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Representation of Algebraic Integers as Sum of Units over the Real Quadratic Fields

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Abstract:

In this paper we generalize Jacobsons results by proving that any integer α in $Q(\sqrt{d}),(d>0,d$ is a square-free integer), belong toW_t. All units of $Q(\sqrt{d})$ are generated by the fundamental unit $\varepsilon^n,(n\geq 0)$ having the forms

$$\varepsilon = t+\sqrt{d}, d\not\equiv 1(mod4)$$
$$\varepsilon = \frac{\left[(2t-1)+\sqrt{d}\right]}{2}, d\equiv 1(mod4)$$

our generalization build on using the conditions

$$t+1=\varepsilon\pm\varepsilon^{-1}+(1-t),$$
$$t=\varepsilon\pm\varepsilon^{-1}+(1-t).$$

This leads us to classify the real quadratic fields $Q\sqrt{d}$ into the sets W_1, W_2, W_3 ... Jacobsons results shows that $Q\sqrt{2}, Q\sqrt{5}\in W_1$ and Sliwa confirm that $Q\sqrt{2}$ and $Q\sqrt{5}$ are the only real quadratic fields in W_1.

Keywords: Fundamental units of real quadratic field, Integers of real quadratic field as sum of finite units, Real quadratic fields.

Introduction:

Thereal quadratic fields $Q(\theta)$ of degree two over the rational numbers with $\theta = d$, where $d>0$ and d is a square free integer for if $\theta=\sqrt{d}$ is a root of the quadratic polynomial

$$x^2+2ax+b=0$$

This leads to state that any integer $\alpha\in Q(\sqrt{d})$ has the forms

$$\alpha = a+b\sqrt{d}, d\not\equiv 1(mod4)$$
$$or \quad \alpha=(a+b\sqrt{d})/2 \ , d\equiv 1(mod4)$$

The fundamental unit of $Q(\sqrt{d})$ has also the forms

$$\varepsilon = t+\sqrt{d}\ , d\not\equiv 1(mod4)$$
$$or \quad \varepsilon=\left[(2t-1)+\sqrt{d}\right)/2, d\equiv 1(mod4)$$

The base of any integer $\alpha\in Q(\sqrt{d}), d\not\equiv 1(mod4)$ is of the form $<1, b\sqrt{d}>$ and the base of any integer $\alpha\in Q(\sqrt{d})$ with $d\equiv 1(mod4)$ is$<1, b(1+\sqrt{d})/2>$.

According to $d\not\equiv 1(mod4) or\ d\equiv 1(mod4)$, where norm$(\varepsilon)=N(\varepsilon)=\varepsilon\bar{\varepsilon}=\pm 1$ and $\bar{\varepsilon}$ is the conjugate of ε.

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B. Jacobsons (1) used the conditions

$$2=\varepsilon_1-\varepsilon_1^{-1}$$
$$2=\varepsilon_2+\varepsilon_2^{-2} \qquad ...(1)$$

and proved that any integer $\alpha\in Q(\sqrt{2})$ and $Q(\sqrt{5})$ are written as a sum of distinct units, where $\varepsilon_1=1+\sqrt{2}$, $\varepsilon_2=(1+\sqrt{5})/2$ are the fundamental units of $Q(\sqrt{2})$ and $Q(\sqrt{5})$ respectively on the other words $Q(\sqrt{2})$, $Q(\sqrt{5})\in W_1$.Sliwa (2) proved that no other real quadratic fields belong to W_1. Belcher (3) and (4) showed that there are seven real pure cubic fields $Q(\theta),\vartheta^3=d=m^3\pm 1$, with negative discriminate belong toW_1 and infinitely many quantic fields in $W_t, t\geq 1$. Goldsmith. Pabst and Scott (5) proved a similar result without the property of being distinct units. The sum of unit of numbers of rings of quadratic field also discussed by Ashrafi and Vamos (6). Further result given by Christopher (7) showing that for every positive integer N, there exists elements of the ring R which cannot be expressed as sum of at most N units. Our result depends on using the conditions

$$t+1=\varepsilon\pm\varepsilon^{-1}+(1-t)$$
$$t=\varepsilon\pm\varepsilon^{-1}+(1-t)$$

and proving that any real quadratic field $Q(\sqrt{d})$ belong to the set $W_t, t \geq 1$.

