The Composition operator on hardy space H^2 Induced by $\varphi(z)=sz+t$ where $|s|\le 1$, $|t|<1$ and $|s|+|t|\le 1$.

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Abstract:

We study in this paper the composition operator of induced by the function $\varphi(z)=sz+t$ where $|s|\le 1$, $|t|<1$ and $|s|+|t|\le 1$.

We characterize the normal composition operator C_φ on Hardy space H^2 and other related classes of operators. In addition to that we study the essential normality of C_φ and give some other partial results which are new to the best of our knowledge.

Key words: composition operators, Hardy spaces.

Introduction:

Let U denote the unite ball in the complex plane, the Hardy space H^2 is the collection of holomorphic (analytic) functions $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ with $\hat{f}(n)$ denoting the n-th Taylor coefficient of f such that $\sum_{n=0}^{\infty}\left|\hat{f}(n)\right|^2<\infty$.

More precisely, $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in H^2 \Leftrightarrow$

$\|f\|^2=\sum_{n=0}^{\infty}\left|\hat{f}(n)\right|^2<\infty$.

The inner product inducing the H^2 norm is given by $\langle f,g\rangle=\sum_{n=0}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}$ ($f,g\in H^2$).

The particular importance of H^2 is due to the fact that it is a Hilbert space. Let ψ be a homomorphic function that take the unit ball U into itself (which is called homomrphic self-map of U). To each holomorphic self-map ψ of U, we associate the composition operator C_ψ defined for all $f\in H^2$ by $C_\psi f=f\circ\psi$.

In this paper, we discuss some links between the function theory and the operator theory and investigate the relationship between the properties of the function φ and the operator C_φ. Composition operators have been studied in many different contexts. A good source of references on the properties of composition operators on H^2 can found in [1].

We state very loosely some basic facts on composition operator on H^2.

Theorem 1, [1] : Every composition operator C_ψ is bounded.

Theorem 2, [1]: C_ψ is normal if and only if $\psi(z)=\lambda z$, $|\lambda|\le 1$.

Theorem 3, [1]: $C_\sigma C_\psi = C_{\psi o \sigma}$

Theorem 4, [1]: C_ψ is an identity operator if and only if ψ is the identity map.

For each $\alpha\in U$, the reproducing kernel at α, denoted by k_α is defined by Shapiro [2] as follows

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$k_\alpha(z)=\dfrac{1}{1-\overline{\alpha}z}$.

He proved [2] that for each α∈U and f∈ H^2, $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ that $\langle f,k_\alpha\rangle$ $=\sum_{n=0}^{\infty}\hat{f}(n)\alpha^n=f(\alpha)\cdot$

The reproducing kernels for H^2 will play an important role in this paper. Shapiro gives the following formula for the adjoint C_ψ^* of a composition operator C_ψ on the family $\{k_\alpha\}_{\alpha\in U}$.

Theorem 5, [1]***:*** Let ψ be a homomorphic self map of U, then for all α∈U $C_\psi^* k_\alpha = k_{\psi(\alpha)}$.

Next, Cowen gave an exact value of the of composition operator induced by φ(z)=sz+t.

Theorem 6, [3]***:*** Let φ(z)=sz+t, then the norm of C_φ on H^2 is defined as follows

$$\|C_\varphi\|=\sqrt{\frac{2}{1+|s|^2-|t|^2+\sqrt{(1-|s|^2+|t|^2)^2-4|t|^2}}}.$$

This paper consists of two sections. In section one, we characterize the normal composition operator C_φ on H^2 and other related classes of operators. In section tow, we characterize the essential normality of C_φ. To the best of our knowledge, these results are seemed to be new.

1. The Characterization for normality of C_φ

In this section we give a characterization of normal composition operator C_φ on Hardy space H^2, induced by φ(z)=sz+t where $|s|\le 1$, $|t|<1$ and $|s|+|t|\le 1$. Moreover, we study other related classis of operators.

Recall that an operator T on a Hilbert space H is said to be normal if $TT^*=T^*T$ where T^* is the adjoint of T. Also, T is said to be isometric if $T^*T=I$, where I is the identity operator. Moreover T is called unitary if $TT^*=T^*T=I$ [4]. We start this section by the following result.

