Representation of Algebraic Integers as Sum of Units over the Real Quadratic Fields

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Abstract:
In this paper we generalize Jacobson’s results by proving that any integer \( \alpha \) in \( \mathbb{Q}(\sqrt{d}) \), \((d > 0, d \) is a square-free integer), belong to \( W_\alpha \). All units of \( \mathbb{Q}(\sqrt{d}) \) are generated by the fundamental unit \( \varepsilon^n, (n \geq 0) \) having the forms

\[
\varepsilon = t + \sqrt{d}, d \not\equiv 1(\text{mod}4)
\]

\[
\varepsilon = \frac{(2t - 1) + \sqrt{d}}{2}, d \equiv 1(\text{mod}4)
\]

our generalization build on using the conditions

\[ t + 1 = \varepsilon \pm \varepsilon^{-1} + (1 - t), \]

\[ t = \varepsilon \pm \varepsilon^{-1} + (1 - t). \]

This leads us to classify the real quadratic fields \( \sqrt{d} \) into the sets \( W_1, W_2, W_3 \) ... Jacobson’s results shows that \( Q(\sqrt{2}), Q(\sqrt{5}) \in W_1 \) and Sliwa confirm that \( Q(\sqrt{2}) \) and \( Q(\sqrt{5}) \) are the only real quadratic fields in \( W_1 \).

Keywords: Fundamental units of real quadratic field, Integers of real quadratic field as sum of finite units, Real quadratic fields.

Introduction:
Thereal quadratic fields \( \mathbb{Q}(\theta) \) of degree two over the rational numbers with \( \theta = d \), where \( d > 0 \) and \( d \) is a square free integer for if \( \theta = \sqrt{d} \) is a root of the quadratic polynomial

\[ x^2 + 2ax + b = 0 \]

This leads to state that any integer \( \alpha \in \mathbb{Q}(\sqrt{d}) \) has the forms

\[ \alpha = a + b\sqrt{d}, d \not\equiv 1(\text{mod}4) \]

or \( \alpha = (a + b\sqrt{d})/2, d \equiv 1(\text{mod}4) \)

The fundamental unit of \( \mathbb{Q}(\sqrt{d}) \) has also the forms

\[ \varepsilon = t + \sqrt{d}, d \not\equiv 1(\text{mod}4) \]

or \( \varepsilon = [(2t - 1) + \sqrt{d}]/2, d \equiv 1(\text{mod}4) \)

The base of any integer \( \alpha \in \mathbb{Q}(\sqrt{d}), d \not\equiv 1(\text{mod}4) \)

is of the form \( < 1, b\sqrt{d} > \) and the base of any integer \( \alpha \in \mathbb{Q}(\sqrt{d}) \) with \( d \equiv 1(\text{mod}4) \) is \( < 1, b(1 + \sqrt{d})/2 > \).

According to \( d \not\equiv 1(\text{mod}4) \) or \( d \equiv 1(\text{mod}4) \),

where \( \text{norm}(\varepsilon) = N(\varepsilon) = \varepsilon\bar{\varepsilon} = \pm 1 \) and \( \bar{\varepsilon} \) is the conjugate of \( \varepsilon \).

B. Jacobson’s (1) used the conditions

\[ 2 = \varepsilon_3 - \varepsilon_1^{-1} \]

\[ 2 = \varepsilon_2 + \varepsilon_2^{-2} \]

\[ \cdots (1) \]

and proved that any integer \( \alpha \in \mathbb{Q}(\sqrt{2}) \) and \( \mathbb{Q}(\sqrt{5}) \) are written as a sum of distinct units, where

\( \varepsilon_1 = 1 + \sqrt{2}, \varepsilon_2 = (1 + \sqrt{5})/2 \) are the fundamental units of \( \mathbb{Q}(\sqrt{2}) \) and \( \mathbb{Q}(\sqrt{5}) \) respectively on the other words \( \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5}) \in W_1 \).

