Oscillation Criteria for Solutions of Neutral Differential Equations of Impulses Effect with Positive and Negative Coefficients

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Abstract:

In this paper, some necessary and sufficient conditions are obtained to ensure the oscillatory of all solutions of the first order impulsive neutral differential equations. Also, some results in the references have been improved and generalized. New lemmas are established to demonstrate the oscillation property. Special impulsive conditions associated with neutral differential equation are submitted. Some examples are given to illustrate the obtained results.

Key words: Impulsive Neutral Differential Equations, Oscillation, Variable Delays, Variable Coefficients.

Introduction

The oscillatory theory of impulsive delay differential equations is appearing as an important field of investigation, because it is much richer than the theory of delay differential equations without impulses effects. The differential equations with impulses effects describe the process of evolution that rapidly changes their state at certain moments; therefore this type of differential equations is suitable for the mathematical simulation of the evolutionary process in which the parameters are subject to relatively long periods of smooth variation followed by rapid short-term change.

The wide possibility of applications determines the increasing interest in impulsive differential equations. The importance of the need to study differential equations with impulses is due to the fact that these equations are more comprehensive in their use of mathematical modeling where gaps in the model can be addressed by limiting these gaps in specific points called the points of impulses (which are not continuous points) in many real processes and phenomena studied in control theory, biology, mechanics, medicine, electronic, economic, etc.

For instance, there are a lot of applications of impulsive differential equations in neural networks (1–4), n control theory (5), in biology (6) and economics (7).

Mohamad et al. (8) obtained sufficient conditions for the oscillation of all solutions of neutral differential equations with variable delays, while H. Chen et al (9) studied the existence of solutions for impulsive differential equations. Isaac (10) classifies non-oscillatory solutions of impulsive differential equations of the second order. So many papers focused on studying the oscillation of impulsive neutral differential equations and impulsive systems (11, 12).

The main purpose of this paper is to study one of the important properties of the solution for the impulsive neutral differential equations with positive and negative coefficients, it is the oscillation property. The sufficient conditions to guarantee the oscillation of all solutions for the impulsive neutral differential equation with positive and negative coefficients have been obtained.

Consider the impulsive neutral differential equation of form:

\[
\begin{align*}
&[x(t) - P(t)x(\tau(t))]' + Q(t)x(\sigma(t)) \\
&- R(t)x(\alpha(t)) = 0, \\
&\quad t \neq t_k, \quad k = 1, 2, ... \\
&x(t_k^+) + b_kx(t_k) = a_kx(t_k), \\
&\quad t = t_k, \quad k = 1, 2, ...
\end{align*}
\]

(1)

Where \( t_k \) are the moments of impulses effect. The numbers \( a_k \) and \( b_k \) are positive real numbers, \( k = 1, 2, ... \), \( P \in PC([t_0, \infty), R^+) \), \( PC([t_0, \infty), R^+) \) denotes the set of all functions \( f: [t_0, \infty) \to R^+ \) such that \( f \) is continuous for \( t \neq t_k, \quad k = 1, 2, ... \) and \( f(t_k) = \lim_{t \to t_k^-} f(t) \).

Let \( Q, R \in C([t_0, \infty), R^+) \), and \( \tau, \sigma, \alpha \) are continuous strictly increasing functions with \( \lim_{t \to \infty} \tau(t) = \infty, \lim_{t \to \infty} \alpha(t) = \infty \).
The functions \( \sigma^{-1}(t), \sigma^{-1}(t), \sigma^{-1}(t) \) are the inverse of the functions \( \tau(t), \sigma(t), \sigma(t) \) respectively, \( \tau(t) < t \).

The initial function has defined:
\[ x(t) = \omega(t), t \in [\rho(t_0), t_0]. \]
\[ \rho(t) = \min \tau(t), \alpha(t), \sigma(t), \]
where \( \omega(t) \in P(C(t_0, R)). \)

**Definition 1:** A function \( f \) is piecewise continuous on \([a, b]\) if
(a) \( f(x_0^+) \) exists for all \( x_0 \) in \([a, b]\);
(b) \( f(x_0^-) \) exists for all \( x_0 \) in \([a, b]\);
(c) \( f(x_0^+) = f(x_0^-) = f(x_0) \) for all but finitely many points \( x_0 \) in \([a, b]\);

If (c) fails to hold at some \( x_0 \) in \([a, b]\), \( f \) has a jump discontinuity at \( x_0 \). Also, \( f \) has a jump discontinuity at \( a \) if \( f(a^+) \neq f(a) \) or at \( b \) if \( f(b^+) \neq f(b) \).