Theorem 1.1: Let φ(z)=sz+t where $|s|\le 1$, $|t|<1$ and $|s|+|t|\le 1$. If $|s|=1$, then C_φ is an isometric operator on H^2.

Proof:

Assume that $|s|=1$. But $|s|+|t|\le 1$, then it is clear that t=0. Therefore φ(z)=sz. To prove that C_φ is isometric, it is enough that to show that $C_\varphi^* C_\varphi = I$. Let α∈U, then

$$\begin{aligned} C_\varphi^* C_\varphi k_\alpha(z) &= C_\varphi^* k_\alpha(\varphi(z)) \\ &= k_{\varphi(\alpha)}(\varphi(z)) \\ &= \frac{1}{1-\overline{\varphi(\alpha)}\,\varphi(z)} \\ &= \frac{1}{1-\overline{s\alpha}\,sz} \\ &= \frac{1}{1-|s|^2\overline{\alpha}\,z} \\ &= \frac{1}{1-\overline{\alpha}\,z} \\ &= k_\alpha(z). \end{aligned}$$

Hence $C_\varphi^* C_\varphi k_\alpha(z)=k_\alpha(z)$ for each α∈U. But it is well known that the span of the family $\{k_\alpha\}_{\alpha\in U}$ is dense subset in H^2 . This implies that $C_\varphi^* C_\varphi = I$ on H^2. So, C_φ is an isometric operator on H^2 ■

The following theorem gives the necessary and sufficient condition for the normality of C_φ .

Theorem 1.2: Let φ(z)=sz+t where where $|s|\le 1$, $|t|<1$ and $|s|+|t|\le 1$.

Then C_φ is a normal operator on H^2 if and only if t=0.

Proof:

Assume that C_φ is normal. Trivially case when C_φ is the identity operator, then by *theorem (4)* we have φ is the identity self-map of U, hence φ(z)=z. Thus t=0. Therefore we may assume that is not the identity operator, then by *theorem (4)* φ is the not identity self-map of U. To prove that t=0, we suppose that t≠0. Thus φ(0)=t≠0.

Since is C_φ normal then $C_\varphi C_\varphi^* = C_\varphi^* C_\varphi$. It follows that $C_\varphi C_\varphi^* k_0(z) = C_\varphi^* C_\varphi k_0(z)$.

But $C_\varphi k_0 = k_0$, $C_\varphi^* k_0 = k_{\varphi(0)}$ (by *theorem (5)*).Thus $C_\varphi k_{\varphi(0)}(z) = C_\varphi^* k_0(z)$. This implies that $k_{\varphi(0)}(\varphi(z)) = k_{\varphi(0)}(z)$. Hence, $\dfrac{1}{1-\overline{\varphi(0)}\,\varphi(z)} = \dfrac{1}{1-\overline{\varphi(0)}\,z}$. Thus $\overline{\varphi(0)}\,\varphi(z) = \overline{\varphi(0)}\,z$. But $\varphi(0) \neq 0$, then φ(z)=z, which a contradiction. Therefore, t=0.

Conversely, if t=0, then $\varphi(z) = sz, \quad |s| \le 1$. So, by *theorem (2)* we have C_φ is normal■

The following consequence gives the description of the unitary operator C_φ on Hardy space H^2.

Corollary 1.3: Let φ(z)=sz+t where where $|s| \le 1$, $|t| < 1$ and $|s| + |t| \le 1$. Then C_φ is a unitary operator on H^2 if and only if |s|=1.

Proof:

Assume that is C_φ unitary, then $C_\varphi C_\varphi^* = C_\varphi^* C_\varphi = I$. But every unitary operator is normal, then by (1.2) we have t=0. This implies that $\varphi(z) = sz, \quad |s| \le 1$. Hence, it is enough to show that |s|=1. Let α∈U, then

$$C_\varphi^* C_\varphi k_\alpha(z) = C_\varphi^* k_\alpha(\varphi(z)) = k_{\varphi(\alpha)}(\varphi(z)) = \frac{1}{1-\overline{\varphi(\alpha)}\,\varphi(z)}$$

But

$$C_\varphi^* C_\varphi k_\alpha(z) = I(k_\alpha(z)) = k_\alpha(z).$$

Hence $\dfrac{1}{1-\overline{\varphi(\alpha)}\,\varphi(z)} = \dfrac{1}{1-\overline{\alpha}\,z}$.