Preliminary Results and Definitions

Our main result depend on generalizing conditions in (1) to be

$2t = \varepsilon \pm \varepsilon^{-1}, t \geq 1$.

which becomes

$t + 1 = \varepsilon \pm \varepsilon^{-1} + (1 - t),$ … (2)

when$d \not\equiv 1(mod 4)$ and

$t = \varepsilon \pm \varepsilon^{-1} + (1 - t),$ … (3)

when$d \equiv 1(mod 4)$.

Def 2.1: We define $W_t, t \geq 1$ to be the set of all real quadratic fields $Q(\sqrt{d})$ in which every integer $\alpha \in Q(\sqrt{d})$ is represented as a sum of units with at most t- repititions.

Def 2.2: An integer $\alpha \in Q(\sqrt{d})$ is called *t*-integer, if α can be written as a finite sum of units of $Q(\sqrt{d})$ with at most *t*-repetitions.

Def 2.3: An integer $\alpha \in Q(\sqrt{d})$ is called a strong *t*-integer if it is a *t*-integer and expressible as a sum of at most (*t* +1) units of $Q(\sqrt{d})$.

Def 2.4: We define the set $W_t, (t \geq 1)$ to be the set of all real quadratic fields $Q(\sqrt{d})$ such that every integer of $Q(\sqrt{d})$ is a *t*-integer.

Def 2.5: Any positive integer $n \geq 1$ is called square free integer if n is of the form

$$n = p_1.p_2 \dots p_r$$

where$p_1, p_2, \dots p_r$ are distinct prime numbers.

Theorem 2.1 (4): If $Q(\theta)$ is a real algebraic field with one fundamental $\varepsilon > 1$ say, then a necessary condition for $Q(\theta)$ to be in W_1 is that $\varepsilon < 3$.

More general than theorem (2.1) (4) is the following result.

Theorem 2.2: If $Q(\theta)$ is a real field and having one fundamental unit $\varepsilon > 1, \ni Q(\theta) \in W_t$ then $\varepsilon < 2t + 1$.

Proof: Since any integer in W_t is a *t*-integer, then we can write (*t* +1) in the form

$\mathrm{t} + 1 = \mathrm{c_p}\varepsilon^{\mathrm{p}} + \cdots + \mathrm{c_q}\varepsilon^{\mathrm{q}},$ … (4)

where p, q $\in Z$ and $|c_i| \leq t$ for all i, (p $\leq i \leq$ q). The form (1) is the polynomial

$b_n\varepsilon^n + b_{n-1}\varepsilon^{n-1} + \cdots + b_0 = 0$ … (5)

with$b_n, b_0 \neq 0, b_i \in Z$ and at most one of the $b_i = \pm(2t + 1)$ and the rest of the coefficients less than or equal to t. Therefore (5) implies that

$$\varepsilon^n \leq b_n\varepsilon^n \leq (2t + 1)\varepsilon^{n-1} + \frac{t\varepsilon^{n-1}}{\varepsilon - 1}$$

$$\varepsilon \leq (2t + 1) + \frac{t}{\varepsilon - 1}$$

$$\varepsilon(\varepsilon - 1) \leq (2t + 1)(\varepsilon - 1) + t$$

$$\varepsilon(\varepsilon - 1) < (2t + 1)\varepsilon - (2t + 1) + t$$

$$\varepsilon - 1 < (2t + 1) - \frac{(t + 1)}{\varepsilon}$$

$$\varepsilon < 2t + 2 - \frac{(t + 1)}{\varepsilon}$$

since$t + 1 \geq 2, then -\frac{(t+1)}{\varepsilon} \leq \frac{-2}{\varepsilon},\ therefore$

$$\varepsilon < (2t + 2) - \frac{(t + 1)}{\varepsilon} \leq 2t + \left(2 - \frac{2}{\varepsilon}\right)$$

where as $\varepsilon > 1, then\ \frac{2}{\varepsilon} \cong 1,$

$\therefore \varepsilon < 2t + 1$

The result of theorem (2.1) (4) above follows when $t = 1$ i.e $\varepsilon < 3$ and $\varepsilon < 2t + 1$ is valid for all $Q(\sqrt{d}) \in W_t, t \geq 1$.

