exists a sequence $z_n = r_n e^{i\theta}$with $r_n \to \infty$ as $n \to \infty$, and finite deficient value a_{j_0}, where $r_n \in E_1 - (E_2 \cup [0,1])$and $\theta \in (\theta_j, \theta_{j+1}), j = 2,4, \dots, 2p$ such that

$$\left|A(r_n e^{i\theta}) - a_{j_0}\right| < \exp(-CT(r_n, A)) \qquad (17)$$

$$\left|B(r_n e^{i\theta})\right| > \exp(r_n{}^{\rho(B)-\varepsilon}) \qquad (18)$$

and

$$\left|\frac{f^{(k)}(r_n e^{i\theta})}{f(r_n e^{i\theta})}\right| \le r_n{}^{2\rho(f)}, \qquad k = 1,2 \qquad (19)$$

hold.

Next, according to Lemma 17, we consider two cases. We consider one of the sectors $S_j, 1 \le j \le 2p$, say $S_1 = S(\theta_1, \theta_2)$. This implies

$$log\frac{1}{\left|A(r_n e^{i\theta}) - a_{j_0}\right|} > CT(r_n, A) \qquad (20)$$

holds for all $z = r_n e^{i\theta} \in S_1$and sufficiently large n:

Case 1 Suppose that the ray $arg\, z = \theta$ is not Borel direction of $B(z)$, where $\theta_1 < \theta < \theta_2$. By Lemma 17, there exist a constant $\sigma \in (0, \pi/8\ )$ such that

$$\lim_{\substack{|z| \to \infty \\ z \in (S(\theta-\sigma, \theta+\sigma) - E_3)}} sup\frac{loglog|B(z)|}{log|z|} = \rho(B),$$

where $E_3 \subseteq S(\theta - \sigma, \theta + \sigma)$ satisfying

$$\lim_{r \to \infty} m(S(\theta - \sigma, \theta + \sigma; r, \infty) \cap E_3) = 0$$

Let$A = \{z: argz = \psi \in E_2\}$. Then there exists a sequence $z_n \to \infty$ as $n \to \infty$,$\{z_n\} \subseteq (S(\theta - \sigma, \theta + \sigma) - E_3) \cap (S_1 - \Delta)$ such that

$$\lim_{n \to \infty} \frac{loglog|B(z_n)|}{log|z_n|} = \rho(B), \qquad (21)$$

Combining (18), (19), (20), (21) and Eq. (1) we get

$$\begin{aligned}\exp(r_n{}^{\rho(B)-\varepsilon}) &< \left|B(r_n e^{i\theta})\right| \\ &\le \left|\frac{f''(r_n e^{i\theta})}{f(r_n e^{i\theta})}\right| \\ &+ \left|A(r_n e^{i\theta})\right| \left|\frac{f'(r_n e^{i\theta})}{f(r_n e^{i\theta})}\right| \\ &\le r_n{}^{2\rho(f)}\left(1 + \left|A(r_n e^{i\theta})\right|\right) \\ &\le r_n{}^{2\rho(f)}\left(1 + \left|a_{j_0}\right| \right. \\ &\left. + \exp(-CT(r_n, A))\right)\end{aligned}$$

holds for all sufficiently large n. This is a contradiction.

Case 2 Suppose that the ray$argz = \theta$is Borel direction of $B(z)$, where $\theta_1 < \theta < \theta_2$. By using Lemma 18, there exists an angular domain $S(\theta_1, \theta_2)$ with $\beta - \alpha \ge \pi/\rho(B)$, such that

$$\lim_{r \to \infty} sup\frac{loglog\left|B(re^{i\varphi})\right|}{logr} = \rho(B) \qquad (22)$$

for any $\theta_1 < \varphi < \theta_2$ where $0 \le \theta_1 < \theta_2 \le 2\pi$.