Therefore $\dfrac{1}{1-\overline{st}\,sz} = \dfrac{1}{1-\overline{\alpha}\,z}$. Hence $\dfrac{1}{1-|s|^2\overline{\alpha}\,z} = \dfrac{1}{1-\overline{\alpha}\,z}$. This implies that |s|=1.

Conversely, if |s|=1, then t=0 (since $|s| + |t| \le 1$). Therefore by (1.2) we get C_φ is normal. On the other hand, since |s|=1, then by (1.1) we have C_φ is isometric. Thus it is clear that $C_\varphi C_\varphi^* = C_\varphi^* C_\varphi = I$. It follows that C_φ is a unitary operator on H^2■

2. The characterization of essential normality of C_φ

Recall that an operator T on a Hilbert space H is said to be compact if it maps every bounded set into a relatively compact one (one whose closure in H is a compact set). Moreover, T is called essentially normal if $T^*T - TT^*$ is compact [5].

To study the compactness and essential normality of C_φ we need some preliminaries.

Let that T be a bounded operator on a Hilbert space H. The norm of T is defined as follows $\|T\| = \sup\{\ \|Tf\|\ \mid f \in H, \|f\| = 1\}$.

Calculating the exact value of the norm of a composition operator can be difficult. Cowen [3] gave an exact value of the norm of C_φ (see *theorem (6)*).

Recall that a holomorphic self-map ψ is called an inner function if $|\psi(z)| = 1$ *a.e. on* ∂U. So one can get the following consequence.

Proposition 2.1: φ is an inner function if and only if $|s| + |t| = 1$.

Proof:

Assume that φ is an inner function, then $|\varphi(z)| = 1$ *a.e. on* ∂U.

Hence |sz+t|=1. So, $1 = |sz+t| \le |s||z| + |t| = |s| + |t| \le 1$. This implies that $|s| + |t| = 1$. On the other hand the converse is clear■

The norm of composition operator induced by inner function is computed by Nordgren [6].

Theorem 2.2: ψ is an inner function if and only if $\|C_\psi\|^2 = \dfrac{1+|\psi(0)|}{1-|\psi(0)|}$.

Corollary 2.3: $\|C_\varphi\|^2 = \dfrac{1+|t|}{1-|t|}$ if and only if $|s| + |t| = 1$.

Proof:

By (2.1) we have φ is an inner function if and only if $|s| + |t| = 1$. Therefore by (2.2) that $\|C_\varphi\|^2 = \dfrac{1+|\varphi(0)|}{1-|\varphi(0)|}$ if and only if $|s| + |t| = 1$. ■

Corollary 2.4: Let $|s| + |t| = 1$, then

(1) $\|C_\varphi\| = 1$ if and only if t=0.

(2) $\|C_\varphi\| > 1$ if and only if $t \neq 0$.

Proof:

(1) Follows immediately from (2.3).

(2) Follows from (2.3) $\|C_\varphi\|^2 = \dfrac{1+|t|}{1-|t|}$. Since $1 - |t| < 1 + |t|$, then it clear that $\|C_\varphi\| > 1$■

Recall that the spectrum of an operator T on a Hilbert space H, denoted by σ(T) is the set of all complex numbers λ for which T-λI is not invertible.The spectral radius of T, denoted by r(T) is defined as $r(T) = \sup\{\ |\lambda| : \lambda \in \sigma(T)\}$.

The right spectrum of an operator T on a Hilbert space H, denoted by $\sigma_r(T)$ is the set of all complex numbers λ for which T-λI is not right invertible. The left spectrum of an operator T on a Hilbert space H, denoted by $\sigma_l(T)$ is the set of all complex numbers λ for which T-λI is not left invertible [7].