The Main Results

Our main result of this paper is given by the following theorem:

Theorem 3.1: If $Q(\sqrt{d})$ has the fundamental unit $\varepsilon = t + \sqrt{d}\ or\ [(2t - 1) + \sqrt{d}]/2$ according to $d \not\equiv 1(mod 4) or\ d \equiv 1(mod 4)$, then $Q(\sqrt{d}) \in W_t, (t \geq 1)$

Proof: The proof is built on using the conditions (2), (3) given above.

Assume that α any integer in $Q(\sqrt{d}\ ) \ni$

$\alpha = \sum_{i=p}^{q} a_i\varepsilon^i$ … (6)

with$a_i, p, q \in Z$, (p $\leq i \leq$ q), $a_i = 0$ if $i \notin$ [p, q], we consider the case (I)

(i) $d \not\equiv 1(mod 4), N(\varepsilon) = 1$ and (t+1) is a strong *t*-integer in $Q(\sqrt{d})$ which is

$t + 1 = \varepsilon + \varepsilon^{-1} + (1 - t)$.

Suppose that α in (6) is such that

$\sum_{i=p}^{q}|a_i| = m,\ \ m \geq 1,$

and$m$ minimal, we shall proceed by induction on m, if m=1, then

$$\sum_{i=p}^{q}|a_i| = 1,$$

and $a_i = 0, \pm1$, therefore α is 1-integer (t=1) see definition (1).

Assume that, as in induction hypothesis, that for all $\alpha \in Q(\sqrt{d})$ and

$\alpha = \sum|a_i|, m < M, (M \geq 2),$

and we prove it for all $\alpha \in Q(\sqrt{d})$ with $m = M$, by writing

$(t + 1)\varepsilon^k = \varepsilon^{k-1} + (1 - t)\varepsilon^k + \varepsilon^{k+1}$, (p$\leq k \leq$ q)

and suppose that a_k is the first coefficient in (6) such that $|a_k| \geq t + 1$ and $|a_{k-1}| \leq t$ for all k and apply the condition (2) to the term $a_k\varepsilon^k$, then α becomes

$\alpha = a_p\varepsilon^p + \cdots + (a_{k-1} + 1)\varepsilon^{k-1} + (1 - t)\varepsilon^k + \varepsilon^{k+1} + \cdots + a_q\varepsilon^q$ … (7)

or$\alpha = a_p\varepsilon^p + \cdots + (a_{k-1} - 1)\varepsilon^{k-1} + (t - 1)\varepsilon^k - \varepsilon^{k+1} + \cdots + a_q\varepsilon^q$ … (8)

according as $a_k = \pm(t+1)$. Since$|a_{k-1}| \leq t$, then if necessary we need to repeat the application of condition (2) on (7) or (8), which implies that either
$\alpha = a_p\varepsilon^p + \cdots + (a_{k-2}+1)\varepsilon^{k-2} + (1-t)\varepsilon^{k-1} + (2-t)\varepsilon^k + \varepsilon^{k+1} + \cdots + a_q\varepsilon^q$ … (9)
or $\alpha = a_p\varepsilon^p + \cdots + (a_{k-2}-1)\varepsilon^{k-2} + (t-1)\varepsilon^{k-1} + (t-2)\varepsilon^k - \varepsilon^{k+1} + \cdots + a_q\varepsilon^q$… (10)
According as$a_{k-1} = \pm t$, by taking $-\alpha$ in (7) and (8), if necessary we obtain that$|a_{k-1} \pm 1| \geq t+1$. Again and in similar application of (2) α will be of the form (9) or (10). If we continue
applying (2) to (9) and (10) to the term $(a_{k-2} \pm 1)\varepsilon^{k-2}$, then after that a finite number of applications we get that α of the form
$\alpha = \sum_{j\leq k} b_j\varepsilon^j + \sum_{i\geq k+1} c_i\varepsilon^i$ … (11)
where $0 < |b_j| \leq t$, and $|c_i| \leq t+1$, this reduce every coefficient a_i with $i \leq k$ to the coefficient b_j, $|b_j| \leq t$ and some of the b_j non-zero, this shows that