Let S be the radial angular domain of order$\rho(B)$of$B(z)$. Then, it follows that Borel direction of $B(z)$ either lie inside of S or lie on the boundary of S. Obviously, $S(\theta_1, \theta_2)$ is a radial angular domain of order $\rho(B)$ of $B(z)$. Hence, if $argz = \theta$is on the boundary of $S(\theta_1, \theta_2)$, without loss of generality, say $\theta = \theta_2$, then there exists a constant$\delta > 0$such that$S(\theta - \delta, \theta) \subseteq S(\theta_1, \theta) \cap S(\theta_1, \theta_2)$, and (22) holds for any $\theta - \delta < \varphi < \theta$.

By Lemma 14(i), there is $\varphi_0 \in S(\theta - \delta, \theta)$ and $R = R(\varphi_0) > 1$, such that (14) holds for all $r > R$. Note that (17) holds for $argz = \varphi_0$, and

$$\lim_{r \to \infty} sup\frac{loglog\left|B(re^{i\varphi_0})\right|}{logr} = \rho(B),$$

Thus there is a sequence r_n with $r_n \to \infty$ as $n \to \infty$, such that (14) and (15) hold for $|z| = r = r_n$.

From (14), (18), (19), (20) and Eq. (1) we get a contradiction as above. Therefore the result hold in the case $argz = \theta$ is on the boundary of $S(\alpha, \beta)$.

If $argz = \theta$ lie inside of $S(\alpha, \beta)$, then there exists a constant $\delta > 0$ such that$S(\theta - \delta, \theta + \delta) \subseteq S(\alpha, \beta) \cap S(\alpha_1, \beta_1)$. Using similar procedure used in the case $argz = \theta$ is on the boundary of $S(\alpha, \beta)$, we also get a contradiction. This completes the proof.

In the following, we modified Theorem 12 to obtain the following result in which the condition $\rho(A) \ne \rho(B)$is replaced with$\mu(B) \ne \rho(A)$and the condition $\mu(B) < 1/2$ is deleted:

Theorem 25 Let $A(z)$ be a nontrivial solution of Eq. (2), and let $B(z)$ be a transcendental entire function with $\mu(B) \ne \rho(A)$. Then every nontrivial solution of Eq. (1) has infinite order.

Proof: By Theorema10 and Theorem 11 it is enough to prove the Theorem in case $\frac{1}{2} \le \mu(B) < \rho(A)$. Suppose that there exists a nontrivial solution f of Eq. (1) with $\rho(f) < \infty$. We must get a contradiction. Put $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = \{z: \theta_j < argz < \theta_{j+1}\}$, where $0 \le j \le n+1$ and $\theta_{n+2} = \theta_0 + 2\pi$. We consider two cases according to Lemma 19.

Case1: Suppose that $A(z)$ blows up exponentially in each sector S_j, where $0 \le j \le n+1$; that is, for any $\theta \in (\theta_j, \theta_{j+1})$, we have

$$\lim_{r\to\infty}\frac{loglog\left|A(re^{i\theta})\right|}{logr}=\rho(A)=\frac{n+2}{2}\qquad(23)$$

Then for any given constant $\varepsilon\in(0,\frac{\pi}{4\rho(A)})$ and $\eta\in(0,\frac{\rho(A)-\mu(B)}{4})$, we have

$$|A(z)|\geq\exp\left\{\left(1+o(1)\right)\alpha|z|^{\frac{n+2}{2}-\eta}\right\}\qquad(24)$$

$$|B(z)|\leq\exp(|z|^{\mu(B)+\eta})\leq\exp(|z|^{\rho(A)-2\eta})\leq\exp\left\{o(1)|z|^{\frac{n+2}{2}-\eta}\right\}\qquad(25)$$

as $z\to\infty$ in $S_j(\varepsilon)=\{z:\theta_j+\varepsilon<argz<\theta_{j+1}-\varepsilon\}$, $0\leq j\leq n+1$, where α is a positive constant depending on ε. Combining (24), (25), and Lemma 22, there exist corresponding constants $b\neq0$ such that

$$|f(z)-b|\leq\exp\{-(1+o(1))\alpha|z|^{\frac{n+2}{2}-\eta}\}\qquad(26)$$

as $z\to\infty$ in $S_j(2\varepsilon)$, $0\leq j\leq n+1$. Therefore, f is bounded in the whole complex plane by Lemma 21. So, by Liouville's Theorem, f is a nonzero constant in the whole complex plane. This contradicts the fact that Eq. (1) doesn't have nonzero constant solutions.

