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# Finite Dimensional Convex Fuzzy Normed Space and Its Basic Properties

***Hawraa Yousif Daher*** , ***Jehad R. Kider****

Branch of Mathematics and Computer Applications, Department of Applied Sciences, University of Technology, Baghdad, Iraq.
*Corresponding Author.

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## Abstract

Here the notion of convex fuzzy absolute value space is represented with an example which shows that the existence of such space. The reason behind introducing the definition of convex fuzzy absolute value and does not using the ordinary absolute value is the main definition in this paper will be not correct with ordinary absolute value. After that the main definition of convex fuzzy normed space is recalled with example which shows that the existence of such space. Then other definitions and theorems is recalled that will be used later in the section of main results such as the convex fuzzy norm is convex fuzzy continuous function. So our goal in the section of results and discussion is to prove some properties of finite dimensional convex fuzzy normed space which not true ingeneral for convex fuzzy normed space. Thus the last section contains the following results with proofs if $\mathcal{D}$ is a subspace of the convex fuzzy normed space $\mathcal{U}$ with dim$\mathcal{D} < \infty$ then $\mathcal{D}$ is convex fuzzy complete subspace of $\mathcal{U}$. Moreover when $\mathcal{Y}$ is a convex fuzzy bounded as well as convex fuzzy closed subspace of the c-FNS, $\mathcal{U}$ also dim $\mathcal{U} < \infty$ then $\mathcal{Y}$ is convex fuzzy compact. As well as if dim$\mathcal{U} < \infty$ for a linear space $\mathcal{U}$ then there is a unique convex fuzzy norm on $\mathcal{U}$. Finally if $\mathcal{W}$={$w \in \mathcal{U}$: the convex fuzzy norm of $w$ belongs to I } a convex fuzzy closed subset of $\mathcal{U}$ as well as convex fuzzy compact this implies that dim$\mathcal{U} < \infty$ where $\mathcal{U}$ is convex fuzzy normed space.

**Keywords:** Convex Fuzzy Absolute Value Space, Convex Fuzzy Complete Space, Convex Fuzzy Compact Space, Convex Fuzzy Equivalent Norms, Convex Fuzzy Normed Space.

## Introduction

Bag and Samanta [1] present a comparative study among several types of fuzzy norms on a linear space defined by various authors has been made and they classified them into two types, one of which is Katsaras's type and the other is Felbin's type. Sadeqi and Kia.[2] they proved the equivalence between fuzzy continuity and topological continuity first. Secondly, they introduce the notion of fuzzy seminorm and they introduce some new results.

Golet[3] present a generalization of fuzzy norms on linear space. Also, he proved the fixed-point theorems. Janfada et al[4] they used the operator norm of weakly fuzzy bounded operators to present some properties of the space of all weakly fuzzy bounded linear operators. A counter examples, introduced here to show that the inverse mapping theorem and the Banach-Steinhaus's theorem, are not valid for

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fuzzy setting. Next, they established a Hahn-Banach theorem.

Nădăban et al[5] introduce various definition of fuzzy norms on a linear space, Nădăban et al[6] present a generalization of the concept of fuzzy norm then proved some theorems for fuzzy Banach space. Also they propose another definition of fuzzy normed space. Finally, some important results in fuzzy fixed point theory were highlighted.

Sabre[7] works in the standard fuzzy and she interest in fuzzy compact operators. Sabre and Ahmed[8] present best proximity point theorem for $\tilde{\alpha}-\tilde{\psi}$ −contractive type mapping in fuzzy normed space. Daher and Kider[9] present the notion of convex fuzzy normed spaces then proved important properties of this space.

Here the study of the case when the convex fuzzy normed space have finite dimension is presented.. This article consist of three sections, in the second section the c-FNS and its basic theorems is recalled with their proves. In section three it was supposed the c-FNS, $\mathcal{U}$ have the property $\dim\mathcal{U} < \infty$ then properties of this space is proved.

## Properties of Convex Fuzzy Normed Space:

**Definition 1:**[9]

If $\mathcal{U} \neq \emptyset$, then the function $\mathcal{D}$: $\mathcal{U} \to [0,1]$ a fuzzy set in $\mathcal{U}$ with $0 \le \mathcal{D}(u) \le 1$ for all $u \in \mathcal{U}$.

**Definition 2:**[9]

Let $A_{\mathbb{R}}:\mathbb{R} \to [0, 1]$ be a fuzzy set if $A_{\mathbb{R}}$ satisfies

(i)$A_{\mathbb{R}}(\sigma) \in [0, 1]$ when $\sigma \neq 0$;

(ii) $A_{\mathbb{R}}(\gamma\delta) \le A_{\mathbb{R}}(\gamma) \,.\, A_{\mathbb{R}}(\delta)$; $A_{\mathbb{R}}(\sigma)=0 \Leftrightarrow \sigma=0$;

(iii) $A_{\mathbb{R}}(\sigma+\theta) \le \omega A_{\mathbb{R}}(\sigma) + \mu A_{\mathbb{R}}(\theta)$, $\forall\ \omega, \mu \in [0, 1]$ with $\sigma + \mu = 1$;

(iv) $A_{\mathbb{R}}(\sigma)=0 \Leftrightarrow \sigma=0$;

Hence $(\mathbb{R}, A_{\mathbb{R}})$ is **c-FAVS (convex fuzzy absolute value space )** $\forall\ \sigma, \theta \in \mathbb{R}$.

