Orthogonal Symmetric Higher bi-Derivations on Semiprime $\Gamma$-Rings

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Received 5/6/2018, Accepted 30/5/2019, Published 18/12/2019

Abstract:
Let $M$ be a $\Gamma$-ring. In this paper the concept of orthogonal symmetric higher bi-derivations on semiprime $\Gamma$-ring is presented and studied and the relations of two symmetric higher bi-derivations on $\Gamma$-ring are introduced.

Key words: Higher bi-derivations on a $\Gamma$-ring, Orthogonal symmetric higher bi-derivations on a $\Gamma$-ring, Symmetric bi-derivations on a $\Gamma$-ring.

Introduction:
Let $M$ and $\Gamma$ be two additive abelian groups, $M$ is called a $\Gamma$-ring if the following conditions are satisfied for any $x,y,z\in M$ and $\alpha, \beta \in \Gamma$:

i) $\alpha xy \in M$

ii) $\alpha(x+y) = \alpha x + \alpha y$

iii) $(\alpha \beta )y = \alpha (\beta y)$

The notion of a $\Gamma$-ring was first introduced by Nobusawa and generalized by Barnes 1966 (1) as above definition. It is well known that every ring is $\Gamma$-ring. $M$ is called prime if $\alpha x=0 \implies x=0$ and said to be semiprime if $\alpha x=0 \implies x=0$ for all $x,y \in M$ (2). $M$ is said to be n-torsion free if $nx=0$, $x \in M$ implies that $x=0$ where $n$ is positive integer, (3). Khan A.R, Chaudhry M.A and Javaid I. in (4) defined a derivation on $\Gamma$-ring as follows: An additive mapping $d:M\rightarrow M$ is said to be derivation on $M$ if $d(\alpha xy) = d(x) \alpha y + d(y) \alpha x$ for all $x,y \in M$ and $\alpha \in \Gamma$. Rahman M.M. and Paul A.C. in (5) are defined a Jordan derivation on a $\Gamma$-ring as follows: An additive mapping $d:M\rightarrow M$ is said to be Jordan derivation on $M$ if $d(x\alpha y,z) = d(x,z)\alpha y + x\alpha d(y,z)$ for all $x,y,z \in M$ and $\alpha \in \Gamma$. It is clear that every bi-derivation of a $\Gamma$-ring $M$ is Jordan bi-derivation of $M$. Marir A.M in (8) introduced the concept of higher bi-derivation on $\Gamma$-ring as follows: Let $D=(d_{i})_{i\in N}$ be a family of bi-additive mapping on $M \times M$ into $M$ is said to be higher bi-derivation if $d_{n}(x,y,z) = \sum_{i+j=n}d_{i}(x,y)d_{j}(y,z)$ for all $x,y,z \in M$ and $\alpha \in \Gamma$. Salah M.Salih and Ahmed M.M. in (9) introduced the concept of Jordan higher bi-derivations on prime $\Gamma$-rings as follows: Let $D=(d_{i})_{i\in N}$ be a family of bi-additive mapping on $M \times M$ into $M$ is said to be Jordan bi-derivation if $d_{n}(x\alpha y,z) = \sum_{i+j=n}d_{i}(x,y)d_{j}(y,z)$ for all $x,y \in M$, $\alpha \in \Gamma$.

In this paper the results of the concept of orthogonal symmetric higher bi-derivations on semiprime $\Gamma$-ring are introduced, and some of lemmas and theorem are proved.

Orthogonal Symmetric Higher bi-Derivations on Semiprime $\Gamma$-Rings.
In this section the concept of orthogonal symmetric higher bi-derivations are presented and studied on $\Gamma$-rings and introduced an example and some lemmas.