Cowen [8] gave an easy estimate of the spectral radius of composition operator.

Theorem 2.5: Suppose that ψ is a holomorphic self-map of U and suppose that ψ has a fixed point c, then $r(C_\psi) = 1$ where $|c| < 1$ and $r(C_\psi) = |\psi'(c)|^{-1/2}$ where $|c| = 1$.

By (2.5) we can compute the spectral radius of composition operator C_φ by determine the position of fixed points of φ.

Proposition 2.6:

(1) φ has an interior fixed point in U if and only if $|t| < |1 - s|$ where $|s| + |t| \le 1$.

(2) φ has a boundary fixed point in ∂U if and only if $|t| = |1-s|$ where $|s|+|t|=1$.

(3) φ has no fixed point outside $\overline{U}$ in ¢ (where $\overline{U}$ is the closure of U).

Proof:

Assume that φ has a fixed point c. Thus φ(c)=c. This implies that sc+t=c, then $c = t/(1-s)$. It follows that

(1) φ has an interior fixed point in U if and only if |c|<1. So, φ has an interior fixed point in U if and only if $|t| < |1-s|$ where $|s|+|t| \le 1$.

(2) φ has a boundary fixed point on ∂U if and only if |c|=1. Thus φ has a boundary fixed point on ∂U if and only if $|t| = |1-s|$. Hence it remains to prove that $|s|+|t|=1$. Note that $|t| = |1-s| \ge |1-|s|| \ge 1-|s|$, so $|s|+|t| \ge 1$. But $|s|+|t| \le 1$, then $|s|+|t|=1$.

(3) If φ has fixed point c outside $\overline{U}$. Hence |c|>1, and then $|t| > |1-s|$. So $|t| > |1-s| \ge |1-|s|| \ge 1-|s|$. Thus $|s|+|t|>1$. But $|s|+|t| \le 1$, then we get a contradiction. Hence φ has no a fixed point outside $\overline{U}$ ■

Now we are ready to compute the spectral radius of composition operator C_φ on Hardy space H^2.

Corollary 2.7:

(1) $r(C_\varphi)=1$ if and only if $|s|+|t| \le 1$ where $|t| < |1-s|$.

(2) $r(C_\varphi)=|s|^{-1/2}$ if and only if $|t| = |1-s|$ where $|s|+|t|=1$.

Proof:

(1) The proof follows directly by (2.6)(1) and (2.5).

(2) By (2.6)(2) we have φ has a boundary fixed point on ∂U if and only if $|t| = |1-s|$ where $|s|+|t|=1$. Hence by (2.5) $r(C_\varphi) = |\varphi'(c)|^{-1/2}$ where c is a boundary fixed point of φ on ∂U. Thus it easily compute that $r(C_\varphi)=|s|^{-1/2}$ ■

Now we can give the necessary and sufficient condition for compactness of C_φ. Recall that a holomorphic self-map of U is called linear-fractional if $\psi(z) = \dfrac{az+b}{cz+d}$ where a,b,c and d are complex numbers. Shapiro [1] studied the compactness of the linear-fractional self-map of U.

Theorem 2.8: If ψ linear-fractional self-map of U. Then C_ψ is not compact operator if and only if ψ maps a point of unit circle ∂U into a point of ∂U.

Corollary 2.9: C_φ is a not compact operator if and only if $|s|+|t|=1$.

Proof:

If $|s|+|t|=1$, then by (2.1) we get φ is an inner function. Clearly φ is a linear-fractional self-map of U, thus by the definition of inner function and (2.8) C_φ is not compact.