Sliwa (2) proved that no other real quadratic fields belong to \( W_1 \). Belcher (3) and (4) showed that there are seven real pure cubic fields \( \mathbb{Q}(\theta), \theta^3 = d = m^3 \pm 1 \), with negative discriminate belong to \( W_1 \) and infinitely many quantic fields in \( W_1 \), \( t \geq 1 \). Goldsmith, Pabst and Scott (5) proved a similar result without the property of being distinct units. The sum of unit of numbers of rings of quadratic field also discussed by Ashrafi and Vamos (6).

Further result given by Christopher (7) showing that for every positive integer \( N \), there exists elements of the ring \( R \) which cannot be expressed as sum of at most \( N \) units. Our result depends on using the conditions

\[ t + 1 = \varepsilon \pm \varepsilon^{-1} + (1 - t) \]

\[ t = \varepsilon \pm \varepsilon^{-1} + (1 - t) \]
and proving that any real quadratic field \( Q(\sqrt{d}) \) belong to the set \( W_t, t \geq 1 \).

**Preliminary Results and Definitions**

Our main result depend on generalizing conditions in (1) to be
\[
t = \epsilon \pm \epsilon^{-1}, t \geq 1.
\]
which becomes
\[
t + 1 = \epsilon \pm \epsilon^{-1} + (1 - t), \quad \ldots \quad (2)
\]
whend \( \not \equiv 1 \pmod{d} \) and
\[
t = \epsilon \pm \epsilon^{-1} + (1 - t), \quad \ldots \quad (3)
\]
whend \( \equiv 1 \pmod{d} \).

**Def 2.1:** We define \( W_t, t \geq 1 \) to be the set of all real quadratic fields \( Q(\sqrt{d}) \) in which every integer \( \alpha \in Q(\sqrt{d}) \) is represented as a sum of units with at most \( t \) repetitions.

**Def 2.2:** An integer \( \alpha \in Q(\sqrt{d}) \) is called \( t \)-integer, if \( \alpha \) can be written as a finite sum of units of \( Q(\sqrt{d}) \) with at most \( t \)-repetitions.

**Def 2.3:** An integer \( \alpha \in Q(\sqrt{d}) \) is called a strong \( t \)-integer if it is a \( t \)-integer and expressible as a sum of at most \((t + 1)\) units of \( Q(\sqrt{d}) \).

**Def 2.4:** We define the set \( W_t, (t \geq 1) \) to be the set of all real quadratic fields \( Q(\sqrt{d}) \) such that every integer of \( Q(\sqrt{d}) \) is a \( t \)-integer.

**Def 2.5:** Any positive integer \( n \geq 1 \) is called square free integer if \( n \) is of the form
\[
n = p_1p_2 \cdots p_r
\]
where \( p_1, p_2, \ldots, p_r \) are distinct prime numbers.

**Theorem 2.1 (4):** If \( Q(\theta) \) is a real algebraic field with one fundamental \( \epsilon > 1 \) say, then a necessary condition for \( Q(\theta) \) to be in \( W_1 \) is that \( \epsilon < 3 \).

More general than theorem (2.1) (4) is the following result.

**Theorem 2.2:** If \( Q(\theta) \) is a real field and having one fundamental unit \( \epsilon > 1, \exists Q(\theta) \in W_t \) then \( \epsilon < 2t + 1 \).

**Proof:** Since any integer in \( W_t \) is a \( t \)-integer, then we can write \((t + 1)\) in the form
\[
t + 1 = c_0e^p + \cdots + c_qe^q,
\]
where \( p, q \in Z \) and \( |c_i| \leq t \) for all \( i, (p \leq i \leq q) \).
The form (1) is the polynomial
\[
b_0e^n + b_{n-1}e^{n-1} + \cdots + b_0 = 0 \quad \ldots \quad (5)
\]
where \( n, b_0 \neq 0, b_1 \in Z \) and at most one of the \( b_l = \pm (2t + 1) \) and the rest of the coefficients less than or equal to \( t \). Therefore (5) implies that
\[
\epsilon^n \leq b_ne^n \leq (2t + 1)e^n + \frac{\epsilon e^{n-1}}{\epsilon - 1}
\]
\[
\epsilon \leq (2t + 1) + \frac{t}{\epsilon - 1}
\]
\[
\epsilon(\epsilon - 1) \leq (2t + 1)(\epsilon - 1) + t
\]
\[
\epsilon(\epsilon - 1) < (2t + 1)\epsilon - (2t + 1) + t
\]
\[
\epsilon - 1 < (2t + 1) - \frac{(t + 1)}{\epsilon}
\]
since \( t + 1 \geq 2 \), then \( -\frac{(t + 1)}{\epsilon} \leq -\frac{2}{\epsilon} \), therefore
\[
\epsilon < 2(t + 2) - \frac{(t + 1)}{\epsilon} \leq 2t + \left(2 - \frac{2}{\epsilon}\right)
\]
where as \( \epsilon > 1 \), then \( \frac{2}{\epsilon} \equiv 1 \), \( \therefore \epsilon < 2t + 1 \)
The result of theorem (2.1) (4) above follows when \( t = 1 \) i.e \( \epsilon < 3 \) and \( \epsilon < 2t + 1 \) is valid for all \( Q(\sqrt{d}) \in W_t, t \geq 1 \).

