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The Effect of MHD on a Longitudinal Flow of a Fractional Maxwell Fluid between Two Coaxial Cylinders

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Abstract:

In this paper fractional Maxwell fluid equation has been solved. The solution is in the Mettag-Leffler form. For $\varepsilon = 1$ the corresponding solutions for ordinary Maxwell fluid are obtained as limiting case of general solutions. Finally, the effects of different parameters on the velocity and shear stress profile are analyzed through plotting the velocity and shear stress profile.

Key words: Coaxial cylinders, Finite Hankel transforms, Fractional Maxwell fluid, Laplace transforms.

Introduction:

Fluids are classified into Newtonian and non-Newtonian where in the second case the relation between the rate of strain and shear stress is nonlinear. Newtonian fluids can be describe by Navier-Stokes equations, for more detail see(1,2). A thermodynamic framework has been put into place to develop a rate type model known as Maxwell which is non-Newtonian model, in which the ordinary Maxwell model has been replaced by the Maxwell model with fractional calculus such that the time derivative of an integer order replacing by the so-called Riemann-Liouville fractional differential operator (3,4,5,6).

Recently, Hyder (7) in his paper solved the Fractional Burgers' model for the flow of fluid with viscoelastic property. Zheng (8) discussed the slip effects on MHD flow of a generalized Oldroyd- B fluid with fractional derivative. Exact analytical solutions for a longitudinal flow of a fractional Maxwell fluid between two coaxial cylinders are investigated by Awan (9). Ghada (10) considered the MHD flow of a Fractional Burgers' Model in an Annular Pipe. The unsteady flow of a Maxwell fluid with fractional derivatives in a circular cylinder moving with a nonlinear velocity is discussed by Athar (11). Khan (12) discussed the electroosmotic flow for generalized Burgers fluid in cylindrical domain. Some interesting and recent results for the solution of viscoelastic fluids with fractional derivatives have been presented in (13-15).

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In this paper, we are interested into the effect of MHD on a longitudinal motion of a fractional Maxwell fluid between two infinite coaxial circular cylinders,. The velocity field and associated tangential shear stress are determined by means of Laplace and finite Hankel transforms. The paper ends with drawing the figures of velocity and shear stress profile in the plane.

Governing Equations:

The equations governing the flow of an incompressible fluid include continuity equation and the motion of equations, in the absence of body forces, they are (16)

$$\nabla \vec{V} = 0, \qquad \dots(1)$$

$$\rho \frac{d\vec{V}}{dt} = \nabla \cdot \vec{\tau}, \qquad \dots(2)$$

where ρ is the density, V is the velocity and t is the time .

The Cauchy stress τ of an incompressible fractional Maxwell fluid is given by (16)

$$\tau = -PI + S, \qquad \dots(3)$$

and S determined by the fractional equation

$$(1+\lambda^{\varepsilon}\tilde{D}^{\varepsilon})S = \mu B \qquad \dots(4)$$

where

$$\tilde{D}_t^{\varepsilon} S = D_t^{\varepsilon} S + v \cdot \nabla S - LS - SL^T, \qquad \dots(5)$$

Remember that the Riemann- Liouville fractional derivative of order ε for a function $f(t)$ is (17)

$$D_t^{\varepsilon}[f(t)] = \frac{1}{\Gamma(1-\varepsilon)} \frac{d}{dt}\int_0^t \frac{f(\tau)}{(1-\tau)^{\varepsilon}} d\tau \ ,$$

$$0 \le \varepsilon \le 1 \qquad \ldots(6)$$

Where the fluid flows in z-direction then we suppose that

$$V = w(r,t)e_z \ , \ S = S(r,t) \qquad (7)$$

where $e_z = (0,0,1)$. Using Eq. (7) in Eqs. (2) - (4) and with the initial condition $S(r,0) = \partial_t S(r,0) = 0 \ , \ r > 0$,

$S_{rr} = S_{\theta z} = S_{r\theta} = S_{\theta\theta} = 0$ and $S_{rz} = S_{zr}$, this yields

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial (rS_{rz})}{\partial r} \qquad \ldots(8)$$

$$(1+\lambda^{\varepsilon}D_t^{\varepsilon})S_{rz} = \mu\frac{\partial w}{\partial r} \qquad \ldots(9)$$