**Definition 2:** A function \( x(t) \) is said to be a solution of Eq.(1) which is satisfying the initial condition if

1. \( x(t) = \alpha(t) \) for \( \rho(t_0) \leq t \leq t_0 \) and \( x(t) \) is continuous for \( t \geq t_0 \) and \( t \neq t_k \), \( k = 1,2,\ldots \);
2. \( x(t) = P(t)x(\tau(t)) \) is piecewise continuously differentiable \( t \geq t_0 \) and \( t \neq t_k, \), \( t \neq \tau^{-1}(t_k), t \neq \tau^{-1}(t_k), t \neq \tau^{-1}(t_k), k = 1,2,\ldots \)
3. \( x(t_k^+) = x(t_k^-) \) and \( x(t_k^+) = x(t_k^-) \) satisfies impulsive Eq.(1) where \( x(t_k^+) = \lim_{t \to t_k^+} x(t_k) \) and \( x(t_k^-) = \lim_{t \to t_k^-} x(t_k) \).

**Definition 3:** The function \( f(t):[t_0, \infty) \to R \) is said to be eventually enjoyable the property \( P \) if there exist an interval \([t_0, \infty) \subset [0, \infty) \) in which \( f(t) \) enjoys the property \( P \) for \( t \geq t_0 \). So a function \( x(t) \in C([t_0, \infty), R) \) is said to be eventually positive (negative), if there exist \( t_1 \geq t_0 \) such that \( x(t) > 0 \) \( x(t) < 0 \) for all \( t \geq t_1 \).

**Definition 4:** A regular solution \( x(t) \) of Eq. (1) is said to be oscillatory in \([t_0, \infty) \) if it has arbitrarily large zeros for \( t \geq t_1 \geq t_0 \), that is, there exist a sequence of zeros \( \{t_n\} \) such that \( \lim_{n \to \infty} t_n = \infty \) where \( x(t_n) = 0 \), otherwise \( x(t) \) is said to be nonoscillatory on \([t_1, \infty) \), that is \( x(t) \neq 0 \) for each \( t \geq t_1 \), which means that either \( x(t) \) is eventually positive, or is eventually negative.

**Some Basic Lemmas:**

The following lemmas will be useful to prove the main results:

**Lemma 1:** Suppose that \( g, h:[t_0, \infty) \to R \) are continuous functions, \( g(t) \geq 0 \) eventually, \( h(t) \geq t \) and \( h'(t) \geq 0 \) for \( t \geq t_0 \). If

\[
\lim_{t \to \infty} \sigma(t) = \infty.
\]

Then the inequality \( x'(t) - g(t)x(h(t)) \geq 0 \) has no eventually positive solutions.

**Lemma 2:** Suppose that \( g, h \in [t_0, \infty) \to [0, \infty), h(t) \leq t \) and \( \lim_{t \to \infty} h(t) = \infty \). If

\[
\lim_{t \to \infty} \int_t^{h(t)} g(s)ds > \frac{1}{e}
\]

then the inequality \( x'(t) + g(t)x(h(t)) \leq 0 \) has no eventually positive solutions.

The following lemma is a generalization of the lemma 1.5.4 (14).

**Lemma 3:** Assume that

\( f, g, y, \tau, \gamma \in C([t_0, \infty), R) \), \( f(t) \leq 0 \), \( \lim_{t \to \infty} f(t) \) exist, \( 0 < g(t) \leq 1 \), \( \tau(t) < t, \gamma(t) \geq t, \gamma(t) \geq t_0 \), \( \lim_{t \to \infty} \tau(t) = \infty \) and

\[
y(t) \leq f(t) + g(t)\max\{y(s): \tau(t) \leq s \leq \gamma(t)\}, \quad t \geq t_0
\]

Then \( y(t) \) cannot be positive for \( t \geq t_1 \geq t_0 \).

**Proof:** The part (I) has been proved, proof of (II) is similar and will be omitted. For the sake of contradiction assume that \( y(t) > 0 \), \( t \geq t_0 \), it follows that \( y(t) \) must be bounded, otherwise, there exists a sequence \( \{t_n\} \),

\[
\lim_{n \to \infty} t_n = \infty, \lim_{t \to \infty \gamma(t)} y(t_n) = \infty \quad \text{and} \quad y(t_n) = \max\{y(s): 0 \leq s \leq y(t_n)\}
\]

Which is possible since \( y(t_n) \to \infty \), and there exists \( n_1 \) such that \( \tau(t_n) \geq t_0 \) for \( n \geq n_1 \) the inequality (4) reduces:

\[
y(t_n) \leq f(t_n) + g(t_n)\max\{y(s): \tau(t_n) \leq s \leq y(t_n)\}
\]

By taking superior limit to both sides of the above inequality it follows that \( M_1 < M_1 \). And that is contradiction.

