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Properties of Fuzzy Compact Linear Operators on Fuzzy Normed Spaces

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Abstract:

In this paper the definition of fuzzy normed space is recalled and its basic properties. Then the definition of fuzzy compact operator from fuzzy normed space into another fuzzy normed space is introduced after that the proof of an operator is fuzzy compact if and only if the image of any fuzzy bounded sequence contains a convergent subsequence is given. At this point the basic properties of the vector space FC(V,U)of all fuzzy compact linear operators are investigated such as when U is complete and the sequence (T_n) of fuzzy compact operators converges to an operator T then T must be fuzzy compact. Furthermore we see that when T is a fuzzy compact operator and S is a fuzzy bounded operator then the composition TS and ST are fuzzy compact operators. Finally, if T belongs to FC(V,U) and dimension of V is finite then T is fuzzy compact is proved.

Key words: Complete fuzzy normed space, Fuzzy normed space, Fuzzy bounded operator, Fuzzy compact operator, Relatively compact set.

Introduction:

Some results of fuzzy complete fuzzy normed spaces were studied by Saadati and Vaezpour in 2005 (1). Properties of fuzzy bounded linear operators on a fuzzy normed space were investigated by Bag and Samanta in 2005 (2).The fuzzy normed linear space and its fuzzy topological structure were studied by Sadeqi and Kia in 2009 (3). Properties of fuzzy continuous operators on a fuzzy normed linear spaces were studied by Nadaban in 2015 (4). The definition of the fuzzy norm of a fuzzy bounded linear operator was introduced by Kider and Kadhum in 2017 (5). Fuzzy functional analysis is developed by the concepts of fuzzy norm and a large number of researches by different authors have been published for reference please see (6, 7, 8, 9, 10).

The structure of this paper is as follows:

In section two we recall the definition of fuzzy normed space (11) also some basic definitions and properties of this space, that we will need later in this paper and the definition of three types of fuzzy convergence sequence of operators. The main results can be found in third section. The aim of this paper is to introduce the notion of fuzzy compact operator from a fuzzy normed space to another fuzzy normed space and some basic properties of this type of operators are investigated and proved.

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Results and Dissections:

Properties of Fuzzy Normed Spaces

In this section we will recall basic properties of fuzzy normed space

Definition 1:(1)

Suppose that U is any set, a fuzzy set $\widetilde{A}$ in U is equipped with a membership function, $\mu_{\widetilde{A}}(u)$: U→ [0,1]. Then, $\widetilde{A}$ is represented by $\widetilde{A}=\{(u,\mu_{\widetilde{A}}(u)): u\in U, 0 \le \mu_{\widetilde{A}}(u) \le 1\}$.

Definition 2:(1)

Let $\otimes : [0,1] \times [0,1] \to [0,1]$ be a binary operation then ⊗ is called a continuous **t -norm** (or **triangular norm**) if forall $\alpha,\beta,\gamma,\delta \in [0,1]$it has the following properties

(1)$\alpha\otimes\beta = \beta\otimes\alpha$,

(2)$\alpha\otimes 1 = \alpha$,

(3)$(\alpha\otimes\beta)\otimes\gamma = \alpha\otimes(\beta\otimes\gamma)$

(4)If $\alpha \le \beta$ and $\gamma \le \delta$ then, $\alpha\otimes\gamma \le \beta\otimes\delta$

Remark 3:(5)

(1)If $\alpha > \beta$ then there is γ such that $\alpha\otimes\gamma \ge \beta$

(2)There is δ such that $\delta\otimes\delta \ge \sigma$ where $\alpha,\beta,\gamma,\delta,\sigma \in [0,1]$

Definition 4 :(5)

The triple$(V, L_V, \otimes)$is said to be a **fuzzy normed space** if V is a vector space over the field 𝔽, ⊗is a t-norm and $L_V : V \times [0,\infty) \to [0,1]$ is a fuzzy set has the following properties for all a, b∈ V and $\alpha,\beta > 0$.