**Example 1:**[9]

Define $A^{|.|}$: $\mathbb{R} \to [0, 1]$; by $A^{|.|}(\delta) = \begin{cases} \frac{1}{|\delta|} & if\ \delta \neq 0 \\ 0 & if\ \delta = 0 \end{cases}$

Hence $(\mathbb{R}, A^{|.|})$ is c-FAVS $\forall\ \delta \in \mathbb{R}$.

**Definition 3:**[9]

If $\mathfrak{N}$: $\mathcal{U} \to$I is a fuzzy set satisfies

(1) $\mathfrak{N}(y) \in [0, 1]$ ;

(2) $\mathfrak{N}(y) = 0 \Leftrightarrow y=0$;

(3) $\mathfrak{N}(\sigma y) \le A_{\mathbb{R}}(\sigma)\ \mathfrak{N}(y)$, $\forall\ 0 \neq \sigma \in \mathbb{R}$;

(4)$\mathfrak{N}(y + v) \le \gamma\mathfrak{N}(y) + \delta\mathfrak{N}(v)$, where $\gamma + \delta=1$, $\forall\ y, v \in \mathcal{U}$. Then $(\mathcal{U}, \mathfrak{N})$ is **c-FNS [convex fuzzy normed space]**.

**Example 2:**[9]

If $\mathcal{U}$=C[a, d], define $\mathfrak{N}(g) = max_{\alpha\in[a,d]}\ A_{\mathbb{R}}[g(\alpha)]$ for all $g \in \mathcal{U}$. Then $(\mathcal{U}, \mathfrak{N})$ is c-FNS.

**Definition 4:**[9]

Suppose that $(\mathcal{U}, \mathfrak{N})$ is c-FNS and assume that $(u_m)$ is a sequence in $\mathcal{U}$, then $(u_m)$ is **convex fuzzy converges** to $y \in \mathcal{U}$ when m → ∞ if $\forall\ \alpha \in (0,1)\ \exists$ N $\in \mathbb{N}$ satisfying $\mathfrak{N}(u_m - y) < \alpha$, $\forall$ m≥N. If $(u_m)$ is convex fuzzy converges to $y$ or $\lim_{m\to\infty} u_m = y$ or $u_m \to$u or $\lim_{m\to\infty} \mathfrak{N}(u_m - y)=0$.

**Definition 5:**[9]

The sequence $(u_k)$ in $\mathcal{U}$ where $(\mathcal{U}, \mathfrak{N})$ is c-FNS is **CFB [ convex fuzzy bounded ]** if $\exists\ \omega \in (0,1)$ satisfying $\mathfrak{N}(u_k) < \omega\ \forall$ k$\in \mathbb{N}$.

**Definition 6:**[9]

If $(\mathcal{U}, \mathfrak{N})$ is c-FNS put cfb$(y, \alpha)$= $\{p \in \mathcal{U}$: $\mathfrak{N}(y - p) < \alpha\}$ and cfb$[y, \alpha]$= $\{q \in \mathcal{U}$: $\mathfrak{N}(y - q) < \alpha\}$.Then cfb$(y, \alpha)$ and cfb$[y, \alpha]$ is **convex open as well as convex closed fuzzy ball** with the center $y \in \mathcal{U}$ and radius $\alpha$, $0 < \alpha < 1$.

**Definitions 7:**[9]

If $(\mathcal{U}, \mathfrak{N})$ is c-FNS then

(1) $\mathcal{W} \subseteq \mathcal{U}$ is CFO [**convex Fuzzy open]** if cfb$(w,\alpha) \subseteq \mathcal{W}$, $\forall\ w \in \mathcal{W}$ and $\alpha \in (0, 1)$.

(2)Also $\mathcal{D} \subseteq \mathcal{U}$ is CFC [**convex fuzzy closed]** if $\mathcal{D}^C$ is CFO.

(3) The **convex fuzzy closure** of $\mathcal{Y}$, $\bar{\mathcal{Y}}$ =∩B where B is CFC set contains $\mathcal{Y}$.

(4)Also $\mathcal{D} \subseteq \mathcal{U}$ is CFD [**convex fuzzy dense]** in $\mathcal{U}$ if $\bar{\mathcal{D}} = \mathcal{U}$.

**Definition 8:**[9]

Let $(\mathcal{U}, \mathfrak{N})$ be c-FNS and let $(u_k) \in \mathcal{U}$ then $(u_k)$ is **convex fuzzy Cauchy** in $\mathcal{U}$ whenever $0 < \varepsilon < 1\ \exists$ N satisfying $\mathfrak{N}(u_k - u_m) < \varepsilon$, $\forall$ k, m ≥N.

**Theorem 1:**[9]

If cfb$(u, \alpha)$ is CFO ball in c-FNS $(\mathcal{U}, \mathfrak{N})$ then it is a CFO set.