**Remark 1:** In the following lemma, suppose that:
\[ W(t) = x(t) - P(t)x(\tau(t)) \]
\[ - \int_{a^{-1}(\delta(t))}^{b(t)} R(u)x(\alpha(u))du \]
\[ - \int_{\sigma^{-1}(\delta(t))}^{\tau(t)} Q(u)x(\sigma(u))du \]
\[ (6) \]

Where \( \delta(t) > t, \alpha(t) < t \) and \( \alpha(t) > t, \)
\( t \in (t_k, t_{k+1}] \).

0 < \tau_0 < t_1 < \cdots < t_k \to \infty \) as \( k \to \infty \).

Let \( a^{-1}(\delta(t)) \leq t \) and \( \sigma^{-1}(\delta(t)) > t, \)
in addition to the following assumptions:

H1: \[ R(a^{-1}(\delta(t)))(a^{-1}(\delta(t)))' \]
\[ - Q(\sigma^{-1}(\delta(t)))(\sigma^{-1}(\delta(t)))' \]
\[ \geq 0, t \geq t_0, t \in (t_k, t_{k+1}] \]

H2: There exist positive real numbers \( a_k \) and \( b_k \) such that
\[ a_k - b_k \geq 1, k = 1, 2, \ldots \] and
\[ P(t_k^+) \leq P(t_k) \text{ for } t_k \neq t_i, i < k, \]
\[ P(t_k^+) \leq \frac{1}{a_k - b_k} P(t_k) \text{ for } t_k = t_i, i < k, \]
where \( a_k = a_i, b_k = b_i \) when \( \tau(t_k) = t_i, i < k. \)

Lemma 4: Let \( W(t) \) be defined as in (6), H1-H2 hold.

\[ \liminf_{t \to \infty} \int_{t_k}^{t} \left[ R \left( a^{-1}(\delta(u)) \right) \left( a^{-1}(\delta(u)) \right)' \right. \]
\[ - Q \left( \sigma^{-1}(\delta(u)) \right) \left( \sigma^{-1}(\delta(u)) \right) ' \] \[ du > \frac{1}{e}, \]
\[ t \in (t_k, t_{k+1}] \]. \[ (7) \]

Let \( x(t) \) be an eventually positive solution of Eq.1. Then \( W(t) \) is eventually negative and nondecreasing function.

Proof. Let \( x(t) \) be an eventually positive solution of Eq.1, that is \( x(t) > 0, \)
\( x(\tau(t)) > 0, x(\sigma(t)) > 0 \) and \( x(\alpha(t)) > 0, t \geq t_0. \)

Differentiate Eq.(6) for every interval \( (t_k, t_{k+1}] \) where \( k = 1, 2, \ldots \) and use Eq.1, then:

\[ W'(t) \]
\[ = [x(t) - P(t)x(\tau(t))]' - R(t)x(\alpha(t)) \]
\[ + R(a^{-1}(\delta(t)))(a^{-1}(\delta(t)))' \]
\[ - Q(\sigma^{-1}(\delta(t)))(\sigma^{-1}(\delta(t)))' \]
\[ + Q(t)x(\sigma(t)) \]
\[ = -Q(t)x(\sigma(t)) + R(t)x(\alpha(t)) - R(t)x(\alpha(t)) \]
\[ + R(a^{-1}(\delta(t)))(a^{-1}(\delta(t)))' \]
\[ - Q(\sigma^{-1}(\delta(t)))(\sigma^{-1}(\delta(t)))' \]
\[ + Q(t)x(\sigma(t)) \]
\[ = -Q(t)x(\sigma(t)) \]
\[ \geq 0 \]
\[ (8) \]

Then: \( W'(t) \geq 0 \)

Hence \( W(t) \) is nondecreasing function on \( t_k < t \leq t_k+1, k = 1, 2, 3, \ldots \).