1-$L_V(a,\alpha) > 0$

2-$L_V(a,\alpha) = 1 \Leftrightarrow a = 0$

3-$L_V(ca, \alpha) = L_V\left(a, \frac{\alpha}{|c|}\right)$ for all $c \neq 0 \in \mathbb{F}$
4-$L_V(a, \alpha) \otimes L_V(b, \beta) \leq L_V(a + b, \alpha + \beta)$
5-$L_V(a, .): [0, \infty) \to [0,1]$is continuous
6-$\lim_{\alpha\to\infty} L_V(a, \alpha) = 1$

Remark 5 : (6)

Assume that $(V, L_V, \otimes)$is a fuzzy normed space and let $a \in V, t > 0, 0 < q < 1$. If $L_V(a, t) > (1 - q)$ then there is s with $0 < s < t$ such that $L_V(a, s) > (1 - q)$.

Definition 6:(5)

Suppose that $(V, L_V, \otimes)$is a fuzzy normed space. Put
FB(a ,p, t) = {b∈V: L_V (a−b, t) > (1−p)}
FB[a ,p, t] = {b∈V: L_V (a−b, t) ≥ (1−p)}
Then FB(a, p, t) and FB[a, p, t] are called **open and closed fuzzy ball** with the center ain V and radius p, with $p > 0$.

Definition 7:(6)

Assume that $(V, L_V, \otimes)$is a fuzzy normed space. A ⊆V is called **fuzzy bounded** if we can find $t > 0$ and $0 < q < 1$ such that $L_V(a, t) > (1 - q)$ for each a ∈ A.

Definition 8 :(5)

A sequence(a_n) in a fuzzy normed space$(V, L_V, *)$ is called **converges to** $a \in V$ if for each $q > 0$ and $t > 0$ we can find N∈ ℕ with $L_V[a_n - a, t] > (1 - q)$ for all $n \geq N$. This is equivalent to $\lim_{n\to\infty} L_V[a_n - a, t] = 1$. Or in other word $\lim_{n\to\infty} a_n = a$ or simply represented by $a_n \to a$, the vector a is known as the limit of (a_n).

Definition 9 :(5)

A sequence(a_n)in a fuzzy normed space $(V, L_V, \otimes)$is said to be a **Cauchy sequence** if for all $0 < q < 1$, $t > 0$ there is a number N∈ ℕ with $L_V[a_m - a_n, t] > (1 - q)$ for all m, n≥ N.

Definition 10:(5)

Suppose that $(V, L_V, \otimes)$is a fuzzy normed space and let A be a subset of V**.** Then A is said to be open if for each a in A there is FB(a, p, t) such that FB(a, p, t) ⊆ A. Also a subset B is said to be closed if B^c is an open set in V.

Definition 11:(5)

Suppose that $(V, L_V, \otimes)$is a fuzzy normed space and let A be a subset of V. Then, the **closure ofA** is written by$\overline{A}$or CL(A)and which is $\overline{A}$= ⋂{B⊆V:B is closed and A⊆B}

Definition 12:(5)

Suppose that $(V, L_V, \otimes)$ is a fuzzy normed space and A ⊆ V. Then A is called **dense** in V when $\overline{A} = V$.

Lemma 13:(5)

Assume that $(V, L_V, \otimes)$ is a fuzzy normed space and A is a subset of V. Then, $y \in \overline{A}$ if and only if there is a sequence(y_n) in A with (y_n) converges to y.

Lemma 14:

If A and B are subsets of a fuzzy normed space $(V, L_V, \otimes)$ then $\overline{A+B} = \overline{A} + \overline{B}$.

Proof:

Let a+b ∈ $\overline{A} + \overline{B}$ then by Lemma 2.13 there is a sequence (a_n) in A such that $\lim_{n\to\infty} L_V(a_n - a, t)$=1 and there is a sequence (b_n) in B such that $\lim_{n\to\infty} L_V(b_n - b, s)$=1 for all t, s>0. Now $\lim_{n\to\infty} L_V[(a_n + b_n) - (a + b), t+s]= \lim_{n\to\infty} L_V[(a_n - \alpha) - (b_n - b), t+s]$

$$\geq \lim_{n\to\infty} L_V(a_n - a, t) \otimes \lim_{n\to\infty} L_V(b_n - b, s)$$

= 1⊗ 1= 1.

That is $\lim_{n\to\infty} L_V[(a_n + b_n) - (a + b), t+s]=1$. Therefore $(a_n + b_n)$ is a sequence in A+B converge to (a+b). Hence a+b ∈ $\overline{A+B}$. Thus $\overline{A} + \overline{B} \subseteq \overline{A+B}$.