The term (- $\sigma E_o^2 w$) has been added to the right hand side of Eq. (8) to study the effect of MHD on the flow, then

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial (rS_{rz})}{\partial r} - \sigma E_o^2 w \quad \ldots(10)$$

From Eqs. (9) and (10), we get

$$(1+\lambda^{\varepsilon}D_t^{\varepsilon})\frac{\partial w}{\partial t} = -\frac{1}{\rho}(1+\lambda^{\varepsilon}D_t^{\varepsilon})\frac{\partial P}{\partial z} +$$

$$\upsilon(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})w - M(1+\lambda^{\varepsilon}D_t^{\varepsilon})w \ \ldots (11)$$

where $\upsilon = \frac{\mu}{\rho}$ is the kinematics' viscosity of the fluid and $M = \frac{\sigma E_o^2}{\rho}$ is the magnetic number.

Statement of the Problem

Referring to Eq. (11), the corresponding fractional partial differential equation that describe such flow takes the form

$$(1+\lambda^{\varepsilon}D_t^{\varepsilon})\frac{\partial w}{\partial t} = -A(1+\lambda^{\varepsilon}\frac{t^{-\varepsilon}}{\Gamma(1-\varepsilon)}) +$$

$$\upsilon(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})w - M(1+\lambda^{\varepsilon}D_t^{\varepsilon})w \qquad \ldots (12)$$

where $A = \frac{1}{\rho}\frac{\partial P}{\partial z}$ is the gradient of the pressure which is constant.

The initial and boundary conditions on Eq. (12) are

$$w(r,0) = \frac{\partial w(r,0)}{\partial r} = \frac{\partial^2 w(r,0)}{\partial r^2} = 0 \quad , \ R_0 < r < R_1 \ \ldots(13)$$

$$w(R_0,t) = w(R_1,t) = 0 \quad , \quad t > 0 \ \ldots (14)$$

Where $\begin{pmatrix} R_0 \\ R_1 \end{pmatrix}$ is the radius of $\begin{pmatrix} internal \\ outer \end{pmatrix}$ cylinders.

Calculation of the Velocity Field

Since the equation (12) is linear with constant coefficients, one can solve it by Laplace transform approach as follows

$$S(1+\lambda^{\varepsilon}S^{\varepsilon})\overline{w} = -\frac{A}{S}(1+\lambda^{\varepsilon}S^{\varepsilon}) +$$

$$\upsilon(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})\overline{w} - M(1+\lambda^{\varepsilon}S^{\varepsilon})\overline{w} \qquad \ldots(15)$$

$$\left.\begin{array}{l} \overline{w}(r,0) = \dfrac{\partial \overline{w}(r,0)}{\partial r} = \dfrac{\partial^2 \overline{w}(r,0)}{\partial r^2} = 0 \ , \\ \quad R_0 < r < R_1 \\ \overline{w}(R_0,s) = \overline{w}(R_1,s) = 0 \ , \quad s > 0 \end{array}\right\} \ldots(16)$$

where $\overline{w}(r,s) = L(w(r,t))$.

To apply finite Hankel transform $\overline{w}_H = H(\overline{w}), [H^{-1}(\overline{w}_H) = \overline{w}]$, (18), which is

$$\overline{w}_H = \int_{R_o}^{R_1} r\overline{w}B_o(rk_i)dr \quad , i = 1, 2, 3, \ldots \qquad \ldots (17)$$

And let $J_o(.)$ and $Y_o(.)$ are the Bessel functions of the 1st and 2nd kinds of order zero, then the inverse of finite Hankel is

$$\overline{w} = \frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\overline{w}_H B_o(rk_i)J_o^2(R_1k_i)}{J_o^2(R_ok_i) - J_o^2(R_1k_i)} \qquad \ldots \ (18)$$

where k_i are the positive roots of equation $B_o(R_1k_i) = 0$ and

$$B_o(rk_i) = J_o(rk_i)Y_o(R_0k_i) - Y_o(rk_i)J_o(R_ok_i)$$

Multiplying both sides of Eq. (15) by $rB_o(rk_i)$ and then integrating it with respect to r from R_o to R_1 and using initial and boundary conditions, Eq. (16) , then

$$\overline{w}_H = -\frac{A}{S}\frac{(1+\lambda^{\varepsilon}S^{\varepsilon})}{(S+\lambda^{\varepsilon}S^{\varepsilon+1}+M+M\lambda^{\varepsilon}S^{\varepsilon}+\upsilon k_i^2)} \ \ldots(19)$$