To prove that \( W(t_k^+) \geq W(t_k) \) for \( k = 1, 2, \ldots \) In view of \( a_k - b_k \geq 1 \) and condition H2 when \( \tau(t_k) = t_i, i < k, \) then:
\[ W(t_k^+) = (a_k - b_k)x(t_k) - P(t_k^+)x(t_k) \]
\[ [\int_{t_k}^{t_k^+} R(u)x(\alpha(u))du \]
\[ - \int_{t_k^+}^{t_k} Q(u)x(\sigma(u))du \]
\[ \geq x(t_k) - P(t_k)x(\tau(t_k)) \]
\[ - \int_{t_k^+}^{t_k} R(u)x(\alpha(u))du \]
\[ - \int_{t_k^+}^{t_k} Q(u)x(\sigma(u))du \]
\[ = W(t_k). \]

When \( \tau(t_k) \neq t_i, i < k \) then from H2
\[ W(t_k^+) = (a_k - b_k)x(t_k) - P(t_k)x(\tau(t_k)) \]
\[ - \int_{t_k}^{t_k^+} R(u)x(\alpha(u))du \]
\[ - \int_{t_k^+}^{t_k} Q(u)x(\sigma(u))du \]
\[ \geq x(t_k) - P(t_k)x(\tau(t_k)) \]
\[ - \int_{t_k^+}^{t_k} R(u)x(\alpha(u))du \]
\[ - \int_{t_k^+}^{t_k} Q(u)x(\sigma(u))du \]
\[ = W(t_k). \]

We claim that \( W(t) \leq 0 \) for \( t \in (t_k, t_{k+1}] \).

 Otherwise, there exists \( t^* \in (t_k, t_{k+1}] \) such that \( W(t^*) > 0 \). So \( W(t) \geq W(t^*) > 0, t \geq t^*, \)
(\( W(t) \leq x(\delta(t)) \) and \( W'(t) \)
\[ = [R \left( a^{-1}(\delta(t)) \right) \left( a^{-1}(\delta(t)) \right)'] \]
\[ - Q \left( \sigma^{-1}(\delta(t)) \right) \left( \sigma^{-1}(\delta(t)) \right) ' \]
\[ \geq [R \left( a^{-1}(\delta(t)) \right) \left( a^{-1}(\delta(t)) \right)'] \]
\[ - Q \left( \sigma^{-1}(\delta(t)) \right) \left( \sigma^{-1}(\delta(t)) \right) ' \]
\[ W'(t) - [R \left( a^{-1}(\delta(t)) \right) \left( a^{-1}(\delta(t)) \right)'] \]
\[ - Q \left( \sigma^{-1}(\delta(t)) \right) \left( \sigma^{-1}(\delta(t)) \right) ' \]
\[ \geq 0 \]

In view of condition (7) and Lemma 1, the last inequality cannot exist eventually positive solution,
which is a contradiction. Hence \( W(t) \leq 0 \) for 
\[ t \in [t_k, t_{k+1}], \quad k = l, l + 1, \ldots \]
It remains to prove that \( W(t_k) < 0 \) for 
\[ k = 1, 2, \ldots \] . If it is not true, then there exists 
\( \epsilon_m \in (t_k, t_{k+1}) \), such that \( W(\epsilon_m) = 0 \), by 
integrating (8) on \( (\epsilon_m, t_{m+1}) \), we have:
\[
W(t_{m+1}) = W(t_m) + \int_{\epsilon_m}^{t_{m+1}} [Q(a^{-1}(\delta(t)))\langle a^{-1}(\delta(t)) \rangle']
- R(a^{-1}(\delta(t)))\langle a^{-1}(\delta(t)) \rangle' x(\delta(t)) dt
\geq W(t_m) \geq W(t_m) = 0
\]
This contradiction shows that 
\( W(t_k) < 0 \) for \( k = 1, 2, \ldots \) therefore from (8) 
\( W(t) \leq W(t_{k+1}) < 0 \), where \( t_k \leq t < t_{k+1}, \quad k = 1, 2, \ldots \)
Thus \( W(t) < 0 \) for \( t \geq t_0 \).