**The Main Results**

Our main result of this paper is given by the following theorem:

**Theorem 3.1:** If \( Q(\sqrt{d}) \) has the fundamental unit \( \epsilon = t + \sqrt{d} or [(2t - 1) + \sqrt{d}]/2 \) according to \( d \equiv 1 \pmod{4} \) or \( d \equiv 1 \pmod{4} \), then \( Q(\sqrt{d}) \in W_t, (t \geq 1) \)

**Proof:** We consider the case (1)
\[
(i) \quad d \not\equiv 1 \pmod{4}, N(\epsilon) = t + 1 \text{ and } (t + 1) \text{ is a strong } t \text{-integer in } Q(\sqrt{d}) \text{ which is}
\]
\[
t + 1 = \epsilon + \epsilon^{-1} + (1 - t).
\]
Suppose that \( \alpha \) in (6) is such that
\[
\sum_{i=p}^{q} a_i = m, \quad m \geq 1,
\]
and \( m \) minimal, we shall proceed by induction on \( m \), if \( m = 1 \), then
\[
\sum_{i=p}^{q} a_i = 1,
\]
and \( a_i = 0, \pm 1 \), therefore \( \alpha \) is \( 1 \)-integer \((t=1)\) see definition (1).

Assume that, as in induction hypothesis, that for all \( \alpha \in Q(\sqrt{d}) \) and
\[
\alpha = \sum_{i=p}^{q} a_i, m < M, (M \geq 2),
\]
and we prove it for all \( \alpha \in Q(\sqrt{d}) \) with \( m = M \), by writing
\[
(t + 1)e^k = e^{k-1} + (1 - t)e^k + e^{k+1}, \quad (p \leq k \leq q)
\]
and suppose that \( a_k \) is the first coefficient in (6) such that \( |a_k| \geq t + 1 \) and \( |a_{k-1}| \leq t \) for all \( k \) and apply the condition (2) to the term \( a_k e^k \), then \( \alpha \) becomes
\[
\alpha = a_p e^p + \cdots + (a_{k-1} + 1)e^{k-1} + (1 - t)e^k + e^{k+1} + \cdots + a_q e^q \quad \ldots \quad (7)
\]
or
\[
\alpha = a_p e^p + \cdots + (a_{k-1} - 1)e^{k-1} + (t - 1)e^k - e^{k+1} + \cdots + a_q e^q \quad \ldots \quad (8)
\]
according as \( a_k = \pm (t+1) \). Since \(|a_{k-1}| \leq t\), then if necessary we need to repeat the application of condition (2) on (7) or (8), which implies that either 
\[
\alpha = a_p e^p + \cdots + (a_{k-2} + 1)e^{k-2} + (1-t)e^{k-1} + (2-t)e^k + e^{k+1} + \cdots + a_q e^q \quad \cdots (9)
\]
or 
\[
\alpha = a_p e^p + \cdots + (a_{k-2} - 1)e^{k-2} + (t-1)e^{k-1} + (t-2)e^k - e^{k+1} + \cdots + a_q e^q \quad \cdots (10)
\]
According as \( a_{k-1} = \pm t\), by taking \(-\alpha\) in (7) and (8), if necessary we obtain that \(|a_{k-1} \pm 1| \geq t+1\). Again and in similar application of (2) \( \alpha \) will be of the form (9) or (10). If we continue applying (2) to (9) and (10) to the term \((a_{k-2} \pm 1)e^{k-2}\), then after that a finite number of applications we get that \( \alpha \) of the form