Now, writing Eq.(19) in series form as (using $\frac{1}{z+a} = \sum_{f=0}^{\infty}(-1)^f\frac{z^f}{a^{f+1}}$ and $(1+b)^f = \sum_{q=0}^{f}\frac{f!b^q}{q!(f-q)!}$)

$$\overline{w}_H = -A(1+\lambda^{\varepsilon}S^{\varepsilon})\sum_{f=0}^{\infty}(-1)^f\sum_{q=o}^{f}\frac{1}{q!(f-q)!}\lambda^{-\varepsilon(1+f+q)}M^q$$

$$\frac{f!\,S^{\varsigma}}{(S^{\varepsilon+1}+\frac{M}{\lambda^{\varepsilon}}+\frac{\upsilon k_i^2}{\lambda^{\varepsilon}})^{f+1}} \qquad \ldots(20)$$

where $\varsigma = -1 + f + q(\varepsilon - 1)$

Taking L^{-1} to both sides of Eq.(20) we get

$$w_H = -A\sum_{f=0}^{\infty}(-1)^f \sum_{q=o}^{f}\frac{1}{q!(f-q)!}\ \lambda^{-\varepsilon(1+f+q)}\ M^q\ \ [t^{(\varepsilon-1)f+(\varepsilon+1-\varsigma)-1}\ E^f_{\varepsilon+1,\,\varepsilon+1-\varsigma} - (\frac{1}{\lambda^{\varepsilon}}(\upsilon k_i^2+M)t^{\varepsilon+1}) + t^{(\varepsilon-1)f+(1-\varsigma)-1}\ E^f_{\varepsilon+1,1-\varsigma} - (\frac{1}{\lambda^{\varepsilon}}(\upsilon k_i^2+M)t^{\varepsilon+1})] \quad ...(21)$$

In which $E_{\lambda,\mu}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\lambda m+\mu)}$, $\lambda,\mu>0,$ is the generalized Mittag-Leffler function (17) with

$$E^{(k)}_{\lambda,\mu}(z)=\frac{d^k}{dz^k}\ E_{\lambda,\mu}(z)=\sum_{m=0}^{\infty}\frac{(m+k)!z^m}{m!\Gamma(\lambda m+\lambda k+\mu)}.$$

In obtaining Eq. (21) the generalized Mittag-Leffler function is used which is (17)

$$L^{-1}\{\frac{k!s^{\lambda-\mu}}{(s^{\lambda}\mp l)^{k+1}}\}=t^{\lambda k+\mu-1}E^{(k)}_{\lambda,\mu}\ (\pm lt^{\lambda}),$$

$(\mathrm{Re}(s)>|l|^{1/\lambda})$.

Finally, taking $H^{-1}(w_H)$ to get

$$w(r,t)=-\frac{A\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\,B_o(rk_i)\,J_o^2(R_1k_i)}{J_o^2(R_ok_i)-J_o^2(R_1k_i)}\left[\sum_{f=0}^{\infty}(-1)^f\sum_{q=o}^{f}\frac{1}{q!(f-q)!}\ \lambda^{-\varepsilon(1+f+q)}\ M^q\ [t^{(\varepsilon+1)f+(\varepsilon+1-\varsigma)-1}E^f_{\varepsilon+1,\,\varepsilon+1-\varsigma}-(\frac{1}{\lambda^{\varepsilon}}(\upsilon k_i^2+M)t^{\varepsilon+1}) + \lambda^{\varepsilon}\ t^{(\varepsilon+1)f+(1-\varsigma)-1}\ E^f_{\varepsilon+1,1-\varsigma}-(\frac{1}{\lambda^{\varepsilon}}(\upsilon k_i^2+M)t^{\varepsilon+1})]\right] \quad ...(22)$$