Definition (2-1):
Let $M$ be a $\Gamma$-ring, let $D=(d_{i})_{i\in N}$ and $G=(g_{i})_{i\in N}$ be two symmetric higher bi-derivations on a $\Gamma$-ring $M$. Then $d_{n}$ and $g_{n}$ are said to be orthogonal if for all $x,y,z \in M$ and $\alpha \in N$

$d_{n}(x,y) \Gamma M \Gamma g_{n}(y,z) = (0) = g_{n}(x,y) \Gamma M \Gamma d_{n}(x,y)$

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\begin{align*}
\text{Lemma (2-3):} & \quad \text{Let } M \text{ be a 2-torsion free semiprime } \Gamma \text{-ring and } x, y \text{ elements of } M \text{ such that } \forall \alpha, \beta \in \Gamma \text{, then } \alpha x = y \beta x = 0. \\
\text{Lemma (2-4):} & \quad \text{Let } M \text{ be a 2-torsion free semiprime } \Gamma \text{-ring and } x, y \text{ elements of } M \text{ such that } \forall \alpha, \beta \in \Gamma \text{, then } \alpha x = y \beta x = 0. \\
\text{Lemma (2-5):} & \quad \text{Let } M \text{ be a semiprime } \Gamma \text{-ring and } x, y \text{ elements of } M \text{ such that } \forall \alpha, \beta \in \Gamma \text{, then } \alpha x = y \beta x = 0. \\
\text{Lemma (2-6):} & \quad \text{Let } M \text{ be a 2-torsion free semiprime } \Gamma \text{-ring such that } x\alpha y = \beta z = x y \beta a z \text{ for all } \alpha, \beta \in \Gamma, x, y, z \in M \text{ and two symmetric higher bi-derivations } d_n \text{ and } g_n \text{ for all } n \in N. \\
\text{Proof:} & \quad \text{Suppose } d_n(x, y) = g_n(x, y) = 0 \text{ for all } n \in N. \\
\text{Main Results:} & \quad \text{In this section introduce some theorems of orthogonal symmetric higher bi-derivations on } \Gamma \text{-ring.} \\
\text{Theorem (3-1):} & \quad \text{Let } M \text{ be a 2-torsion free semiprime } \Gamma \text{-ring such that } x\alpha y = \beta z = x y \beta a z \text{ for all } \alpha, \beta \in \Gamma, x, y, z \in M. \text{Two symmetric higher bi-derivations } d_n \text{ and } g_n \text{ for all } n \in N \text{ are orthogonal if and only if } d_n(x, y) = g_n(x, y) = 0 \text{ for all } n \in N. 
\end{align*}
\]
Proof:
Suppose that $d_n(x,y)ag_n(y,z)=0$
\[d_n(x,y)ag_n(y,z) = \sum_{i=1}^{n} d_i(x,y)ag_i(y,z) = 0 \quad \ldots (1)\]
Replace $x$ by $x\beta t$ in (1) for all $t \in M$ and $\beta \in \Gamma$
\[\sum_{i=1}^{n} d_i(x\beta t,y)ag_i(y,z) = 0 \quad \ldots (2)\]
Replace $d_i(t,y)$ by $d_i(y,z)$ for all $m \in M$
\[\sum_{i=1}^{n} d_i(y,z)\beta m a g_i(y,z) = 0 \quad \ldots (3)\]
By Lemma (2-3), $d_n$ and $g_n$ are orthogonal

Similarly, if $g_n(x,y)ad_n(y,z)=0$, $d_n$ and $g_n$ are orthogonal

Conversely, suppose that $d_n$ and $g_n$ are orthogonal
\[\sum_{i=1}^{n} d_i(x,y)\beta m a g_i(y,z) = 0 \quad \ldots (4)\]
By Lemma (2-3)
\[\sum_{i=1}^{n} d_i(x,y)ag_i(y,z) = 0 \quad \ldots (5)\]
Thus $d_n(x,y)ag_i(y,z) = 0$ and by $g_n(x,y)\Gamma M \Gamma g_n(y,z) = 0 (0)$
\[\sum_{i=1}^{n} g_i(x,y)\beta m a g_i(y,z) = 0 \quad \ldots (6)\]
By Lemma (2-3)
\[\sum_{i=1}^{n} g_i(x,y)ad_i(y,z) = 0 \quad \ldots (7)\]
Thus $g_n(x,y)ad_i(y,z) = 0$ and $g_n(x,y)ad_n(y,z) = 0$