Case 2: There is at least one sector of the $n+2$ sectors, such that $A(z)$ decays to zero exponentially, say $S_{j_0}(\varepsilon)=\{z:\theta_{j_0}+\varepsilon<argz<\theta_{j_0+1}-\varepsilon\}$, $j_0\in\{0,1,\ldots,n+1\}$. That is, for any $\theta\in(\theta_{j_0},\theta_{j_0+1})$,we have

$$\lim_{r\to\infty}\frac{loglog\left|\frac{1}{A(re^{i\theta})}\right|}{logr}=\frac{n+2}{2}\qquad(27)$$

Since $\frac{1}{2}\leq\ \mu(B)<\rho(A)$, by Lemma 20 we see that there exists a sector $S(\alpha,\beta)$ with $\beta-\alpha\geq\pi/(\rho(B))$, $0\leq\alpha<\beta\leq2\pi$ such that for all the rays $arg\,z=\theta\in(\alpha,\beta)$ we have

$$\lim_{r\to\infty}sup\frac{loglog\left|B(re^{i\theta})\right|}{logr}\geq\mu(B)\qquad(28)$$

Observe that $\mu(B)<\rho(A)$. Thus there exists a sector $S(\alpha',\beta')$, where $\alpha<\alpha'<\beta'<\beta$, such that (27) and (28) hold for all $\theta\in(\alpha',\beta')$. By using Lemma 23 (i), there exists $\theta\in(\alpha',\beta')$ and $R>1$ such that

$$\left|\frac{f^k(re^{i\theta_0})}{f(re^{i\theta_0})}\right|\leq r^{2\rho(f)},\qquad k=1,2\qquad(29)$$

holds for all $r>R$. Note that (28) holds for $\theta=\theta_0$. Thus there is a sequence (r_n) with $r_n\to\infty$ as $n\to\infty$, such that

$$\left|B(r_ne^{i\theta_0})\right|\geq\exp\left({r_n}^{\mu(B)-\varepsilon}\right)\qquad(30)$$

holds for every $0<\epsilon<\mu(B)$. Therefore, we conclude from (27), (28), (29), (30) and Eq.1 that

$$\begin{aligned}\exp\left({r_n}^{\mu(B)-\varepsilon}\right)&\leq\left|B(r_ne^{i\theta_0})\right|\\&\leq\left|\frac{f''(r_ne^{i\theta_0})}{f(r_ne^{i\theta_0})}\right|+\left|A(r_ne^{i\theta_0})\right|\left|\frac{f'(r_ne^{i\theta_0})}{f(r_ne^{i\theta_0})}\right|\\&\leq{r_n}^{2\rho(f)}\left(1+o(1)\right)\end{aligned}\qquad(31)$$

holds. Obviously, for all sufficiently large n this is a contradiction. This completes the proof.

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حول نمو الحلول للمعادلات التفاضلية الخطية العقدية والتي معاملاتها تحقق شروط معينة

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الخلاصة:

في هذا البحث ندرس نمو الحلول للمعادلات التفاضلية العقدية الخطية من الرتبة الثانية $f'' + A(z)f' + B(z)f = 0$ ونبرهن ان اي حلول غير تافهة لها رتبة غير منتهية. بفرض ان تحقق المعاملات الشرط الرأسي لمتراجحة يانك والشرط الرأسي لتخمين دينجوي. الشرط الاخر هو ان احد المعاملات هو نفسه حل للمعادلة التفاضلية $f'' + P(z)f = 0$.

الكلمات المفتاحية: تخمين دينجوي، الدالة الكلية، رتبة النمو، رتبة النمو السفلى، دالة يانك الرأسية.