Calculation of the Shear Stress

On taking Laplace transform to both sides of Eq. (9) with $S(r,0)=\partial_t S(r,0)=0$, $r>0$, we find that

$$\bar{\tau}\ (r,s)=\mu\frac{1}{1+\lambda^{\varepsilon}s^{\varepsilon}}\frac{\partial\,\bar{w}(r,s)}{\partial r} \qquad ...\ (23)$$

where $\bar{\tau}(r,s)=\bar{S}_{rz}(r,s)$. The image function $\bar{w}(r,s)$ of $w(r,t)$ can immediately be obtained through Eq.(22). Consequently, evaluating $\frac{\partial\,\bar{w}(r,s)}{\partial r}$ from the mentioned equation and introducing it into Eq. (23), results in

$$\tau(r,t)=-\frac{A\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\,B_o(rk_i)\,J_o^2(R_1k_i)}{J_o^2(R_ok_i)-J_o^2(R_1k_i)}\left[\sum_{p=0}^{\infty}\ \sum_{f=0}^{\infty}(-1)^{f+p}\sum_{q=o}^{f}\frac{1}{q!(f-q)!}\ \lambda^{-\varepsilon(1+f+q-p)}\ M^q\,[t^{(\varepsilon+1)f+(\varepsilon+1-\ell)-1}\,E^f_{\varepsilon+1,\,\varepsilon+1-\ell}-(\frac{1}{\lambda^{\varepsilon}}(\upsilon k_i^2+M)t^{\varepsilon+1}) + \lambda^{\varepsilon}\ t^{(\varepsilon+1)f+(1-\ell)-1}\ E^f_{\varepsilon+1,1-\ell}-(\frac{1}{\lambda^{\varepsilon}}(\upsilon k_i^2+M)t^{\varepsilon+1})]\right] \quad ...(24)$$

where $\ell=\varsigma+p\varepsilon$.

The Special Case ($\varepsilon\to1$)

Put $\varepsilon\to1$ in Eqs. (22) and (24), to give the velocity field and the shear stress corresponding to an ordinary Maxwell fluid performing the same motion.

$$w(r,t)=-\frac{A\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\,B_o(rk_i)\,J_o^2(R_1k_i)}{J_o^2(R_ok_i)-J_o^2(R_1k_i)}\left[\sum_{f=0}^{\infty}(-1)^f\sum_{q=o}^{f}\frac{1}{q!(f-q)!}\ \lambda^{-(1+f+q)}\ M^q\ \ [t^{2f+(2-\varsigma)-1}\ E^f_{2,2-\varsigma}-(\frac{1}{\lambda}(\upsilon k_i^2+M)t^2) + \lambda\ \ t^{2f+(1-\varsigma)-1}\ E^f_{2,1-\varsigma}-(\frac{1}{\lambda}(\upsilon k_i^2+M)t^2)]\right] \quad ...(25)$$

$$\tau(r,t) = -\frac{A\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^3\,B_o(rk_i)\,J_o^2(R_1k_i)}{J_o^2(R_ok_i)-J_o^2(R_1k_i)}\left[\sum_{p=0}^{\infty}\ \sum_{f=0}^{\infty}(-1)^{f+p}\sum_{q=o}^{f}\frac{1}{q!(f-q)!}\right.$$
$$\lambda^{-(1+f+q-p)}\ M^{q}\ [t^{2f+(2-\ell)-1}\ E^{f}_{2,2-\ell}-(\frac{1}{\lambda}(\upsilon k_i^2+M)t^2)$$
$$\left.+\lambda\ \ t^{2f+(1-\ell)-1}\ E^{f}_{2,1-\ell}-(\frac{1}{\lambda}(\upsilon k_i^2+M)t^2)]\right] \qquad ...(26)$$

Results and Discussion:

The main results of this work are the effect of the MHD on the flow described by Eqs. (12)-(14). The exact solution for the velocity field and shear stress is evaluated by using the two transforms Laplace and finite Hankel. The solution of ordinary Maxwell fluid is a special case ($\varepsilon \rightarrow 1$) of our model. All figures are plotted by using Mathematica to illustrate the effect of the parameters in our solutions.

Figure (1) is plotted to illustrate the effect of the relaxation time λ. It is observed that $w(r,\theta,t)$ decreases with increasing λ. In Fig. (2), showns the influence of the fractional parameter ε. It is clearly seen that the velocity is decreasing with increasing ε.