Theorem (3-2) Let $d_n$ and $g_n$ be two symmetric higher bi-derivations of a 2-torsion free semiprime $\Gamma$-ring $M$ and where $d_n$ is commuting and $g_n$ is commuting for all $n \in N$. Then the following conditions are equivalent for every $n \in N$:

i) $d_n$ and $g_n$ are orthogonal.
ii) $d_n g_n = 0$
iii) $g_n d_n = 0$
iv) $d_n g_n + g_n d_n = 0$
v) $g_n d_n$ is symmetric higher bi-derivation.
vi) $g_n d_n$ is symmetric higher bi-derivation.

Proof: (i) $\Rightarrow$ (ii)
Suppose that $d_n$ and $g_n$ are orthogonal
\[g_n(x,y)\Gamma M \Gamma g_n(y,z) = 0 \quad \ldots (8)\]
\[d_n(g_n(x,y)\Gamma M \Gamma d_n(y,z), m_1) = 0 \quad \forall m_1 \in M \quad \ldots (9)\]
By $\Gamma M \Gamma$ in (9)
\[\sum_{i=1}^{n} d_i(g_i(x,y), m_1)\alpha d_i(m, m_1)\beta d_i(d_i(x,y), m_1) = 0 \quad \ldots (10)\]
Replace $d_i(d_i(x,y), m_1)$ by $d_i(g_i(x,y), m_1)$ in (10)
\[\sum_{i=1}^{n} d_i(g_i(x,y), m_1)\alpha d_i(m, m_1)\beta d_i(g_i(x,y), m_1) = 0 \quad \ldots (11)\]
Since $M$ is semiprime $\Gamma$-ring
\[\sum_{i=1}^{n} d_i(g_i(x,y), m_1) = 0 \quad \ldots (12)\]
\[d_n g_n = 0 \quad \ldots (13)\]
Conversely, suppose that $d_n g_n = 0$
\[d_n(g_n(x,y), m) = 0 \quad \forall m \in M \quad \ldots (14)\]
Replace $x$ by $x \alpha z$ in (14)
\[\sum_{i=1}^{n} d_i(g_i(x,y), m) + g_i(d_i(x,y), m) = 0 \quad \ldots (15)\]
Replace $x$ by $x \alpha z$ in (15) we get
\[\sum_{i=1}^{n} d_i(g_i(x,y), m) + g_i(d_i(x,y), m) = 0 \quad \ldots (16)\]
\[ \sum_{i=1}^{n} d_i(g_i(x,y), m)ad_i(g_i(z,y), m) + g_i(d_i(x,y), m)ag_i(d_i(z,y), m) = 0 \]

\[ \sum_{i=1}^{n} d_i(g_i(x,y), m)ag_i(d_i(z,y), m) + g_i(d_i(x,y), m)ad_i(g_i(z,y), m) = 0 \]  

Replacing \( d_i(g_i(x,y), m) \) by \( d_i(x,y) \) and

\( g_i(d_i(z,y), m) \) by \( g_i(y,z) \) and

\( g_i(d_i(x,y), m) \) by \( g_i(y) \) and

\( d_i(g_i(z,y), m) \) by \( d_i(y,z) \) in (8)

\[ \sum_{i=1}^{n} d_i(x,y)ag_i(y,z) + g_i(x,y)\alpha_{d_i(y,z)} = 0 \]

\[ d_n(x,y)ag_n(y,z) + g_n(x,y)\alpha_{d_n(y,z)} = 0 \]

By Lemma (2-5) \( d_n \) and \( g_n \) are orthogonal

**Proof:** (i) \( \iff \) (v)

To prove \( (d_n g_n)(xaz, y) = (d_n g_n)(x)\alpha(d_n g_n)(z, y) \)

\[ (d_n g_n)(x, y)\alpha(d_n g_n)(z, y) = d_n(g_n(x, y), m)d_n(g_n(y, z), m) \]  

for all \( m \in M \)

\[ = \sum_{i=1}^{n} d_i(g_i(x, y), m)ad_i(g_i(z, y), m) \]  