**Remark 2:** In the following lemma, suppose that:
Let \( a^{-1}(\delta(t)) \leq t \) and \( t \leq a^{-1}(\delta(t)) \), and 
\[ H4: \langle Q(a^{-1}(\delta(t)))\langle a^{-1}(\delta(t)) \rangle' \rangle \]
\[ - R(a^{-1}(\delta(t)))\langle a^{-1}(\delta(t)) \rangle' x(\delta(t)) dt \geq 0, \quad t \geq t_0, \quad t \in (t_k, t_{k+1}) \]
\[ H5: \text{There exists nonnegative real numbers } a_k \text{ and } b_k, \quad 0 < a_k - b_k \leq 1 \text{ , } \]
\[ k = 1, 2, \ldots \text{ such that } \]
\[ P(t_k^+) \geq P(t_k) \text{ for } \tau(t_k) \neq t_i, i < k, \]
\[ P(t_k^+) \geq \frac{1}{a_k - b_k} P(t_k) \text{ for } \tau(t_k) = t_i, i < k, \]
where 
\( a_k = a_i, b_k = b_i \) when \( \tau(t_k) = t_i, \quad i < k \),
\( \lim_{k \to \infty} W(t_k) = W(t_k) \)
\[ \text{When } \tau(t_k) \neq t_i, \quad i < k \text{ then } \]
\( \text{from } (6) \) with regard to the condition H5:
\[ W(t_k^+) = (a_k - b_k)x(t_k) - P(t_k^+)x(\tau(t_k)) \]
\[ \leq (a_k - b_k)x(t_k) - P(t_k)x(\tau(t_k)) \]
\[ \leq W(t_k) \]
\[ \text{When } \tau(t_k) = t_i, \quad i < k \text{ then } \]
\[ W(t_k^+) = (a_k - b_k)x(t_k) - P(t_k)x(\tau(t_k)) \]
\[ \leq (a_k - b_k)x(t_k) - P(t_k)x(\tau(t_k)) \]
\[ \leq W(t_k) \]
\[ \text{By } (1) \text{ and } \]
\[ \lim_{t \to \infty} \int_{\delta(t)^-}^{\delta(t)} Q(a^{-1}(\delta(t)))\langle a^{-1}(\delta(t)) \rangle' dx \geq 0, \quad t \geq t_0, \]
\[ t \in (t_k, t_{k+1}) \]
\[ \text{Where } \delta(t) < t, \quad t \in (t_k, t_{k+1}), \quad 0 < t_0 < t_1 < \]
\[ \cdots < t_k \to \infty \text{ as } k \to \infty. \]
\[ \text{Let } W(t) \text{ be defined as } \]
\[ \text{in } (6). \text{ Then } W(t) \text{ is eventually negative and } \]
nonincreasing function.

**Proof.** Let \( x(t) \) be an eventually positive solution of equation that is 
\( x(t) > 0, \quad x(\sigma(t)) > 0 \) and \( x(\alpha(t)) > 0, \quad t \geq t_0. \)
Differentiate (6) for every interval \( (t_k, t_{k+1}] \) where 
\( k = 1, 2, \ldots \text{ and use Eq.(1), then:} \]
\[ W'(t) = [x(t) - P(t)x(\tau(t))]' - R(t)x(\alpha(t)) \]
\[ + R(a^{-1}(\delta(t)))x(\delta(t))x(\alpha(t)) - a^{-1}(\delta(t))x(\delta(t))x(\alpha(t))' \]
\[ - Q(a^{-1}(\delta(t)))x(\delta(t)x(\alpha(t))') \]
\[ + Q(t)x(\sigma(t)) \]
\[ = -Q(t)x(\sigma(t)) + R(t)x(\alpha(t)) - R(t)x(\alpha(t)) \]
\[ + R(a^{-1}(\delta(t)))x(\delta(t))x(\alpha(t))' \]
\[ - Q(a^{-1}(\delta(t)))x(\delta(t))x(\alpha(t))' \]
\[ - Q(t)x(\sigma(t)) \]
\[ = -Q(t)x(\sigma(t)) + R(t)x(\alpha(t)) \]
\[ + R(a^{-1}(\delta(t)))x(\delta(t))x(\alpha(t))' \]
\[ - Q(a^{-1}(\delta(t)))x(\delta(t))x(\alpha(t))' \]
\[ + Q(t)x(\sigma(t)) \]
\[ \leq W(t_k) \]
\[ \text{By } (12) \]
\[ \text{W(t) is nonincreasing on } [t_0, \infty), \]
\[ -\infty \leq \lim_{t \to \infty} W(t) = L < \infty. \text{ Where } \]
\[ |L| = \sup\{W(t_k^+) \}, \quad k \to \infty, W(t_k^+), \quad t \in [t_i, \infty) \]
\[ \text{Suppose that } W(t) \leq 0 \text{ for } t \in (t_k, t_{k+1}], \quad k = l, l + 1, \ldots \text{ Otherwise, there exists } \]
\[ t^* \in (t_k, t_{k+1}] \text{ such that } \]
\[ W(t^*) > 0. \text{ So } W(t) \geq W(t^*) > 0, \]
\[ t_k < t \leq t^* \]
\[ W(t) \leq x(t) \text{ for } t \in (t_k, t_{k+1}] \]
\[ W(\delta(t)) \leq x(\delta(t)) \text{ and } \]
\[ W'(t) = -[Q\left(\sigma^{-1}(\delta(t))\right)\sigma^{-1}(\delta(t))]' - R\left(\alpha^{-1}(\delta(t))\right)\left(\sigma^{-1}(\delta(t))\right)'x(\delta(t)) \]
\[ W'(t) \leq -[Q\left(\sigma^{-1}(\delta(t))\right)\sigma^{-1}(\delta(t))]' - R\left(\alpha^{-1}(\delta(t))\right)\left(\sigma^{-1}(\delta(t))\right)'W(\delta(t)) \]
\[ = W(t^{+}) + \int_{t^{-}}^{t} Q(u)\sigma^{-1}(\delta(t))\sigma^{-1}(\delta(t))'x(\delta(t))dt \]
\[ \leq W(t^{-}) \]