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**Definition 9:**[9]

A c-FNS $(\mathcal{U}, \mathfrak{N})$ is convex fuzzy complete if $\forall$ $(u_k)$ fuzzy Cauchy in $\mathcal{U}$ we must have $u_k \rightarrow y \in \mathcal{U}$.

**Theorem 2**:[9]

In c-FNS $(\mathcal{U}, \mathfrak{N})$ if $u_k \rightarrow y \in \mathcal{U}$ then $(u_k)$ is convex fuzzy Cauchy.

**Theorem 3:**[9]

If c-FNS $(\mathcal{U}, \mathfrak{N})$ and $\mathcal{D} \subset \mathcal{U}$ then $d \in \overline{\mathcal{D}} \Leftrightarrow \exists (d_k) \in \mathcal{D}$ satisfying $d_k \rightarrow d$.

**Theorem 4:**

Let $(\mathcal{U}, \mathfrak{N})$ be a convex fuzzy complete c-FNS and $\mathcal{W}$ is a subspace of $\mathcal{U}$ then

(1)$\mathcal{W}$ is convex fuzzy complete

(2)$\mathcal{W}$ is CFC

Are equivalent

**Proof:**

(1)⟹(2);

Assume that $\mathcal{W}$ is convex fuzzy complete and let $w \in \overline{\mathcal{W}}$ then by using Theorem 2.14 there is $(w_k) \in \mathcal{W}$ satisfying $w_k \rightarrow w$ this implies that $(w_k)$ is convex fuzzy Cauchy by Theorem 2.13. Using $\mathcal{W}$ is convex
fuzzy complete $w_k \rightarrow w \in \mathcal{W}$ therefore $\overline{\mathcal{W}} \subseteq \mathcal{W}$ together with $\mathcal{W} \subseteq \overline{\mathcal{W}}$ thus $\overline{\mathcal{W}} = \mathcal{W}$ hence $\mathcal{W}$ is CFC.

(2)⟹(1);

Assume that $\mathcal{W}$ is CFC and let $(w_k) \in \mathcal{W}$ be a convex fuzzy Cauchy. Then $w_k \rightarrow w \in \mathcal{U}$
This implies that $w \in \overline{\mathcal{W}}$ this implies that $w \in \mathcal{W}$ because $\overline{\mathcal{W}} = \mathcal{W}$. But $(w_k)$ was any convex fuzzy Cauchy in $\mathcal{W}$ hence $\mathcal{W}$ is convex fuzzy complete.

**Theorem 5:**[9]

If c-FNS $(\mathcal{U}, \mathfrak{N})$ and $\mathcal{D} \subset \mathcal{U}$ then $\overline{\mathcal{D}} = \mathcal{U} \Leftrightarrow \forall$ u $\in \mathcal{U}$ $\exists$ d∈D satisfying $\mathfrak{N}(u - d) < \alpha$, $0 < \alpha < 1$.

**Definition 10:**[9]

If $(\mathcal{U}, \mathfrak{N}_{\mathcal{U}})$ and $(\mathcal{V}, \mathfrak{N}_{\mathcal{V}})$ are two c-FNS. Then $\mathcal{T}:\mathcal{U} \rightarrow \mathcal{V}$ is **convex fuzzy continuous at $u$** $\in \mathcal{U}$.

If $\forall$ $\alpha \in (0, 1)$, $\exists$ $\beta \in (0, 1)$, with $\mathfrak{N}_{\mathcal{V}}[\mathcal{T}(u) - \mathcal{T}(v)] < \alpha$ for any $v \in \mathcal{U}$ satisfying $\mathfrak{N}_{\mathcal{U}}(u - v) < \beta$.

Also $\mathcal{T}$ is **convex fuzzy continuous on $\mathcal{U}$** if it is convex fuzzy continuous at every $y \in \mathcal{U}$.

**Theorem 6:**[9]

The functional $\mathfrak{N}$: $\mathcal{U} \rightarrow$I is a convex fuzzy continuous if $(\mathcal{U}, \mathfrak{N})$ is c-FNS.

**Theorem 7:**

Let $(\mathcal{U}, \mathfrak{N})$ be a c-FNS and $\mathcal{W} \subseteq \mathcal{U}$ and $\mathcal{U}$ is convex fuzzy complete then

(1)$\mathcal{W}$ is convex fuzzy complete

(2) $\mathcal{W}$ is convex fuzzy closed.

are equivalent.

**Proof:**

(1)⟹ (2);

Suppose that $\mathcal{W}$ is convex fuzzy complete then for every z $\in \overline{\mathcal{W}}$ there is $(z_k) \in \mathcal{W}$ such that $z_k \rightarrow$z. This implies that $(z_k)$ is fuzzy Cauchy in $\mathcal{W}$ using $\mathcal{W}$ is convex fuzzy complete thus $(z_k)$ is convex fuzzy converge in $\mathcal{W}$ since limit is unique so z $\in \mathcal{W}$ it follows that $\overline{\mathcal{W}} \subseteq \mathcal{W}$. Hence $\mathcal{W} = \overline{\mathcal{W}}$ so $\mathcal{W}$ is convex
fuzzy closed.