Figures (3 , 4) are depicted to show the behavior of the magnetic parameter *M* for small as well as for long time. It is observed that for short time t = 0.1 the increase in magnetic field *M* will decrease the velocity profile, while quite the opposite effect is observed for long time t = 0.4 , i.e., the increase in magnetic field *M* will strongly increase the velocity profile.

Figure (5) illustrates the variation of the velocity profiles for different values of time t , in which as t increases there is a strong increasing in velocity.

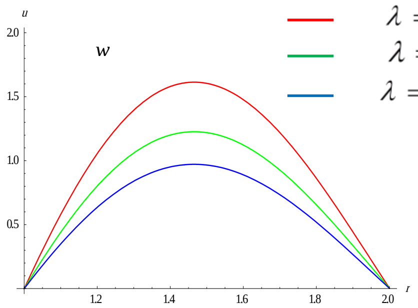

Figure 1. Velocity profile for *A*=1, ε =0.3, υ =0.004, *M*=3, *t* = 0.1 and different values of λ.

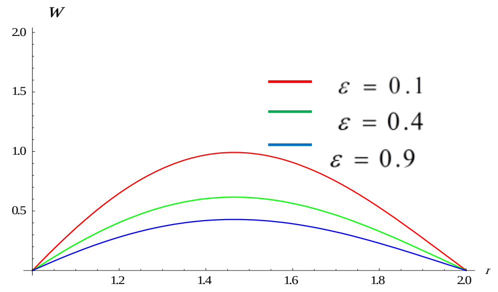

Figure 2. Velocity profile for A=1, λ=2, υ =0.004, M=3, t = 0.1 and different values of

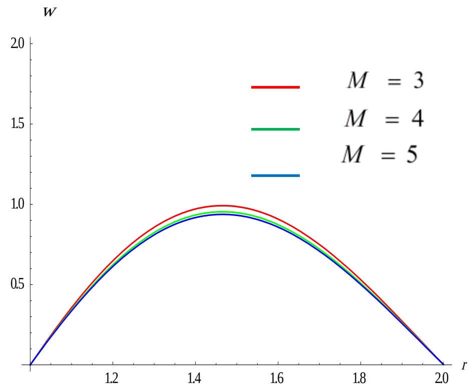

Figure 3. Velocity profile for A=1, λ=2, λ =0.004, υ =0.1, t = 0.1 and different values of M.

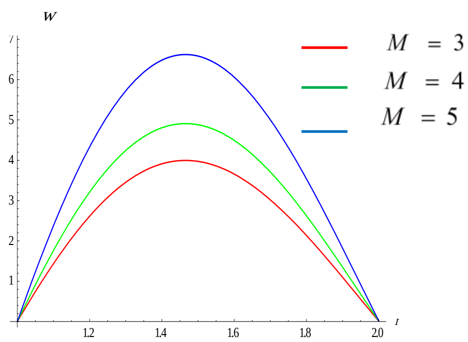

Figure 4. Velocity profile for A=1, λ=2, λ =0.004, ε =0.1, t=0.4 and different values of M.

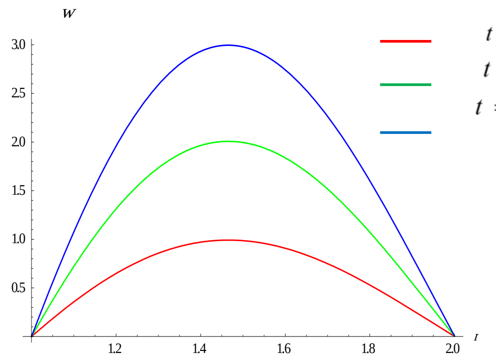

Figure 5. Velocity profile for A=1, λ=2, λ =0.004, υ =0.1, M=3 and different values of t.

Figures (6 to10) show the effect of different parameters upon the shear stress. The parameters λ and υ as they increase they have the same effect upon the shear stress. It is clear that they have the opposite influence to that on velocity, see (Figures 6 and 7). Finally, the effect of the magnetic parameter M and the time t on shear stress have the behavior similar with velocity. Of course, these results entirely agree with those resulting from Figures (8 to 10).