Similarly, we can prove that $\overline{A+B} \subseteq \overline{A} + \overline{B}$. Hence $\overline{A+B} = \overline{A} + \overline{B}$.

Theorem 15:(5)

Suppose that$(V, L_V, \otimes)$ is a fuzzy normed space and A is a subset of V. Then A is dense in Vif and only if for every $x \in V$there is $a \in A$ such that$L_V[x - a, t] > (1 - \varepsilon)$for some $0 < \varepsilon < 1$ and $t > 0$.

Definition 16:(1)

A fuzzy normed space $(V, L_V, \otimes)$ is said to be **complete** if every Cauchy sequence in V converges to a point in V.

Definition 17:(2)

Suppose that $(V, L_V, \otimes)$ and$(W, L_W, \odot)$ are two fuzzy normed spaces .The operator $S: V \to W$ is said to be **fuzzy continuous at v_0** $\in V$ if for all $t > 0$ and for all $0 < \alpha < 1$there is s and there is β with, $L_V[v - v_0, s] > (1 - \beta)$ we have $L_W[S(v) - S(v_0), t] > (1 - \alpha)$ for all v ∈V.

Theorem 18:(5)

Suppose that $(V, L_V, \otimes)$and$(U, L_U, \odot)$are two fuzzy normedspaces. The operator $T: V \to U$ is fuzzy continuous at a ∈V if and only if $a_n \to a$ in V implies $T(a_n) \to T(a)$ in U.

Definition 19:(5)

Suppose that $(V, L_V, \otimes)$ and$(W, L_W, \odot)$ are two fuzzy normed spaces. An operator $T: D(T) \to W$ is said to be **fuzzy bounded** if there exists r, $0 < r < 1$ such that $L_W(Tx, t) \geq (1 - r) \otimes L_V(x, t)$, for each $x \in D(T) \subseteq X$ and t> 0 where ⊗ is a continuous t-norm and D(T) is the domain of T.

Theorem 20:(5)

Suppose that $(V, L_V, \otimes)$ and$(W, L_W, \odot)$ are two fuzzy normed spaces. The operator S:D(S) →W is fuzzy bounded if and only if S(A) is fuzzy bounded for every fuzzy bounded subset A of D(S).

Put FB(V,W) ={S:V→W, S is a fuzzy bounded operator} when $(V, L_V, \otimes)$ and $(W, L_W, \odot)$ are two fuzzy normed spaces (5).

Theorem 21:(5)

Suppose that $(V, L_V, \otimes)$ and $(W, L_W, \odot)$ are two fuzzy normed spaces. Define $\mathbf{L(T, t)} = \mathbf{inf}_{x\in D(T)} \mathbf{L_Y(Tx, t)}$ for all $T \in FB(V, W)$ and $t > 0$ then $(FB(V, W), L, *)$ is fuzzy normed space.

Theorem 22 :(5)

Suppose that $(V, L_V, \otimes)$ and $(W, L_W, \odot)$ are two fuzzy normed spaces with S:D(S) →W is a linear operator where D(S) ⊆V. Then, S is fuzzy bounded if and only if S is fuzzy continuous.

Corollary 23 :(5)

Suppose that $(V, L_V, \otimes)$ and $(W, L_W, \odot)$ are two fuzzy normed spaces. Assume that $T: D(T) \to W$ is a linear operator where $D(T) \subseteq V$. Then, T is a fuzzy continuous if T is a fuzzy continuous at $x \in D(T)$.

Definition 24:(6)

Suppose that $(V, L_V, \otimes)$ is a fuzzy normed space and W ⊆ V then, it is said to be **totally fuzzy bounded** if for any σ ∈ (0, 1), $t > 0$ we can find $W_\sigma = \{a_1, a_2, \dots, a_n\}$ in W with any v ∈ V there is some $a_i \in \{a_1, a_2, \dots, a_k\}$ with $L_V(v - a_i, t) > (1 - \sigma)$. Then, W_σ is called ***σ* –fuzzy net**.