\[
\alpha = \sum_{j \leq k} b_j e^j + \sum_{|z| \leq k+1} c_i e^i \quad \cdots (11)
\]
where \(|b_j| \leq t\), and \(|c_i| \leq t+1\), this reduces every coefficient \(a_i\) with \( i \leq k \) to the coefficient \(b_j, |b_j| \leq t\) and some of the \(b_j\) non-zero, this shows that

\[
\sum_{j \leq k} |b_j| \leq \sum_{i \geq k+1} |c_i|
\]
and

\[
\sum_{j \leq k} |b_j| + \sum_{|z| \leq k+1} |c_i| \leq M
\]
and because \( M \) is minimal then

\[
\sum_{j \leq k} |b_j| + \sum_{|z| \leq k+1} |c_i| = M
\]
since \( \sum_{j \leq k} |b_j| > 0 \), then \( \sum_{|z| \leq k+1} |c_i| < M \). Apply the induction to

\[
\alpha^* = \sum_{i \geq k+1} c_i e^i
\]
thenz \( \alpha^* \) will be a \( t \)-integer.

(ii) If \( N(\varepsilon) = -1 \), then condition (2) consider to be 
\( t+1 = \varepsilon - \varepsilon^{-1} + (1-t) \), and a similar procedure of (i) will lead to show that \( \alpha \) is a \( t \)-integer.

(II) If \( d \equiv 1(\text{mod}4) \) and (i) \( N(\varepsilon) = 1 \) and considering condition (3), which is 
\( t+1 = \varepsilon + \varepsilon^{-1} + (2-t) \), and apply (\( q \)) on \( \alpha \in \mathbb{Q}(\sqrt{d}) \) \( \nexists \)

\[
\alpha = \sum_{i \leq q} a_i e^i
\]
Then by following the induction steps on \( \sum |a_i| = m \), \( m \) minimal and writing 
\( (t+1)e^k = e^{k+1} + (2-t)e^k + e^{k+1}, (p \leq k \leq q) \) 
\( \cdots (12) \)
In a similar procedure of case (I) after the application of condition (3), \( \alpha \) will be either of the form

\[
\alpha = a_p e^p + \cdots + (a_{k-1} + 1)e^{k-1} + (2-t)e^k + e^{k+1} + \cdots + a_q e^q \quad \cdots (13)
\]
Or

\[
\alpha = a_p e^p + \cdots + (a_{k-1} - 1)e^{k-1} + (t-2)e^k - e^{k+1} + \cdots + a_q e^q \quad \cdots (14)
\]
According as \( a_k = \pm (t+1) \). This implies that

\[
\alpha = \sum_{j \leq k} d_j e^j + \sum_{i \geq k+1} e_i e^i
\]
with \(|d_j| \leq t, \sum_{|i| \leq k+1} e_i| \geq t+1 \) and some of the \( d_j \) non-zero and hence

\[
\sum_{j \leq k} |d_j| + \sum_{i \geq k+1} |e_i| = M
\]
and this completes the induction steps. Therefore \( \alpha \) is a \( t \)-integer.

(ii) A similar procedure can be followed by considering the condition (3) with \( N(\varepsilon) = -1 \) and applying

\[
t+1 = \varepsilon - \varepsilon^{-1} + (2-t)
\]
which also, imply that \( \alpha \) is a \( t \)-integer too.

As an immediate consequence of theorem (3.1).

**Corollary 3.1:** Any integer of \( \mathbb{Q}(\sqrt{d}), d > 0 \) expressible as a \( t \)-integer, \( t \geq 1 \).

**Corollary 3.2:** If \( \mathbb{Q}(\sqrt{d}) \subseteq W_{t+1} \setminus W_t \), then every integer of \( \mathbb{Q}(\sqrt{d}) \) is a \( t \)-integer plus a single unit.