This contradiction shows that \( W(t_{k}) < 0 \) for \( k = 1, 2, \ldots \), therefore from (11) it follows that \( W(t) \leq W(t_{0}) < 0 \), where
\[ t_{k} < t \leq t_{k+1}, \quad k = l, l + 1, \ldots. \]
Thus \( W(t) < 0 \) for \( t \geq t_{0} \).

**Main results**

**Remark 3:** In the following theorems, if \( y(t) = x(t) \) almost everywhere, where \( y(t) \in C(t_{0}, \infty); R \) and \( x(t) \in PC(\rho(t_{0}), R) \).

The next results provide sufficient conditions for the oscillation of all solutions of Eq. (1):

**Theorem 1.** Let \( W(t) \) be defined as in (6) and the assumptions H1 – H2 and (7) hold, in addition to the condition
\[ P(t) + \int_{\alpha^{-1}(\delta(t))}^{\sigma^{-1}(\delta(t))} R(u)du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)du \]
\[ \leq 1, \quad t \geq t_{0} \] (13)

Where \( t_{k} < t < \delta(t) \leq t_{k+1}, \)
\( t_{k} < t \leq \tau^{-1}(\delta(u)) < t \leq t_{k+1}, \quad k = l, l + 1, \ldots, k_{l} < t \leq \gamma^{-1}(\delta(t)) \leq t_{k+1}. \)

Then every bounded solution of Eq. (1) oscillates.

**Proof.** Assume that \( x(t) \) be bounded eventually positive solution of Eq. (1). By lemma 4 it follows that \( W(t) \) is eventually negative and nondecreasing, from (6) we get:

\[ x(t) = W(t) + P(t)x(\tau(t)) \]
\[ + \int_{\alpha^{-1}(\delta(t))}^{t} R(u)x(\alpha(u))du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)x(\sigma(u))du \]
\[ \leq W(t) \]
\[ + \left[ P(t) + \int_{\alpha^{-1}(\delta(t))}^{t} R(u)du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)du \right] \max\{x(s); \rho(t) \leq s \}
\[ \leq W(t) \]
\[ \leq W(t_{1}) \]
\[ + \left( P(t) + \int_{\alpha^{-1}(\delta(t))}^{t} R(u)du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)du \right) \max\{x(s); \rho(t) \leq s \} \leq \sigma^{-1}(\delta(t)) \]

By lemma 3, we get \( x(t) \) is eventually negative, a contradiction.

**Theorem 2.** Let \( W(t) \) be defined as in (6), and let \( H1 - H5, (10) \) and (13) hold, then every bounded solution of Eq. (1) oscillates.

**Proof.** Assume that \( x(t) \) is eventually positive solution of Eq. (1.1). From lemma 5, it follows that \( W(t) \) is eventually negative and nonincreasing. From (6) for \( t \geq t_{1} \geq t_{0} \) we get:

\[ x(t) = W(t) + P(t)x(\tau(t)) \]
\[ + \int_{\alpha^{-1}(\delta(t))}^{t} R(u)x(\alpha(u))du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)x(\sigma(u))du \]
\[ \leq W(t_{1}) \]
\[ + \left( P(t) + \int_{\alpha^{-1}(\delta(t))}^{t} R(u)du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)du \right) \max\{x(s); \rho(t) \leq s \} \leq \sigma^{-1}(\delta(t)) \]

Consider \( \frac{\strut x(t)}{\strut \rho(t)} \) and \( \frac{\strut t_{1}}{\strut \tau(t)} \) and \( \frac{\strut t}{\strut \alpha^{-1}(\delta(t))} \), max\{x(s); \rho(t) \leq s \} at \( (10) \) and \( (13) \) hold, Then every bounded solution of Eq. (1) oscillates.

**Example 1:** Consider the impulsive neutral differential equation
\[
\begin{align*}
[x(t) - e^{-\pi}(1 - e^{-t})x(t - 2\pi)]' + e^{-t}x(t - 2\pi) &= 0, \\
(1 - e^{-\pi} + e^{-\pi}e^{-t})x(t + \frac{\pi}{2}) &= 0,
\end{align*}
\]

Let
\[
x(t) = \begin{cases} 
sin t, & t \neq t_k \\
1 + \frac{1}{k}, & t = t_k \\
k, & t = t_k
\end{cases}
\]

where \(a_k = \frac{k+2}{k}\) and \(b_k = \frac{1}{k}\).