Definition 25:(11)

Suppose that $(V, L_V, \otimes)$ is a fuzzy normed space and W is a subset of V. Assume that Ψ = { A ⊆ V: A is open sets in V } where $W \subseteq \cup_{A\in\Psi} A$. Then, Ψ is said to be an **open cover** or open covering of W. If $\Psi = \{A_1, A_2, \dots, A_k\}$ with W= $\cup_{i=1}^{k} A_i$ then, Ψ is known as a finite **sub covering** of W.

Definition 26:(11)

A fuzzy normed space $(V, L_V, \otimes)$ is called **compact** if V $=\cup_{A\in\Psi} A$ where Ψ is an open covering then, we can find $\{A_1, A_2, \dots, A_n\} \subset \Psi$ with V= $\cup_{i=1}^{k} A_i$.

Theorem 27:(11)

The fuzzy normed space $(V, L_V, \otimes)$ is compact if and only if every (v_n) in V contains (v_{n_k}) with $v_{n_k} \to$v.

Lemma 28:

If A and B are two compact subsets of a fuzzy normed space $(V, L_V, \otimes)$ then, A+B is compact.

Proof:

Let { G_i: i∈I} be an open covering for A+B then there are $I_1 \subseteq$I and $I_2 \subseteq$I such that{ G_i: i∈ I_1 } is an open covering for A and { G_k: k∈ I_2 } is an open covering for B. But A and B are compact so, there is a finite sub covering Ψ_1 of { G_i: i∈ I_1 } for A and a finite sub covering Ψ_2 of { G_k: k∈ I_2} for B. Hence $\Psi_1 \cup \Psi_2$ is a finite sub covering of { G_i: i∈I } for A+B. Hence, A+B is compact.

Proposition 29:(6)

Let $(V, L_V, \otimes)$ be a fuzzy normed space if V is totally fuzzy bounded then, V is fuzzy bounded.

Proposition 30:(6)

If the fuzzy normed space $(V, L_V, \otimes)$ is compact then, it is totally fuzzy bounded.

Definition 31 :(5)

A linear functional f from a fuzzy normed space $(V, L_V, \otimes)$ into the fuzzy normed space $(F, L_F, *)$ is said to be **fuzzy bounded** if there exists r, $0 < r < 1$ such that $L_F[f(x), t] \geq (1 - r) \otimes L_V[x, t]$ for all $x \in D(f)$ and $t > 0$. Furthermore, the fuzzy norm of f is
$L(f, t) = \inf L_F(f(x), t)$ and $L_F(f(x), t) \geq L(f, t) \otimes L_V(x, t)$.

Definition 32 :(5)

Suppose that $(V, L_V, \otimes)$ is a fuzzy normed space. Then, the vector space FB(V,𝔽) ={f:V→ 𝔽, f is fuzzy bounded linear function } with a fuzzy norm defined by $\mathbf{L(f, t) = inf\, L_F\,(f(x), t)}$ form a fuzzy normed space which is called the fuzzy dual space of V.

Definition 33:(5)

A sequence (v_n) in a fuzzy normed space $(V, L_V, \otimes)$ is said to **fuzzy weakly convergent** if we can find v ∈ V with every h ∈ FB(V,R) $\lim_{n\to\infty} h(v_n) = h(v)$. This is written $v_n \to^{w} v$ the element v is said to be the weak limit to (v_n) and (v_n) is said to be fuzzy converges weakly to v.

Definition 34:(6)

Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces. A sequence (T_n) operators $T_n \in$ FB(V, U) is said to be

1.**Fuzzy uniformly operator convergent** if there is $T: V \to U$ with $L[T_n - T, t] \to 1$ for any $t > 0$ and n≥N.

2.**Fuzzy strong operator convergent** if $(T_n v)$ converges in U for every v∈ V that is that is there is $T: V \to U$ with $L_U[T_n v - Tv, t] \to 1$ for every $t > 0$ and $n \geq N$.

3.**Fuzzy weakly operator convergent** if for every v ∈ V there is $T: V \to U$ with $L_R[f(T_n v) - f(Tv), t]$ for every $t > 0$, f ∈ FB(U, ℝ) and n≥N.