$$\sum_{j\leq k}|b_j| \leq \sum_{i\geq k+1}|c_i|$$

and$\sum_{j\leq k}|b_j| + \sum_{i\geq k+1}|c_i| \leq M$
and because M is minimal then

$$\sum_{j\leq k}|b_j| + \sum_{i\geq k+1}|c_i| = M$$

since $\sum_{j\leq k}|b_j| > 0$, then $\sum_{i\geq k+1}|c_i| < M$. Apply the induction to

$$\alpha^* = \sum_{i\geq k+1} c_i\varepsilon^i$$

thenα will be a t-integer.

(ii) If $\mathrm{N}(\varepsilon) = -1$, then condition (2) consider to be
$t+1 = \varepsilon - \varepsilon^{-1} + (1-t)$,
and a similar procedure of (i) will lead to show that α is a t-integer.

(II) If $d \equiv 1(mod 4)$ and (i) $\mathrm{N}(\varepsilon) = 1$ and considering condition (3), which is
$t+1 = \varepsilon + \varepsilon^{-1} + (2-t)$,
and apply (ε) on $\alpha \in Q(\sqrt{d}) \ni$

$$\alpha = \sum_{p}^{q} a_i\varepsilon^i$$

Then by following the induction steps on $\sum|a_i| = m$, m minimal and writing
$(t+1)\varepsilon^k = \varepsilon^{k-1} + (2-t)\varepsilon^k + \varepsilon^{k+1}$, (p$\leq k \leq$ q) … (12)
In a similar procedure of case (I) after the application of condition (3), α will be either of the form
$\alpha = a_p\varepsilon^p + \cdots + (a_{k-1}+1)\varepsilon^{k-1} + (2-t)\varepsilon^k + \varepsilon^{k+1} + \cdots + a_q\varepsilon^q$ … (13)
Or $\alpha = a_p\varepsilon^p + \cdots + (a_{k-1}-1)\varepsilon^{k-1} + (t-2)\varepsilon^k - \varepsilon^{k+1} + \cdots + a_q\varepsilon^q$ … (14)
According as $a_k = \pm(t+1)$. This implies that

$$\alpha = \sum_{j\leq k} d_j\varepsilon^j + \sum_{i\geq k+1} e_i\varepsilon^i$$

with$|d_j| \leq t, \sum_{i\geq k+1}|e_i| \geq t+1$ and some of the d_j non-zero and hence

$$\sum_{j\leq k}|d_j| + \sum_{i\geq k+1}|e_i| = M$$

and this completes the induction steps. Therefore α is a t-integer.

(ii) A similar procedure can be followed by considering the condition (3) with $\mathrm{N}(\varepsilon) = -1$ and applying
$t+1 = \varepsilon - \varepsilon^{-1} + (2-t)$,
which also, imply that α is a t-integer too.
As an immediate consequence of theorem (3.1).

Corollary 3.1: Any integer of $Q(\sqrt{d}), d > 0$ expressible as a t-integer, $t \geq 1$.

Corollary 3.2: If $Q(\sqrt{d}) \in W_{t+1}\backslash W_t$, then every integer of $Q(\sqrt{d})$ is a t-integer plus a single unit.