(2)⟹ (1);

Suppose that $\mathcal{W}$ is convex fuzzy closed and let $(z_k)$ be convex fuzzy Cauchy sequence in $\mathcal{W}$ so $(z_k)$ be convex fuzzy Cauchy sequence in $\mathcal{U}$ using $\mathcal{U}$ is convex fuzzy complete thus $z_k \rightarrow$z $\in \mathcal{U}$ but $\mathcal{W}$ is convex fuzzy closed so z∈ $\mathcal{W}$. Hence $\mathcal{W}$ is convex fuzzy complete.

**Definition 11:**

A c-FNS $(\mathcal{U},\mathfrak{N})$ is said to be **convex fuzzy compact** if $\forall$ CFO covering $\mathfrak{G} = \{\mathfrak{B}_j : j \in J\}$ of $\mathcal{U}$ there is $\{\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \ldots, \mathfrak{B}_k\} \subseteq \mathfrak{C}$ such that $\mathcal{U} = \bigcup_{j=1}^{k} \mathfrak{B}_j$.

**Theorem 8:**[9]

The c-FNS $(\mathcal{U},\mathfrak{N})$ is convex fuzzy compact $\Leftrightarrow \forall$ $(y_k)$ in $\mathcal{U}$ contains a subsequence $(y_{k_j})$ satisfying $y_{k_j} \rightarrow y \in \mathcal{U}$.

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## Results and Discussion

### Finite Dimension c-FNS:

Here the study of the case when the vector space $\mathcal{U}$ has the property dim $\mathcal{U} < \infty$ with convex fuzzy norm.

All results in this section depends on the following Theorem which is very important in studying c-FNS $(\mathcal{U}, \mathfrak{N})$ when dim $\mathcal{U}$=k with k$\in \mathbb{N}$.

**Theorem 9:**

Suppose that $(\mathcal{U}, \mathfrak{N})$ is c-FNS and $(\mathbb{R}, A_{\mathbb{R}})$ is c-FAVS. If $\{u_1, u_2, \ldots, u_k\}$ is linearly independent set in $\mathcal{U}$ . Then $\exists$ $0< \sigma < 1$ such that

$$\mathfrak{N}[\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_k u_k] \geq \sigma[\delta_1 A_{\mathbb{R}}(\alpha_1) + \delta_2 A_{\mathbb{R}}(\alpha_2) + \ldots + \delta_k A_{\mathbb{R}}(\alpha_k)] \quad 1$$

Where $\delta_1 + \delta_2 + \ldots + \delta_k = 1$.

**Proof:**

Put $[\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2 A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k A_{\mathbb{R}}(\alpha_k)]=\varepsilon$ and $\sigma.\varepsilon=\mu$ for some $0< \mu < 1$. Thus instead of (1) it must prove that

$$\mathfrak{N}[\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_k u_k] \geq \mu.$$

Suppose that this is not true, then it can be find a sequence $(w_m)$ in $\mathcal{U}$ where $w_m=\alpha_{1m}u_1+ \alpha_{2m}u_2+\ldots+\alpha_{km}u_k$ such that $\mathfrak{N}(w_m) \to 0$ as $m \to \infty$.

Now $\forall$ fixed j, $\alpha_{jm}=(\alpha_{j1}, \alpha_{j2} \ldots, \alpha_{jm}, \ldots)$ is CFB since $A_{\mathbb{R}}(\alpha_{jm}) \in [0, 1]$ so $(\alpha_{jm})$ has a subsequence $(\alpha_{jm_n})$ . Suppose that $\alpha_{jm_n} \to \alpha_j$ $\forall$ j=1, 2,…,k and assume that $(w_{jm}) \subseteq (w_m)$

Where $w_{jm}=\alpha_{1m_1}u_1+ \alpha_{2m_2}u_2+\ldots+\alpha_{km_k}u_k$.

Consider $w= \sum_{j=1}^{k} \alpha_j u_j$ then $(w_{jm}) \subseteq (w_m)$ and $w_{jm} \to w$ now using the assumption $\{u_1, u_2, \ldots, u_k\}$ is linearly independent set implies $w\neq 0$.

Again $w_{jm} \to w$ implies $\mathfrak{N}(w_{jm}) \to \mathfrak{N}(w)$ by Theorem 6. But $\mathfrak{N}(w_m) \to 0$ as $m \to \infty$ and $(w_{jm}) \subseteq (w_m)$. Therefore $\mathfrak{N}(w_{jm}) \to 0$. Thus $\mathfrak{N}(w)=0$ hence w=0. This contradicts $w\neq 0$.

**Theorem 10:**

Suppose that $(\mathcal{U}, \mathfrak{N})$ is c-FNS then $\mathcal{Y}$ is convex fuzzy complete if $\mathcal{Y}$ is a subspace of $\mathcal{U}$ with dim$\mathcal{Y}$ =k $< \infty$.