Definition 35:(6)

Let $(V, L_V, \otimes)$ be a fuzzy normed space. A sequence (h_n) of functional $h_n \in FB(V, \mathbb{R})$ is called

1) **Fuzzy strong converges** in the fuzzy norm on FB(V, R) that is there is h ∈ FB(V, ℝ) with $L[h_n - h, t] \to 1$ for all $t > 0$ this written $h_n \to h$

2) **Fuzzy weak converges** in the fuzzy norm on R that is there is h ∈ FB(V,ℝ) with $h_n(v) \to f(v)$ for every v ∈ V written by $\lim_{n\to\infty} h_n(v) = h(v)$.

Fuzzy Compact linear operators

Definition 36:

Let $(V, L_V, \otimes)$ be a fuzzy normed space and $A \subset V$. Then, A is called **relatively compact** if $\overline{A}$ is compact

Definition 37:

Let $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ be two fuzzy normed spaces. An operator $T: V \to U$ is called **fuzzy compact** linear operator if T is linear and if for every fuzzy bounded subset $E \subset V$ the image $T(E)$ is relatively compact.

Lemma 38:

If $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces then, every fuzzy compact linear operator $T: V \to U$ is fuzzy bounded and hence, fuzzy continuous

Proof:

The set E = FB(0, 1, t) with $t > 0$ is fuzzy bounded since it is fuzzy open ball. By assumption T is fuzzy compact so $\overline{T(E)}$ is compact which implies that it is totally fuzzy bounded by Propositions 30 so, it is fuzzy bounded by Propositions 29. Now $L_U[T(v), t] > (1 - r)$ for every $v \in E$ so $L[T, t] = \inf L_U[T(v), t] > (1 - r)$ hence T is fuzzy bounded by Theorem 20. Hence, by Theorem 22 T is fuzzy continuous.

Theorem 39:

Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces and let $S: V \to U$ be a linear operator. Then, S is fuzzy compact if and only if for every fuzzy bounded sequence (v_n) in V then $(S(v_n))$ has convergent subsequence in U.

Proof:

Let S be a fuzzy compact operator and let (v_n) be fuzzy bounded sequence in V so $(S(v_n))$ is relatively compact by Definition 37 that is the closure of $(S(v_n))$ is compact in U by definition 36 hence, by Theorem 27, $(S(v_n))$ contains a convergent subsequence.

Conversely, suppose that every fuzzy bounded sequence (v_n) in V contains a subsequence (v_{n_k}) such that $(S(v_{n_k}))$ converges in U. Let E be any fuzzy bounded subset of V and let (u_n) be any sequence in $S(E)$. Then $u_n = S(v_n)$ for some $v_n \in E$ and (v_n) is fuzzy bounded since E is fuzzy bounded. Now by assumption $(S(v_n))$ contains a convergent subsequence. Hence $\overline{S(E)}$ is compact. But (u_n) in $S(E)$ was arbitrary, therefore S is fuzzy compact operator.

Lemma 40:

If $(V, L_V, \otimes)$ is a fuzzy normed spaces and A is a compact subset of V then, αA is compact for every $\alpha \neq 0 \in \mathbb{R}$.

Proof:

Let (αa_n) be a sequence in αA then (a_n) is a sequence in A but A is compact so by Theorem 27 (a_n) has a subsequence (a_{n_k}) such that $a_{n_k} \to a \in A$. Hence $\alpha a_{n_k} \to \alpha a \in \alpha A$ that is (αa_n) has a convergent subsequence (αa_{n_k}). Thus αA is compact.

Lemma 41:

If $(V, L_V, \otimes)$ is a fuzzy normed space and A, B are relatively compact subset of V then A + B and αA are relatively compact.

Proof:

Suppose that A, B are two relatively compact subsets of V then, $\overline{A}$ and $\overline{B}$ are compact by Definition 36. Now by using lemma 28 we have $\overline{A} + \overline{B}$ is compact and $\overline{A} + \overline{B} = \overline{A + B}$ by Lemma 14 so $\overline{A + B}$ is compact. Hence A + B is relatively compact. Similarly we can prove that αA is relatively compact by using Lemma 38.

Theorem 42:

Suppose that $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces then

FC(V, U) = { S: S:V→U is a fuzzy compact linear operator } is a vector space over the field $\mathbb{F}$ (where $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$).