Corollary (2) implies that \( \mathbb{Q}(\sqrt{13}) \) belong to \( W_3 \) and not in \( W_2 \) since the fundamental unit of \( \mathbb{Q}(\sqrt{13}) \) is 
\( \varepsilon = (3 + \sqrt{13})/2, N(\varepsilon) = -1 \) and
\( 3 = \varepsilon - \varepsilon^{-1} \) \( \cdots (15) \)
\( 2 = \varepsilon - \varepsilon^{-1} - 1 \) \( \cdots (16) \)
and by theorem (2), \( \varepsilon = [(2t-1) + \sqrt{13}]/2, \) with \( t = 2 \) for if \( \alpha \in \mathbb{Q}(\sqrt{13}) \) \( \exists \)

\[
\alpha = -\varepsilon^{-1} + 2 + \varepsilon
\]
Apply (16), then \( \alpha \) becomes

\[
\alpha = -\varepsilon^{-1} + (\varepsilon - \varepsilon^{-1} - 1) + \varepsilon
\]
\[
\alpha = -2\varepsilon^{-1} + 2 \varepsilon - 1
\]
again apply (16), then

\[
\alpha = (-\varepsilon - \varepsilon^{-1} - 1) + (\varepsilon - \varepsilon^{-1} - 1) - \varepsilon - 1
\]
by simplifying this, then

\[
\alpha = \varepsilon^{-2} + \varepsilon^{-1} - \varepsilon + \varepsilon^2 - 3
\]
put\( 3 = \varepsilon - \varepsilon^{-1}, \) then

\[
\alpha = \varepsilon^{-2} + \varepsilon^{-1} - \varepsilon + \varepsilon^2 - (\varepsilon - \varepsilon^{-1})
\]
\[
\alpha = \varepsilon^{-2} + 2\varepsilon^{-1} - 2\varepsilon + \varepsilon^2
\]
\[
\alpha = \varepsilon^{-2} + 2\varepsilon^{-1} - (\varepsilon - \varepsilon^{-1} - 1)\varepsilon + \varepsilon^2
\]
\[
\alpha = \varepsilon^{-2} + 2\varepsilon^{-1} + \varepsilon + 1
\]
\[
\alpha = (\varepsilon^{-2} + \varepsilon^{-1} + \varepsilon + 1) + \varepsilon^{-1}
\]
This last form of \( \alpha \) shows that \( \mathbb{Q}(\sqrt{13}) \in W_{t+1}/W_t \) and \( t = 1 \).

**Theorem 3.2:** Suppose that \( \mathbb{Q}(\sqrt{d}), d > 0 \) with fundamental unit \( \varepsilon = t + s\sqrt{d} \) or \( \varepsilon = [(2t-1) + s\sqrt{d}]/2 \), then \( \mathbb{Q}(\sqrt{d}) \subseteq W_{t+1} \setminus W_{t} \), \( t > 1 \).

**Proof:** If \( d \equiv 1(\text{mod}4), \varepsilon = t + s\sqrt{d}, s,t \in N \), then the proof followed if we show that any integer \( \alpha \in \mathbb{Q}(\sqrt{d}) \) is not \((t-1)\)-integer by writing
\[ t = \sum_{i=0}^{m} c_i e^i \]

with \(|c_i| \leq t - 1, k, m \in \mathbb{Z}\). If \(m \geq 1\), then

\[ e^m \leq |c_m| e^m \leq \sum_{i=k}^{m-1} c_i e^i + t < (t - 1) \sum_{i=-\infty}^{m-1} e^i + t \]

\[ < (t - 1) \frac{e^{m-1}}{1 - \frac{\varepsilon}{t}} + t \]

\[ < (t - 1) \frac{e^m}{t - \varepsilon} + t e^m \]

\[ e^m \leq \frac{(t-1)e^m}{\varepsilon - 1} + t e^{m-1} \]

where \(m \geq 1, \varepsilon^{m-1} > 1\). Therefore

\[ \frac{1}{\varepsilon} - t - 1 + \frac{\varepsilon}{t} \]

\[ \varepsilon^2 - 2t \varepsilon + t \leq 0 \]

since \(\varepsilon = t + \sqrt{t^2 + 1}\), then

\[ \varepsilon \leq t + \sqrt{t^2} - t < t + \sqrt{t^2} - 1 \leq \varepsilon \]

a contradiction. By the same way and if \(d \equiv 1 (mod 4)\), we get

\[ 2\varepsilon \leq (2t - 1) + \sqrt{(2t-1)^2 - 4t} \]

\[ < (2t - 1) + \sqrt{(2t-1)^2 - 4} \]

\[ = 2t \]

also, a contradiction. Hence \(Q(\sqrt{d}) \in W_{t-1}, (t > 1)\).