**Proof:**

If $(d_m)$ is a convex fuzzy Cauchy in $\mathcal{Y}$ and B=$\{u_1, u_2, \ldots, u_k\}$ is basis for $\mathcal{Y}$ then each $d_m$ can be represented uniquely by $d_m= \alpha_{1m}u_1+ \alpha_{2m}u_2 +\ldots+\alpha_{km}u_k$. Now using $(d_m)$ is convex fuzzy Cauchy thus $\forall$ $0< \alpha < 1$ $\exists$ N $\in \mathbb{N}$ satisfying $\mathfrak{N}(d_m - d_j) < \alpha$ $\forall$ m, j $\geq$N. Using Theorem 9, $\exists$ $\sigma \in(0, 1)$ satisfying

$$\sigma[\delta_1 A_{\mathbb{R}}(\alpha_{1m} - \alpha_{1j}) + \delta_2 A_{\mathbb{R}}(\alpha_{2m} - \alpha_{2j})+\ldots+\delta_k A_{\mathbb{R}}(\alpha_{km} - \alpha_{kj})]$$

$$\leq \mathfrak{N}(d_m - d_j) = \mathfrak{N}\left[\sum_{i=1}^{k}(\alpha_{im} - \alpha_{ij})u_i\right] \leq \alpha$$

dividing by $\sigma$ implies $A_{\mathbb{R}}(\alpha_{im} - \alpha_{ij}) \leq \frac{\alpha}{\sigma}$

This show that $(\alpha_{i1}, \alpha_{i2}, \ldots)$ is convex fuzzy Cauchy in $\mathbb{R}$.

Hence $\alpha_{im} \to \alpha_i$ $\forall$ $1\leq i \leq k$ . Now consider $b = \sum_{j=1}^{k} \alpha_j u_j$ this implies that b $\in \mathcal{Y}$.

Moreover $\forall$ m>N

$$\mathfrak{N}(d_m - b)= \mathfrak{N}\left[\sum_{i=1}^{k}(\alpha_{im} - \alpha_i)u_i\right]$$

$$\leq \delta_1 A_{\mathbb{R}}(\alpha_{1m} - \alpha_1)\mathfrak{N}(u_1) + \delta_2 A_{\mathbb{R}}(\alpha_{2m} - \alpha_2)\mathfrak{N}(u_2)+\ldots+ \delta_k A_{\mathbb{R}}(\alpha_{km} - \alpha_k)\mathfrak{N}(u_k)$$

Limiting both sides as $m \to \infty$ implies $\lim_{m\to\infty} \mathfrak{N}(d_m - b) \leq 0$.

Thus $\lim_{m\to\infty} \mathfrak{N}(d_m - d)=0$.

Therefore $d_m \to b$ and this proves that $\mathcal{Y}$ is convex fuzzy complete.

From Theorem 10 and Theorem 7 implies the next result

**Theorem 11:**

Suppose that $(\mathcal{U}, \mathfrak{N})$ is c-FNS then $\mathcal{D}$ is CFC if $\mathcal{D}$ is a subspace of $\mathcal{U}$ with dim$\mathcal{D}$ =k $< \infty$.

**Definition 12:**

If $(\mathcal{U}, \mathfrak{N}_1)$ is c-FNS and $(u_k) \in \mathcal{U}$ then the c-FN $\mathfrak{N}_2$ is said to be CFE [**convex fuzzy equivalent]** to $\mathfrak{N}_1$if

(1)$u_k \to y \in \mathcal{U}$ in $(\mathcal{U}, \mathfrak{N}_1)$

(2)$u_k \to y \in \mathcal{U}$ in $(\mathcal{U}, \mathfrak{N}_2)$.

Are equivalent.

**Theorem 12:**

If $(\mathcal{U}, \mathfrak{N}_1)$ is c-FNS and $\mu\mathfrak{N}_2(u) \leq \mathfrak{N}_1(u) \leq \varepsilon\mathfrak{N}_2(u)$ for some $\mu$, $\varepsilon$ in (0, 1) then the c-FN $\mathfrak{N}_2$ is CFE to $\mathfrak{N}_1$.

**Proof:**

If $u_k \to y \in \mathcal{U}$ in $(\mathcal{U}, \mathfrak{N}_1)$ thus $\forall$ $0< \alpha < \mu <1$ $\exists$ N$\in \mathbb{N}$ satisfying $\mathfrak{N}_1(u_k - y)< \alpha$ $\forall$ k$\geq$N.

Now since $\mu\mathfrak{N}_2(u_k - y) \leq \mathfrak{N}_1(u_k - y) < \alpha$ by dividing both sides by $\mu$ since $\mu \neq 0$ implies

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$\mathfrak{N}_2(u_k - \mathcal{y}) < \frac{\alpha}{\mu}$ for all k≥N. put $\frac{\alpha}{\mu}=\theta$ where $0<\theta<1$. Therefore $\mathfrak{N}_2(u_k - \mathcal{y}) < \theta$ for all k≥N.
Thus $u_k \to \mathcal{y} \in \mathcal{U}$ in $(\mathcal{U}, \mathfrak{N}_2)$.
On the other hand if $u_k \to \mathcal{y} \in \mathcal{U}$ in $(\mathcal{U}, \mathfrak{N}_2)$ so $\lim_{k\to\infty} \mathfrak{N}_2(u_k - \mathcal{y})=0$. Now
$\lim_{k\to\infty} \mathfrak{N}_1(u_k - \mathcal{y}) \le \lim_{k\to\infty} \varepsilon\mathfrak{N}_2(u_k - \mathcal{y})=\varepsilon.0=0$.
Therefore $u_k \to \mathcal{y} \in \mathcal{U}$ in $(\mathcal{U}, \mathfrak{N}_1)$.