Corollary (2) implies that $Q(\sqrt{13})$ belong to W_2 and not in W_1 since the fundamental unit of $Q(\sqrt{13})$ is $\varepsilon = (3+\sqrt{13})/2$, $N(\varepsilon) = -1$ and
$3 = \varepsilon - \varepsilon^{-1}$ … (15)
$2 = \varepsilon - \varepsilon^{-1} - 1$ … (16)
and by theorem (2),$\varepsilon = [(2t-1)+\sqrt{13}]/2$, with t = 2 for if $\alpha \in Q(\sqrt{13}) \ni$

$$\alpha = -\varepsilon^{-1} + 2 + \varepsilon$$

Apply (16), then α becomes

$$\alpha = -\varepsilon^{-1} + (\varepsilon - \varepsilon^{-1} - 1) + \varepsilon$$
$$\alpha = -2\varepsilon^{-1} + 2\varepsilon - 1$$

again apply (16), then

$$\alpha = -(\varepsilon - \varepsilon^{-1} - 1)\varepsilon^{-1} + (\varepsilon - \varepsilon^{-1} - 1)\varepsilon - 1$$

by simplifying this , then

$$\alpha = \varepsilon^{-2} + \varepsilon^{-1} - \varepsilon + \varepsilon^2 - 3$$

put$3 = \varepsilon - \varepsilon^{-1}, then$

$$\alpha = \varepsilon^{-2} + \varepsilon^{-1} - \varepsilon + \varepsilon^2 - (\varepsilon - \varepsilon^{-1})$$
$$\alpha = \varepsilon^{-2} + 2\varepsilon^{-1} - 2\varepsilon + \varepsilon^2$$
$$\alpha = \varepsilon^{-2} + 2\varepsilon^{-1} - (\varepsilon - \varepsilon^{-1} - 1)\varepsilon + \varepsilon^2$$
$$\alpha = \varepsilon^{-2} + 2\varepsilon^{-1} + \varepsilon + 1$$
$$\alpha = (\varepsilon^{-2} + \varepsilon^{-1} + \varepsilon + 1) + \varepsilon^{-1}$$

This last form of α shows that $Q(\sqrt{13}) \in W_{t+1}/W_t$ and t = 1.

Theorem 3.2: suppose that $Q(\sqrt{d}), d > 0$ with fundamental unit $\varepsilon = t + s\sqrt{d}$or $\varepsilon = [(2t-1) + s\sqrt{d}\,]/2$, then $Q(\sqrt{d}) \in W_t\backslash W_{t-1}, (t > 1)$.

Proof: If $d \not\equiv 1(mod 4), \varepsilon = t + s\sqrt{d},\ s,t \in N$, then the proof followed if we show that any integer $\alpha \in Q(\sqrt{d})$ is not (t-1)-integer by writing

$$t = \sum_{i=k}^{m} c_i \varepsilon^i$$

with $|c_i| \leq t-1,\ k,m \in Z$. If $m \geq 1$, then

$$\varepsilon^m \leq |c_m|\varepsilon^m \leq \sum_{i=k}^{m-1} c_i\varepsilon^i + t < (t-1)\sum_{i=-\infty}^{m-1} \varepsilon^i + t$$

$$< (t-1)\frac{\varepsilon^{m-1}}{1-\frac{1}{\varepsilon}} + t$$

$$< (t-1)\frac{\varepsilon^m}{\varepsilon-1} + t\varepsilon^m$$

$\varepsilon^m \leq \frac{(t-1)\varepsilon^m}{\varepsilon-1} + t\varepsilon^{m-1}$,

where $m \geq 1, \varepsilon^{m-1} > 1$. Therefore

$$1 \leq \frac{t-1}{\varepsilon-1} + \frac{t}{\varepsilon}$$

$$\varepsilon^2 - 2t\varepsilon + t \leq 0$$

since $\varepsilon = t + \sqrt{t^2 \pm 1}$, then

$$\varepsilon \leq t + \sqrt{t^2 - t} < t + \sqrt{t^2 - 1} \leq \varepsilon$$

a contradiction. By the same way and if $d \equiv 1(mod4)$, we get

$$2\varepsilon \leq (2t-1) + \sqrt{(2t-1)^2 - 4t}$$

$$< (2t-1) + \sqrt{(2t-1)^2 - 4}$$

$$= 2t$$

also, a contradiction. Hence $Q(\sqrt{d}) \in W_{t-1}, (t > 1)$.