Proof:

Let T_1, $T_2 \in$ FC(V,U) and $\alpha \neq 0 \in \mathbb{R}$. Suppose that E is a fuzzy bounded subset of V then, $T_1(E)$ and $T_2(E)$ are relatively compact. Now by using lemma 41, $T_1(E) + T_2(E) = (T_1 + T_2)(E)$ is relatively compact. So, $T_1 + T_2 \in$ FC(V, U). Also by using Lemma 41, we see that $\alpha T_1(E)$ is relatively compact so, $\alpha T_1 \in$ FC(V, U). Hence FC(V, U) is a vector space over the field $\mathbb{R}$.

Theorem 43:

Let $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ be two fuzzy normed spaces and $T: V \to U$ be a linear operator. Then

(1) If T is fuzzy bounded and dim T(V) $< \infty$ then, the operator T is fuzzy compact

(2) If dim V $< \infty$ then, the operator T is fuzzy compact.

Proof (1):

Let (v_n) be any fuzzy bounded sequence in V. Then, by using $L_U[Tv_n, t] \geq L[T, t] \otimes L_V[v_n, t]$ we see that (Tv_n) is fuzzy bounded. Hence, (Tv_n) is relatively compact since $\dim T(V) < \infty$. It follows that $(T(v_n))$ has a convergent subsequence. But (v_n) was an arbitrary fuzzy bounded sequence in V hence, by Theorem 27 the operator T is fuzzy compact.

Proof (2):

Since $\dim V < \infty$ so by [6,Theorem 2.5] it follows that T is fuzzy bounded and since dim T(V) $\leq$ dim V so, using part (1) it follows that T is fuzzy compact.

Theorem 44:

Let $(V, L_V, \otimes)$ be a fuzzy normed space and let $(U, L_U, \odot)$ be a complete fuzzy normed space. If $T_n \in FC(V, U)$ and (T_n) is fuzzy uniformly operator convergent to T then, $T \in FC(V, U)$

Proof:

Let(v_m) be a fuzzy bounded sequence in V, sinceT_1 is fuzzy compact$(T_1 v_m)$ has a subsequence $(T_1 v_{1m})$ such that$(T_1 v_{1m})$is a Cauchy sequence. Similarly(v_{1m}) has a subsequence(v_{2m}) such that$(T_2 v_{2m})$ is a Cauchy sequence. Continuing in this way we obtain$(y_m) = (v_{mm})$ is a subsequence of(v_m) such that for every fixed $n \in \mathbb{N}$ the sequence $(T_n y_m)_{m=1}^{\infty}$ is a Cauchy sequence. Since(v_m) is fuzzy bounded so there is $\sigma \in$ (0, 1) with$L_V[v_m, t] \geq (1 - \sigma)$for all m. Hence, $L_V[y_m, t] \geq (1 - \sigma)$ for any m. Since$T_m \to T$ there is m=p such that $L[T - T_p, t] > (1 - r)$for some 0<r<1. Also since $(T_p y_m)_{m=1}^{\infty}$ is a Cauchy sequence, there is an N$\in \mathbb{N}$ with $L_U[T_p y_i - T_p y_k, t] > (1 - r)$.

Now we can find $0 < \varepsilon < 1$ such that

$(1 - r)\otimes(1 - \sigma)\odot(1 - r)\odot(1 - r)\otimes(1 - \sigma) > (1 - \varepsilon)$.

Hence for $j, k \geq N$, we obtain

$$L_U[Ty_j - Ty_k, t] \geq L_U\left[Ty_j - T_p y_j, \frac{t}{3}\right] \odot L_U\left[T_p y_j - T_p y_k, \frac{t}{3}\right] \odot L_U\left[T_p y_k - Ty_k, \frac{t}{3}\right]$$
$$\geq L\left[T - T_p, \frac{t}{3}\right] \otimes L_V\left[y_j, \frac{t}{3}\right] \odot L_U\left[T_p y_j - T_p y_k, \frac{t}{3}\right] \odot L\left[T_p - T, \frac{t}{3}\right] \otimes L_V\left[y_k, \frac{t}{3}\right]$$

$\geq (1 - r)\otimes(1 - \sigma)\odot(1 - r)\odot(1 - r)\otimes(1 - \sigma) > (1 - \varepsilon)$.