**Theorem 13:**
Consider dim$\mathcal{U}$=k for a linear space $\mathcal{U}$ if $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are two c-FN on $\mathcal{U}$ then $\mathfrak{N}_1$ is CFE to $\mathfrak{N}_2$.

**Proof:**
Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for $\mathcal{U}$. $\forall\ \mathcal{y} \in \mathcal{U}$, $\mathcal{y}=\sum_{j=1}^{k} \alpha_j u_j$. Now

$\mathfrak{N}_1(\mathcal{y})=\mathfrak{N}_1\left(\sum_{j=1}^{k} \alpha_j u_j\right)$
$\le \delta_1\mathfrak{N}_1(\alpha_1 u_1)+\delta_2\ \mathfrak{N}_1(\alpha_2 u_2)+\ldots+\delta_k\ \mathfrak{N}_1(\alpha_k u_k)$
$\le \delta_1 A_{\mathbb{R}}(\alpha_1)\,\mathfrak{N}_1(u_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)\,\mathfrak{N}_1(u_2)+\ldots+\delta_k A_{\mathbb{R}}(\alpha_k)\,\mathfrak{N}_1(u_k)$
$\le \lambda[\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)]$

Where $\lambda=\max\{\mathfrak{N}_1(u_1), \mathfrak{N}_1(u_2), \ldots, \mathfrak{N}_1(u_k)\}$ that is $\mathfrak{N}_1(\mathcal{y}) \le \lambda[\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)]$ or dividing by $\lambda$

$\frac{1}{\lambda}\mathfrak{N}_1(\mathcal{y}) \le [\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)]$ 2

Now by Theorem 9
$\mathfrak{N}_2(\mathcal{y})=\ \mathfrak{N}_2\left(\sum_{j=1}^{k} \alpha_j u_j\right) \ge \sigma[\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)]$
or dividing by $\sigma$

$\frac{1}{\sigma}\mathfrak{N}_2(\mathcal{y}) \ge [\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)]$ 3

Now eqs 2 and 3 implies
$\frac{1}{\lambda}\mathfrak{N}_1(\mathcal{y}) \le [\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)] \le \frac{1}{\sigma}\mathfrak{N}_2(\mathcal{y})$ or
$\frac{1}{\lambda}\mathfrak{N}_1(\mathcal{y}) \le \frac{1}{\sigma}\mathfrak{N}_2(\mathrm{u})$ this implies that $\mathfrak{N}_1(\mathcal{y}) \le \frac{\lambda}{\sigma}\mathfrak{N}_2(\mathcal{y})$ put $\frac{\lambda}{\sigma}=\varepsilon$ we have

$\mathfrak{N}_1(\mathcal{y}) \le \varepsilon\ \mathfrak{N}_2(\mathcal{y})$ 4

In similar way
$\mathfrak{N}_2(\mathcal{y})=\mathfrak{N}_2\left(\sum_{j=1}^{k} \alpha_j u_j\right)$
$\le \delta_1\mathfrak{N}_2(\alpha_1 u_1)+\delta_2\ \mathfrak{N}_2(\alpha_2 u_2)+\ldots+\delta_k\ \mathfrak{N}_2(\alpha_k u_k)$
$\le \delta_1 A_{\mathbb{R}}(\alpha_1)\,\mathfrak{N}_2(u_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)\,\mathfrak{N}_2(u_2)+\ldots+\delta_k A_{\mathbb{R}}(\alpha_k)\,\mathfrak{N}_2(u_k)$
$\le \theta[\delta_1 a_{\mathbb{R}}(\alpha_1)+\delta_2\ a_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ a_{\mathbb{R}}(\alpha_k)]$

Where $\theta=\max\{\mathfrak{N}_2(u_1), \mathfrak{N}_2(u_2), \ldots, \mathfrak{N}_2(u_k)\}$ that is $\mathfrak{N}_2(\mathcal{y}) \le \theta[\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)]$ or dividing by t

$\frac{1}{\theta}\mathfrak{N}_2(\mathcal{y}) \le [\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)]$ 5

Now by Theorem 9
$\mathfrak{N}_1(\mathcal{y})=\ \mathfrak{N}_1\left(\sum_{j=1}^{k} \alpha_j u_j\right) \ge \sigma[\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)]$
or dividing by $\sigma$

$\frac{1}{\sigma}\mathfrak{N}_1(\mathcal{y}) \ge [\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)]$ 6

Now eqs 4 and 5 implies
$\frac{1}{\theta}\ \mathfrak{N}_2(\mathcal{y}) \le [\delta_1 A_{\mathbb{R}}(\alpha_1)+\delta_2\ A_{\mathbb{R}}(\alpha_2)+\ldots+\delta_k\ A_{\mathbb{R}}(\alpha_k)] \le \frac{1}{\sigma}\mathfrak{N}_1(\mathcal{y})$
or
$\frac{1}{\theta}\mathfrak{N}_2(\mathcal{y}) \le \frac{1}{\sigma}\mathfrak{N}_1(\mathcal{y})$ this implies that $\frac{\sigma}{\theta}\mathfrak{N}_2(\mathcal{y}) \le \mathfrak{N}_1(\mathcal{y})$ put $\frac{\sigma}{\theta}=\mu$ we have