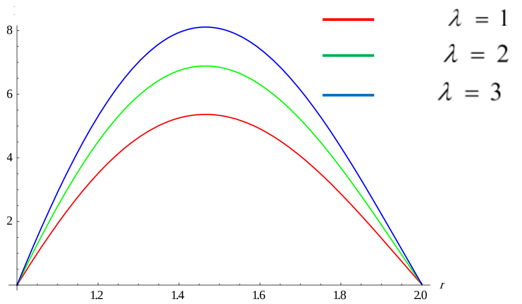

Figure 6. Shear stress for A=1, υ =0.1, λ =0.004, t =0.1, M=3 and different values of λ.

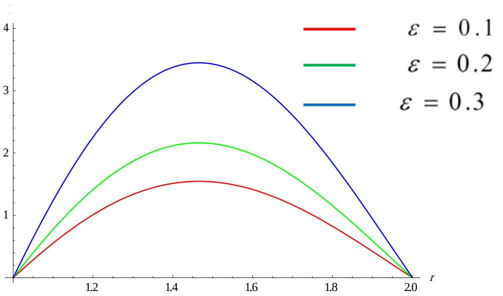

Figure 7. Shear stress for A=1, λ=2 , λ =0.004, t =0.1, M=3 and different values of υ .

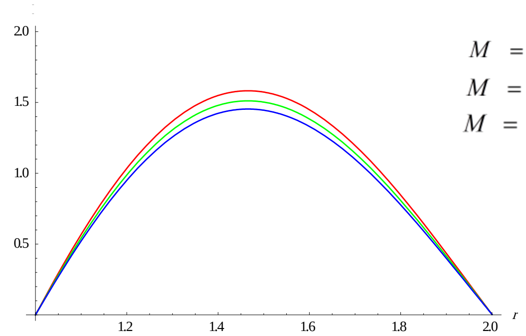

Figure 8. Shear stress for A=1, λ=2 , λ =0.004, t =0.1, ε =2 and different values of M.

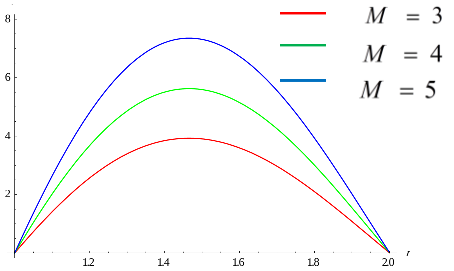

Figure 9. Shear stress for A=1, λ=2 , λ =0.004, t =0.4, ε =2 and different

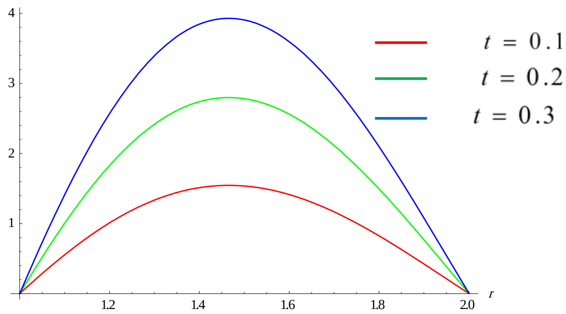

Figure 10. Shear stress for A=1, λ=2 , λ =0.004, M=3, ε =2 and different values of t.

Conflicts of Interest: None.

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تأثير الحقل المغناطيسي الهيدروديناميكي (MHD) في جريان طولي لمائع ماكسويل الكسري بين اسطوانتين محوريتين

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الخلاصة:

قمنا في هذا البحث بحل معادلة مائع ماكسويل التفاضلية ذات الرتبة الكسرية .كان الحل بصيغة دالة ميتاج- لفلر (Mettag-Leffler) . في حالة $\varepsilon = 1$ فان حلول مائع ماكسويل غير الكسرية حصلنا عليها كحالة محددة من الحل العام. اخيراً ، تاثير المعلمات المختلفة في حقل السرعة واجهاد القص تم تحليلها من خلال رسم السرعة واجهاد القص.

الكلمات المفتاحية: اسطوانات محورية، تحويلات هانكل المنتهية، مائع ماكسويل الكسري، تحويلات لابلاس.