This shows that(Ty_m) is a Cauchy sequence in U and converges since U is complete notice that$(y_m)$is a subsequence of the arbitrary fuzzy bounded sequence$(v_m)$and using Theorem 3.4 this proves that the operator T is fuzzy compact.

Lemma 45:

Suppose that $(V, L_V, \otimes)$ is a fuzzy normed space then, every relatively compact subset B of V is totally fuzzy bounded.

Proof:

Assume that B is relatively compact set and let $0 < r < 1$ be given. If $B = \emptyset$ then $\emptyset$ is an r-fuzzy net. If $B \neq \emptyset$ take any$m_1 \in B$. If $L_V[m_1 - b, t] > (1 - r)$ for all $t > 0$ and for all $b \in B$ then$\{m_1\}$ is an r-fuzzy net for B. Otherwise, let$m_2 \in B$ such that $L_V[m_1 - m_2, t] \leq (1 - r)$. If for all $b \in B$,$L_V[m_1 - b, t] > (1 - r)$ and$L_V[m_2 - b, t] > (1 - r)$ for $t > 0$ then, $\{m_1, m_2\}$ is an r-fuzzy net for B, otherwise let$m_3 \in B$ be a point such that$L_V[m_3 - m_1, t] \leq (1 - r)$ and$L_V[m_3 - m_2, t] \leq (1 - r)$. If for all $b \in B$, $L_V[m_j - b, t] > (1 - r)$ for $j = 1,2$ and 3 then$\{m_1, m_2, m_3\}$ is an r-fuzzy net for B. We continue by selecting an$m_4 \in$ B,etc. we obtained after k steps the set $\{m_1, m_2, \dots, m_k\}$ is an r-fuzzy net for B. Otherwise we have a sequence (m_j) satisfying $L_V[m_j - m_k, t] \leq (1 - r)$ bviously,(m_j) could not have a subsequence which is a Cauchy sequence hence(m_j) could not have a subsequence which converges in V. But this contradicts the relatively compactness ofB because(m_j)lies in B by the construction. Hence, there must be a finite r-fuzzy net for B. Since $0 < r < 1$ was arbitrary, we conclude that B is totally fuzzy bounded.

Lemma 46:

Let $(V, L_V, \otimes)$ be a complete fuzzy normed space and$B \subset V$. IfB is totally fuzzy bounded set then, B is relatively compact

Proof:

We consider any sequence(v_n) in B and will show that it has a subsequence which converges in V so thatB will be then relatively compact set. By assuming that B has a finite r-fuzzy net for r =0. Hence B is contained in the union of finitely many open fuzzy balls of radius 1 from these fuzzy balls we pick a fuzzy ballB_1which contains infinitely many terms of(v_n). Let$(v_{1,n})$ be the subsequence of (v_n) which lies inB_1.Again by assumption B is also contained in the union of finitely many fuzzy balls of radius $r = \frac{1}{2}$ from these balls we can pick a fuzzy ballB_2which contains a subsequence $(v_{2,n})$of the subsequence$(v_{1,n})$. We continue by choosing $r = \frac{1}{3}, \frac{1}{4}, \dots$ and setting$y_n = v_{n,n}$.Then for every given$0 < r < 1$ there is an N (depending on r) such that ally_n with$n > N$ lies in the fuzzy ball of radius r. Hence(y_n) is Cauchy. It converges in V say$y_n \to y \in V$, since V is a complete space. Also$y_n \in B$ implies$y \in \overline{B}$.Now by the definition of the closure for every sequence(z_n) in$\overline{B}$ there is a sequence (v_n) in B which satisfies$L_V[v_n - z_n, t] > \left(1 - \frac{1}{n}\right)$ for every n. Since (v_n) is in B, it has a subsequence which converges in$\overline{B}$ as we have just shown. Hence, (z_n) also has a subsequence which converges in$\overline{B}$since$L_V[v_n - z_n, t] \geq \left(1 - \frac{1}{n}\right)$ so that, $\overline{B}$is compact, and B is relatively compact.

Proposition 47:

Let $(V, L_V, \otimes)$ be a fuzzy normed space and let $B \subset V$. IfB is totally fuzzy bounded thenB is separable set.