Theorem 3.3: There are a finite number of $Q(\sqrt{d}) \in W_t,\ t \in N$.

Proof: If $d \not\equiv 1(mod4)$ and $\varepsilon = t + \sqrt{d}$ is the fundamental unit of $Q(\sqrt{d}), d > 0$ the proof followed by showing $t \in \left(\frac{\varepsilon-1}{2}, \frac{\varepsilon+1}{2}\right)$. from theorem (2.2) we have that

$$\varepsilon < 2t + 1, (t \geq 1)$$

By writing

$$2t = \varepsilon \pm \varepsilon^{-1}$$

$$2t - 1 < \varepsilon < 2t + 1$$

$$\frac{\varepsilon-1}{2} < t < \frac{\varepsilon+1}{2}$$

Therefore,

$$t - 1 < \sqrt{d} < t + 1$$

and if $d \equiv 1(mod4)$ we get also and by the same way

$$t - 1 < \frac{1+\sqrt{d}}{2} < t + 1$$

Thus, therefore finite number of $Q(\sqrt{d}) \in W_t$.

Conclusion:

The conclusion of our result is declared by some of the real quadratic fields below by applying theorem (3.1) and considering the fundamental unit $\varepsilon \in Q(\sqrt{d})$, where both $Q(\sqrt{3}), Q(\sqrt{13}) \in W_2$, follows from corollary (3.2).

$$Q(\sqrt{d}) \in W_1, \qquad d = 2,5$$

$Q(\sqrt{d}) \in W_2, d = 3, 13$

$Q(\sqrt{d}) \in W_3, d = 10, 21, 29$

$Q(\sqrt{d}) \in W_4, d = 15, 53$

$$Q(\sqrt{d}) \in W_5, d = 26, 77, 82, 85$$

$Q(\sqrt{d}) \in W_6, d = 35$

As an open question there is a possibility to represent integers of imaginary quadratic fields $Q(\sqrt{d}), d < 0, d = -1, -2, -3$ as sum of units of this field of certain repetition $t, t \geq 1$. this may also classifying the imaginary fields in W_t.

Conflicts of Interest: None.

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تمثيل الأعداد الجبرية للحقل التربيعي الحقيقي كمجموع لوحدات الحقل الأساسية

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الخلاصة:

في هذا البحث تم تعميم نتائج الباحث جاكوبسن بأستخدام الوحدات الأساسية للحقل التربيعي الحقيقي ممثلة في الشرطين التاليين:

$$t + 1 = \varepsilon \pm \varepsilon^{-1} + (1 - t), t \geq 1$$

عندما $d \not\equiv 1(mod4)$، $\varepsilon = t + \sqrt{d}$ والشرط الثاني هو $t = \varepsilon \pm \varepsilon^{-1} + (1 - t), t \geq 2$، وبأستخدام هذه الشروط أستطعنا أثبات أن الحقل التربيعي $d \equiv 1(mod4)$, $\varepsilon = [(2t - 1) + \sqrt{d}]/2$ عندما الحقيقي $Q(\sqrt{d})$ في الجموعة W_t حيث أن المجموعة W_t تمثل مجموعة كل الحقول التربيعية التي تمثل اعدادها كتجمع لوحدات بتكرار عدده $t \geq 1, t$. وبذلك يتم تصنيف الحقول $Q(\sqrt{d})$ وفق للقيم t في المجاميع $W_1, W_2, W_3, \ldots$ حيث أن جاكوبسن برهن أن الحقلين التربيعيين $Q(\sqrt{5}), Q(\sqrt{2})$ ينتميان للمجموعة W_1 كذلك برهن الباحث (J.Silwa) بأن $Q(\sqrt{5}), Q(\sqrt{2})$ هما الحقلان الوحيدان في المجموعة W_1.

الكلمات المفتاحية: الاعداد الجبرية الحقيقة كمجموع منتهي لوحدات الحقل التربيعي الاساسية، الوحدات الاساسية للحقل التربيعي الحقيقي، الحقل التربيعي الحقيقي