And

\[ (d_n g_n)(xaz, y) = d_n(g_n(xaz, y), m) = \sum_{i=1}^{n} d_i(g_i(xaz, y), m) \]

\[ = \sum_{i=1}^{n} d_i(g_i(x, y)ag_i(y, z), m) \]

\[ = \sum_{i=1}^{n} d_i(g_i(x, y)\alpha_{d_i(y, z)}, m) \]

By (9) and (10) \( d_n g_n \) is higher bi-derivation

Conversely, to prove \( d_n \) and \( g_n \) are orthogonal

\[ d_n(x, y)\Gamma Mg_n(y, z) = d_n(x, y)\Gamma Mg_n(y, z, t) \text{ for all } t \in M \]

\[ = \sum_{i=1}^{n} d_i(x, y)ag_i(y, z, t) \text{ for all } \alpha, \beta \in \Gamma \]

and \( m \in M \)

\[ = \sum_{i=1}^{n} d_i(g_i(x, y), t)ad_i(g_i(y, z), t) \]

Replace \( g_i(y, z) \) by \( g_i(x, y) \) in (11)

\[ = \sum_{i=1}^{n} d_i(g_i(x, y), t)ad_i(g_i(y, z), t) \]

Since \( R \) is semiprime \( \Gamma \)-ring and by Theorem (3-2) (ii)

\[ 0 = \sum_{i=1}^{n} d_i(g_i(x, y), t) \]

we get \( d_n g_n = 0 \)

Thus \( d_n(x, y)\Gamma Mg_n(y, z) = (0) \) by lemma (2-3)

\( d_n \) and \( g_n \) are orthogonal

**Proof:** (i) \( \iff \) (v)

Similar way used in the proof of (v)

**Theorem (3-3):** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring, \( d_n \) and be two symmetric higher bi-derivations for all \( m \in N \). Suppose that \( d_n = g_n \), then \( d_n - g_n \) and \( d_n + g_n \) are orthogonal.

**Proof:**

Suppose that \( d_n = g_n \), then for all \( x, y \in M \):

\[ \sum_{i=1}^{n} d_i(g_i(x,y), m)ad_i(g_i(z,y), m) + g_i(d_i(x,y), m)ag_i(d_i(z,y), m) = 0 \]

\[ \sum_{i=1}^{n} d_i(g_i(x,y), m)ag_i(d_i(z,y), m) + g_i(d_i(x,y), m)ad_i(g_i(z,y), m) = 0 \]

Replacing \( d_i(g_i(x,y), m) \) by \( d_i(x,y) \) and

\( g_i(d_i(z,y), m) \) by \( g_i(y,z) \) and

\( g_i(d_i(x,y), m) \) by \( g_i(y) \) and

\( d_i(g_i(z,y), m) \) by \( d_i(y,z) \) in (8)

\[ \sum_{i=1}^{n} d_i(x,y)ag_i(y,z) + g_i(x,y)\alpha_{d_i(y,z)} = 0 \]

\[ d_n(x,y)ag_n(y,z) + g_n(x,y)\alpha_{d_n(y,z)} = 0 \]

By Lemma (2-5) \( d_n \) and \( g_n \) are orthogonal

**Conflicts of Interest:** None.

**References:**


تعامد المشتقات الثنائية العليا المتناظرة على الحلقات شبه الأولية من النمط –\( \Gamma \)

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الخلاصة:
لكن حلقة من النمط –\( \Gamma \). في هذا البحث قمنا بدراسة تعامد المشتقات الثنائية العليا المتناظرة على الحلقات شبه أولية النمط –\( \Gamma \) ودرسنا العلاقات بين المشتقات الثنائية العليا المتناظرة على الحلقات شبه أولية من النمط –\( \Gamma \) ونجد خواص مكافئة للتعامد.

الكلمات المفتاحية:
المشتقات الثنائية العليا المتناظرة على الحلقة شبه أولية النمط –\( \Gamma \)، تعامد المشتقات الثنائية العليا المتناظرة على الحلقة شبه أولية النمط –\( \Gamma \)