Proof:

SinceB is totally fuzzy bounded set then, the setB contains a finite r-fuzzy net$M_{\frac{1}{n}}$for itself, where $r = r_n = \frac{1}{n}$, $n = 1,2, \ldots$. Then put$M = \bigcup_{n=1}^{\infty} M_{\frac{1}{n}}$soM is countable and M is dense in B in fact for any given$0 < r < 1$ there is an$n$ such that$\frac{1}{n} < r$, hence for anyb $\in$ B there is anm $\in M_{\frac{1}{n}} \subset$ M such that$L_V[b - m, t] > (1 - r)$.This shows thatBis separable set.

Theorem 48:

Let $(V, L_V, \otimes)$ and $(U, L_U, \odot)$ be two fuzzy normed spaces. Let$T: V \to U$be fuzzy compact linear operator. Then, T(V) is separable set.

Proof:

Let the fuzzy balls$B_n = FB\left(0, \frac{1}{n}, t\right) \subset V$ since T is fuzzy compact then the image $T(B_n) = C_n$ is relatively compact butC_n are separable by lemma 46. The fuzzy norm of any v is $L_V[v, t] > \left(1 - \frac{1}{n}\right)$, hence $v \in B_n$ thus, $V = \bigcup_{n=1}^{\infty} B_n$ and $T(V) = \bigcup_{n=1}^{\infty} T(B_n) = \bigcup_{n=1}^{\infty} C_n$ sinceC_n are separable by lemma 46, it has a countable dense subsetD_n and $D = \bigcup_{n=1}^{\infty} D_n$ is countable. This shows that D is dense in T(V).

Proposition 49:

Let T be a fuzzy compact linear operator on V and let S be a fuzzy bounded linear operator on V where $(V, L_V, \otimes)$ is a fuzzy normed space. ThenTS andSTare fuzzy compact operators.

Proof:

Let B be any fuzzy bounded set in V sinceS is fuzzy bounded operator so, S(B) is a fuzzy bounded set and the set$T[S(B)] = TS[B]$ is relatively compact because S is fuzzy compact. Hence, TS is fuzzy compact linear operator. We now prove ST is fuzzy compact. Let(v_n) be any fuzzy bounded sequence in V then, (Tv_n) has a convergent subsequence $\left(Tv_{n_k}\right)$ by Theorem 38 and$\left(STv_{n_k}\right)$ converge by Theorem 18. Hence, ST is fuzzy compact by Theorem 38.

Conclusion:

The main goal of this paper is to introduce the definition of fuzzy compact linear operator from a fuzzy normed space into another fuzzy normed space in order to investigate the basic properties of this type of operators. We have tried here to translate the basic properties of compact linear operator to fuzzy context and we have succeeded in this situations.

Conflicts of Interest: None.

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خواص المؤثرات الخطية المتراصة الضبابية على فضاء القياس الضبابي

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الخلاصة:

في هذا البحث تعريف فضاء القياس الضبابي تم استعارته وخواصه الاساسية. ثم تعريف المؤثر المتراص منفضاء قياس ضبابي الى فضاء قياس ضبابي اخر تم تقديمه بعد ذلك برهان ان المؤثر يكون متراص ضبابيا اذا وفقط اذا صورة اي متتابعة مقيدة تحتوي على متتابعة جزئية متقاربة تم تقديمه. في هذه المرحلة افضل الخواصالاساسية لفضاء المتجهات FC(V,U) الذي يحتوي على المؤثرات الخطية المتراصة ضبابيا تم بحثها ومنها عندما يكون الفضاء U كامل والمتتابعة (T_n) من المؤثرات الخطية المتراصة ضبابيا تقترب من المؤثر T عندئذ T يجب ان تكون متراصة ضبابيا. بالاضافة الى ذلك عندما يكون المؤثر T متراص ضبابيوالمؤثر S مقيد ضبابيا فان التركيب TS و ST يكون مؤثرا متراص ضبابيا. واخيرا اذا كان المؤثر T ينتمي الى الفضاء FC(V,U) وبعد الفضاء V منتهي عندئذ الموثر T يكون متراص ضبابيا تم برهانها.

الكلمات المفتاحية:فضاء القياس الضبابي التام، فضاء القياس الضبابي، المؤثرات المقيدة ضبابيا، المؤثرات المتراصة ضبابيا، المجموعات المتراصة المترابطة.