$\mu\mathfrak{N}_2(\mathcal{y}) \le \mathfrak{N}_1(\mathcal{y})$ 7

Now eqs 4 and 7 implies, $\mu\ \mathfrak{N}_1(\mathcal{y}) \le\ \mathfrak{N}_1(\mathcal{y})\ \le \varepsilon\ \mathfrak{N}_2(\mathcal{y})$, for all $\mathcal{y} \in\ \mathcal{U}$.
Therefore, by Theorem 12, $\mathfrak{N}_1$ is CFE to $\mathfrak{N}_2$.

**Theorem 14:**
Consider dim$\mathcal{U}$=k where $(\mathcal{U}, \mathfrak{N})$ is c-FNs and $\mathcal{D} \subset \mathcal{U}$. Then $\mathcal{D}$ is convex fuzzy compact if $\mathcal{D}$ is CFC and CFB in $\mathcal{U}$.

**Proof:**
Consider $\{u_1, u_2, \ldots, u_k\}$ is basis for $\mathcal{U}$. If $(d_m)$ in $\mathcal{D}$ then $d_m=\alpha_{1m}u_1+\alpha_{2m}u_2+\ldots+\alpha_{km}u_k$ because $\mathcal{D}$ is CFB so is $(d_m)$ therefore $\mathfrak{N}(d_m) < \alpha\ \forall$ m ∈ ℕ and $\alpha \in (0, 1)$. By using Theorem 9

$\mathfrak{N}[d_m]=\ \mathfrak{N}[\sum_{j=1}^{k} \alpha_{jm}u_j] \ge \sigma[\delta_1 A_{\mathbb{R}}(\alpha_{1m})+\delta_2 A_{\mathbb{R}}(\alpha_{2m})+\ldots+\delta_k A_{\mathbb{R}}(\alpha_{km})]$
Thus $\sigma\ [\delta_1 A_{\mathbb{R}}(\alpha_{1m})+\delta_2 A_{\mathbb{R}}(\alpha_{2m})+\ldots+\delta_k A_{\mathbb{R}}(\alpha_{km})] \le \mathfrak{N}[d_m] <\ \alpha$ or
$A_{\mathbb{R}}(\alpha_{jm}) \le [\delta_1 A_{\mathbb{R}}(\alpha_{1m})+\delta_2 A_{\mathbb{R}}(\alpha_{2m})+\ldots+\delta_k A_{\mathbb{R}}(\alpha_{km})] < \frac{\alpha}{\sigma\delta_k}$
Therefore, for fixed j, $(\alpha_{jm})$ is CFB so $\alpha_{jm} \to \alpha_j\ \forall\ 1<j<k$. It was seen that $(d_m) \subseteq (z_m)$ and $d_m$

$\to z=\sum_{j=1}^{n} \alpha_j u_j$.

But $\mathcal{D}$ is CFC thus $z \in \mathcal{D}$ . Because $(d_m) \in \mathcal{D}$ was arbitrary. Therefore $\mathcal{D}$ is convex fuzzy compact by Theorem 8.

**Lemma 1:**

Suppose that $(\mathcal{U}, \mathfrak{N})$ is a c-FNS. If $\mathcal{B} \subseteq \mathcal{H} \subseteq \mathcal{U}$ where $\mathcal{H}$ is a subspace of $\mathcal{U}$ and $\mathcal{B}$ is CFC subspace of $\mathcal{U}$. Then $\forall\ \alpha \in (0, 1)\ \exists\ p \in \mathcal{H}$ with $\mathfrak{N}(p - q) \geq \alpha$ for all $q \in \mathcal{B}$.

**Proof:**

If $w \in \mathcal{H} - \mathcal{B}$ then put $\theta=\inf_{q\in\mathcal{B}}\mathfrak{N}(w - q)$. Thus $\theta > 0$ since $\mathcal{B}$ is CFC. Choose $\alpha \in (0, 1)$ with $\theta > \alpha$ so the infimum implies that $\exists\ q_0 \in \mathcal{B}$ satisfying $\theta \leq \mathfrak{N}(w - q_0) \leq \frac{\theta}{\alpha}$. Choose $p=w - q_0$.

$\mathfrak{N}(p - q)= \mathfrak{N}(w - q_0 - q)= \mathfrak{N}(w - q_1)$ with $q_1=q_0+q$. Hence $\mathfrak{N}(p - q)= \mathfrak{N}(w - q_1)\geq \theta > \alpha$.

**Theorem 15:**

If $\mathcal{W}=\{w \in \mathcal{U}:\mathfrak{N}(w)\in(0, 1]\ \}$ is a CFC in $\mathcal{U}$ and is convex fuzzy compact then $\dim\mathcal{U} = k$ for some $k\in \mathbb{N}$ where $(\mathcal{U}, \mathfrak{N})$ is a c-FNS.