We can see that
\[
d(t) = t + \pi = \frac{t}{4}.
\]

H1: \([R(\alpha^{-1}(\delta(t)))\alpha^{-1}(\delta(t))]' - Q(\alpha^{-1}(\delta(t)))\alpha^{-1}(\delta(t))]'\]

\[
= R(t - \frac{\pi}{4}) - Q(t + \frac{9\pi}{4})
\]

\[
= (1 - e^{-\pi} + e^{-\pi}e^{-t+\frac{\pi}{4}}) - e^{-\pi}e^{-t+\frac{9\pi}{4}}
\]

\[
= 0.956786 + 0.094744e^{-t} > 0,
\]

\(t \in (t_k, t_{k+1}).\)

\[
P(t_k^+) = P(k^+) = \lim_{t \to k^+} P(t)
\]

\[
= \lim_{t \to k^+} e^{-\pi}(1 - e^{-t}) = e^{-\pi}(1 - e^{-k})
\]

\[
\leq 0.043213
\]

\[
< \frac{k^2}{k + 1} = \frac{1}{a_k - b_k}P(t_k)
\]

for \(k = 2, 3, \ldots\)

\[
P(t_k^+) = 0.043213 < \frac{k+1}{k+2} = k + 1
\]

\[
= (a_k - b_k)P(t_k)
\]

for \(k = 2, 3, \ldots\)

So, H2 holds.

\[
\begin{align*}
\lim_{t \to \infty} \int_t^{t+\frac{\pi}{4}} & \left[ R(\sigma^{-1}(\delta(t)))\alpha^{-1}(\delta(t)) \right]' \\
& - Q(\sigma^{-1}(\delta(t)))\alpha^{-1}(\delta(t)) \right]' du \\
= \lim_{t \to \infty} \int_t^{t+\frac{\pi}{4}} & \left[ R(u - \frac{\pi}{4}) \\
& - Q(u + \frac{9\pi}{4}) \right] du \\
= \lim_{t \to \infty} \int_t^{t+\frac{\pi}{4}} & [0.956786 + 0.094744e^{-u}] du
\end{align*}
\]

\[
= 0.751457 > \frac{1}{e}
\]

\[
[P(t) + \int_t^{t+\frac{\pi}{4}} R(u)du + \int_t^{t+\frac{\pi}{4}} Q(u)du]\]

\[
= [e^{-\pi}(1 - e^{-t})
\]

\[
+ \int_t^{t+\frac{\pi}{4}} (1 - e^{-\pi} + e^{-\pi}e^{-u}) du
\]

\[
+ \int_t^{t+\frac{9\pi}{4}} e^{-\pi}e^{-u} du
\]

\[
= 0.845385 < 1.
\]

Hence all conditions of theorem 1 are fulfilled, then according to theorem 1 all solutions of equation (14) oscillate. It is obvious that

\[
x(t) = \begin{cases} 
sin t, & t \neq t_k \\
1 + \frac{1}{k}, & t = t_k
\end{cases}
\]

is such an oscillatory solution.

Example 2: Consider the impulsive neutral differential equation

\[
[x(t) - e^{-2\pi}(1 - e^{-t})x(t - 2\pi)]' + e^{-\pi}(1 - e^{-t})x(t + \frac{\pi}{2}) = 0,
\]

\(t \neq t_k, k = 1, 2, \ldots\)

\[
x(t_k^+) = \frac{k+1}{k+2}x(t_k),
\]

\(t = t_k
\]

\[
P(t_k^+) = P(k^+) = \lim_{t \to k^+} P(t)
\]

\[
= \lim_{t \to k^+} e^{-2\pi}(1 - e^{-t}) = e^{-2\pi}(1 - e^{-k})
\]

\[
\leq 0.001187
\]

\[
< \frac{k^2}{k + 1} = \frac{1}{a_k - b_k}P(t_k)
\]

for \(k = 2, 3, \ldots\)

\[
P(t_k^+) = 0.001187 < \frac{k+1}{k+2} = k + 1
\]

\[
= (a_k - b_k)P(t_k)
\]

for \(k = 2, 3, \ldots\)

So \(0 \leq P(t_k^+) \leq 0.001187,\)

if \(k = 1\) then \(P(t_1^+) = 0.001187 = \frac{1}{15000(k+2)} \]

\[
= \frac{1}{15000k(k+2)}
\]

If \(k = 1\) then \((a_1 - b_1)P(t_1) = 0.000444, \) and

\[
\frac{1}{(a_k - b_k)}P(t_k) = \frac{1}{k+2} \frac{1}{15000k(k+1)}
\]

If \(k = 1\), then \(\frac{1}{(a_1 - b_1)}P(t_1) = 0.001,\) so H5 holds.