**Theorem 3.3:** There are a finite number of \(Q(\sqrt{d}) \in W_t, t \in \mathbb{N}\).

**Proof:** If \(d \equiv 1 (mod 4)\) and \(\varepsilon = t + \sqrt{d}\) is the fundamental unit of \(Q(\sqrt{d}), d > 0\) the proof followed by showing \(t \in \left(\frac{\varepsilon - 1}{\varepsilon + 1}, \frac{\varepsilon + 1}{\varepsilon - 1}\right)\). From theorem (2.2) we have that

\[ \varepsilon < 2t + 1, (t \geq 1) \]

By writing

\[ 2t = \varepsilon + \varepsilon^{-1} \]

\[ 2t - 1 < \varepsilon < 2t + 1 \]

\[ \frac{\varepsilon - 1}{2} < t < \frac{\varepsilon + 1}{2} \]

Therefore,

\[ t - 1 < \sqrt{d} < t + 1 \]

and if \(d \equiv 1 (mod 4)\) we get also and by the same way

\[ t - 1 < \frac{1 + \sqrt{d}}{2} < t + 1 \]

Thus, therefore finite number of \(Q(\sqrt{d}) \in W_t\).

**Conclusion:**

The conclusion of our result is declared by some of the real quadratic fields below by applying theorem (3.1) and considering the fundamental unit \(\varepsilon \in Q(\sqrt{d})\), where both \(Q(\sqrt{3}), Q(\sqrt{13}) \in W_2\), follows from corollary (3.2).

\[ Q(\sqrt{3}) \in W_1, \quad d = 2, 5 \]

\[ Q(\sqrt{5}) \in W_2, d = 3, 13 \]

\[ Q(\sqrt{13}) \in W_3, d = 10, 21, 29 \]

\[ Q(\sqrt{25}) \in W_4, d = 15, 53 \]

\[ Q(\sqrt{3}) \in W_5, d = 26, 77, 82, 85 \]

As an open question there is a possibility to represent integers of imaginary quadratic fields \(Q(\sqrt{d}), d < 0, d = -1, -2, -3\) as sum of units of this field of certain repetition, \(t, t \geq 1\). This may also classifying the imaginary fields in \(W_t\).

**Conflicts of Interest:** None.

**References:**

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تمثيل الأعداد الجبرية للحقل التربيعي الحقيقي كمجموع لوحدات الحقل الأساسية

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الخلاصة:
في هذا البحث تم تعميم نتائج الباحث جاكوبسن باستخدام الوحدات الأساسية للحقل التربيعي الحقيقي ممثلة في الشروتين التاليين:

\[ t + 1 = \varepsilon \pm \varepsilon^{-1} + (1 - t), \quad t \geq 1 \]

والمشترط الثاني هو \( \varepsilon = t + \sqrt{d} \), \( d \equiv 1 \pmod{4} \) عندما يكون الحقل التربيعي \( Q(\sqrt{d}) \) حيماً في الحقول \( W_1 \) حيث أن العدد \( d \equiv 1 \pmod{4} \) وفق للقيم \( t \). وتم تقسيم الحقول في المجموعة \( W_1 \) بحيث أن جاكوبسن برهن أن الحقول التربيعيين \( Q(\sqrt{5}), Q(\sqrt{2}) \) ينتميان للمجموعة \( W_1 \) كما برهن الباحث J.Silwa.

الكلمات المفتاحية: الأعداد الجبرية الحقيقية كمجموع منتهي لوحدات الحقل التربيعي الأساسية، الوحدات الأساسية للحقل التربيعي الحقيقي.