**Proof:**

If $\dim \mathcal{U}=\infty$ then pick $u_1 \in \mathcal{W}$ and let $\{u_1\}$ be a basis for the subspace $\mathcal{U}_1$ of $\mathcal{U}$. Hence $\mathcal{U}_1$ it is CFC by Theorem 15. But $\mathcal{U}_1 \neq \mathcal{U}$ since $\dim \mathcal{U}=\infty$.

Using Lemma 1 $\exists\ u_2 \in \mathcal{W}$ satisfying $\mathfrak{N}(u_2 - u_1)\geq \alpha=\frac{1}{2}$. Let $\{ u_1, u_2\}$ be a basis for the subspace $\mathcal{U}_2$ of $\mathcal{U}$ since $\mathcal{U}_2 \neq \mathcal{U}$ thus $\exists\ u_3 \in \mathcal{W}$ satisfying $\mathfrak{N}(u_3 - u_2)\geq \frac{1}{2}$ and $\mathfrak{N}(u_3 - u_1)\geq \frac{1}{2}$.

Using induction will because $(u_k) \in \mathcal{W}$ with $\mathfrak{N}(u_m - u_j))\geq \frac{1}{2}$ where $m\neq j$. Therefore $(u_k)$ does not contain $(u_{k_j})$ with $u_{k_j} \to u \in \mathcal{W}$ .This will be cause $\mathcal{W}$ is not convex fuzzy compact thus this is contradiction. Therefore $\dim \mathcal{U} =k$ where $k\in \mathbb{N}$.

## Conclusion

First here new definition of fuzzy normed space is used it was called convex fuzzy normed space this definition does not use the binary operations t-norm and t-conrm but use new addition in the fourth condition which the property of convex so for this reason it was called convex fuzzy normed space also the ordinary absolute value does not work in the third condition of this definition so for this reason the definition of convex fuzzy absolute value is introduced. Here in this paper some properties of c-FNS $(\mathcal{U}, \mathfrak{N})$ when $\dim\mathcal{U} = k$ for some $k\in \mathbb{N}$ is proved and clearly these properties does not hold in c-FNS. Thus, this opens a new line of research for authors.

## Authors' Declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Technology.

## Authors' Contribution Statement

This work was carried out in collaboration between all authors H. Y. D. and J. R. K. read and approved the final manuscript.

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# الفضاء القياسي الضبابي المحدب المنتهي البعد وبعض خواصه الاساسية

**حوراء يوسف طاهر، جهاد رمضان خضر**

فرع الرياضيات وتطبيقات الحاسوب, قسم العلوم التطبيقية, الجامعة التكنولوجية ، بغداد , العراق.

## الخلاصة

هنا في هذه البحث تعريف القيمة المطلقة الضبابية المحدبة تم استدعائه مع مثال يوضح الوجود لهذا الفضاء.السبب الرئيس من وراء استخدام تعريف القيمة المطلقة الضبابية المحدبة بدلا من تعريف القيمة المطلقة هو إن التعريف الأساسي في هذا البحث لا يصح مع تعريف القيمة المطلقة. بعد ذلك التعريف الأساسي والذي هو تعريف فضاء القياس الضبابي المحدب تم استدعائه مع مثال يوضح الوجود لهذا الفضاء. التعاريف الأخرى والمبرهنات تم استدعائها والتي سوف تستخدم لاحقا في فقرة النتائج الأساسية مثلا دالة فضاء القياس الضبابي هي دالة مستمرة ضبابيا محددة. لذا سيكون هدفنا الأساسي في فقرة النتائج والاستنتاجات هو برهان خواص لفضاء القياس الضبابي المحدب المنتهي البعد والتي هي غير متحققة في فضاء القياس الضبابي المحدب وعليه ستكون الفقرة الأخيرة من هذا البحث النتائج التالية مع البراهين إذا كان $\mathcal{D}$ فضاء جزئي من فضاء القياس المحدب $\mathcal{U}$ وبعد $\mathcal{D}$ منتهي عندئذ يكون فضاء جزئي كامل ضبابي محدب. كذلك إذا كان $\mathcal{D}$ فضاء جزئي مغلق ضبابي محدب ومقيد ضبابي محدب من فضاء القياس الضبابي المحدب منهي البعد $\mathcal{U}$عندئذ يكون $\mathcal{D}$ متراص ضبابي محدب. بالإضافة إلى إذا كان فضاء متجهات $\mathcal{U}$ منتهي البعد عندئذ يوجد قياس ضبابي وحيد معرف عليه. وأخيرا إذا كان $\mathcal{W}=\{w \in \mathcal{U}$: the convex fuzzy norm of $w$ belongs to (0, 1] $\}$ هو فضاء مغلق ضبابي محدب في $\mathcal{U}$ ومتراص ضبابي محدب عندئذ يجب إن يكون بعد$\mathcal{U}$ منتهي عندما يكون $\mathcal{U}$ فضاء القياس الضبابي المحدب.

**الكلمات المفتاحية:** فضاء القيمة المطلقة الضبابي المحدب, الفضاء الكامل الضبابي المحدب, الفضاء المتراص الضبابي المحدب, القياسات الضبابية المحدبة المتكافئة, الفضاء القياسي الضبابي المحدب.