Where \(\delta(t) = t - \frac{\pi}{4}\).

H4: \([R(\alpha^{-1}(\delta(t)))\alpha^{-1}(\delta(t))]' - Q(\alpha^{-1}(\delta(t)))\alpha^{-1}(\delta(t))]'\)

\[
= R(t - \frac{5\pi}{4})
\]

\[
- Q(t + \frac{5\pi}{4})
\]

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\[
\begin{align*}
e^{-2\pi}e^{-t\frac{5\pi}{4}} & - (1 - e^{-2\pi} \\
+ e^{-2\pi}e^{-t\frac{\pi}{4}} & = -0.903376 - 0.00851e^{-t} < 0, \\
t & \in (t_k, t_{k+1}].
\end{align*}
\]

H5: \(P(t_k^+) = \frac{1}{2} \leq \frac{k^2}{k+1} \leq \frac{1}{a_k-b_k} P(t_k), t \neq t_k \) for 

\(k = 1, 2, \ldots \)

\(P(t_k^+) = \frac{1}{2} \leq \frac{k+1}{k} = k + 1 = (a_k - b_k)P(t_k), t = t_k \) for \(k = 1, 2, \ldots \)

\[
\liminf_{\delta(t) \rightarrow 0} \int_{\delta(t)}^{t} \left[ Q(\sigma^{-1}(\delta(u)))(\sigma^{-1}(\delta(u)))' \right] du \\
= \liminf_{\delta(t) \rightarrow 0} \int_{t}^{t+\frac{\pi}{4}} \left[ Q(u + \frac{\pi}{4}) - R \left( u - \frac{5\pi}{4} \right) \right] du \\
= \liminf_{\delta(t) \rightarrow 0} \int_{t}^{t+\frac{\pi}{4}} \left[ 0.903376 + 0.00851e^{-u} \right] du \\
= 0.709509 > \frac{1}{e^{2\pi}}
\]

\[
[ P(t) + \int_{t}^{t+\frac{\pi}{4}} R(u) du + \int_{t}^{t+\frac{\pi}{4}} Q(u) du ] \\
e^{-2\pi} \left( 1 - e^{-t} \right) \\
+ \int_{t}^{t+\frac{\pi}{4}} e^{-2\pi}e^{-u} du \\
+ \int_{t}^{t+\frac{\pi}{4}} \left( 1 - e^{-2\pi} \\
+ e^{-2\pi}e^{-u} \right) du \\
= 0.879703 \leq 1
\]

Hence all conditions of theorem 2 are fulfilled, then according to theorem 2 all solutions of equation (15) oscillate. It is obvious that

\[
x(t) = \begin{cases} 
\cos t, & t \neq t_k \\
1 + \frac{1}{5k}, & t = t_k
\end{cases}
\]
is such oscillatory solution with impulsive.

**Conclusion:**
In this paper, the impulsive neutral differential equations are studied. The impulses characteristics of the first order neutral differential equations have been clarified. Some necessary and sufficient conditions have been obtained to ensure that all bounded solutions of the first order neutral differential equations are oscillatory. The lemma 1.5.4 in (14) has been generalized, some new lemmas have been submitted to obtain the main results of the oscillation property. Illustrative examples of the obtained results have been provided.

**Authors' declaration:**
- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

**References:**
معيارية التذبذب لحلول المعادلات التفاضلية المحايدة ذات المعاملات الموجبة والسالبة وتأثير النبضات

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الخلاصة:
في هذا البحث تم دراسة المعادلات التفاضلية المحايدة النابضة. تم توضيح الخواص النابضة لمعادلات تفاضلية من رتبة الأولى. تم الحصول على بعض الشروط الضرورية والكافية لتذبذب جميع الحلول المقيدة لمعادلات تفاضلية محيدة نابضة من الرتبة الأولى. تم تقديم التمهيديات 1.5.4 في (14). تم تطبيق بعض التمهيديات الجديدة للحصول على النتائج الرئيسية لخاصية التذبذب. تم تجهيز مثال لنتائج البحث التي تم الحصول عليها.

الكلمات المفتاحية: معادلات تفاضلية محيدة نابضة، التذبذب، التباطؤات المتغيرة، المعاملات المتغيرة.