Conversely, if C_φ is not compact. To show that $|s|+|t|=1$, we assume that $|s|+|t|<1$. Since C_φ is not compact, then by (2.8) there exists $z_0 \in \partial U$ such that $|\varphi(z_0)|=1$. But $|\varphi(z_0)| = |sz_0+t| \le |s||z_0|+|t| = |s|+|t| < 1$. Hence $|\varphi(z_0)|<1$, which a contradiction ■

Definition 2.10 [8]: Let B(H) be a Banach space of all bounded operators on a Hilbert space H, and K(H) be the ideal of all compact operators on H, then the Calkin algebra is the quotient space B(H)/K(H). If T∈B(H), then the canonical projection $\Pi(T)$ onto B(H)/K(H) will be denoted by $\widetilde{T}$. The essential norm of T is $\|T\|_e = \|\widetilde{T}\|$. The essential spectral radius of T is $r_e(T) = r(\widetilde{T})$. The essential spectrum of T is $\sigma_e(T) = \sigma(\widetilde{T})$. The left essential spectrum of T is $\sigma_{le}(T) = \sigma_l(\widetilde{T})$, or equivalently $\sigma_{le}(T)$={μ | range(T-μ) is not closed, or dim ker(T-μ)=∞}. The right essential spectrum of T is $\sigma_{re}(T) = \sigma_r(\widetilde{T})$, or equivalently $\sigma_{re}(T)$={μ | range(T-μ) is not closed, or dim ker($T^*-\overline{\mu}$)=∞}.

One can show that if T is an essential normal operator on a Hilbert space H if $T^*T - TT^*=0$ in Calkin algebra. It follows easily from the definition of the essentially normal operator that every normal operator and compact operator is essentially normal.

Shapiro [7] proved a holomorphic self-map ψ is inner if and only if $\|C_\psi\|_e = \|C_\psi\|$ and $r_e(C_\psi) = r(C_\psi)$.

Now we end this paper by the following main result.

Theorem 2.11: C_φ is an essentially normal operator on H^2 if and only if $|s|+|t|<1$ or |s|=1.

Proof:

Suppose that C_φ is essentially normal, then by [4] $\sigma_{re}(T) = \sigma_e(T)$.

If we assume that $|s|+|t|=1$, then we must show that |s|=1. Assume the converse, hence |s|<1, then it is clear that t≠0. It follows by (2.4)(2) that $\|C_\varphi\|>1$. But $|s|+|t|=1$, this implies by (2.1) that φ is inner. Therefore $r_e(C_\varphi) = r(C_\varphi)$ and $\|C_\varphi\|_e = \|C_\varphi\| > 1$. Now by (2.6) we have the following cases:

Case 1: φ has an interior fixed point in U. Thus by (2.7)(1) we get that $r(C_\varphi)=1$. But C_φ is essentially normal, then by [6] $r_e(T) = \|T\|_e$.Hence $1 < r_e(T) = \|T\|_e = r(C_\varphi) = 1$, which is a contradiction.

Case 1: φ has a boundary fixed point on ∂U. Thus by (2.7)(2) we get that $r(C_\varphi)=|s|^{-1/2}$. Since |s|<1, then $r(C_\varphi)>1$. But $r_e(C_\varphi) = r(C_\varphi)$, then $r_e(C_\varphi)=|s|^{-1/2}>1$. It was proven in [9] that if C_φ is essentially normal, then for any μ in the interior of $\sigma_e(C_\varphi)$, $C_\varphi - \mu$ is onto. Hence $C_\varphi - \mu$ has a closed range, and then $range(C_\varphi - \mu) = H^2$. But it is well known [4] that $\overline{range(C_\varphi - \mu)} + \ker(C_\varphi^* - \overline{\mu}) = H^2$. Thus it easily seen that $\dim\ker(C_\varphi^* - \overline{\mu}) = 0$. Hence $\mu \notin \sigma_{re}(T)$, this implies that $\sigma_{re}(T) \neq \sigma_e(T)$, which a contradiction. Thus from previous cases we have |s|=1.

Conversely, if $|s|+|t|<1$, then by (2.9) we have C_φ is a compact operator. Hence it is essentially normal. Moreover if |s|=1, then by (1.3) we have C_φ is a unitary operator on H^2 . This implies that C_φ is a normal operator. Hence it is essentially normal■

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المؤثر التركيبي على فضاء هاردي المحتث من الدالة **φ(z)=sz+t** عندما $|s| \le 1, |t| < 1$ و $|s|+|t| \le 1$

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الخلاصة:

في هذا البحث درسنا المؤثر التركيبي على فضاء هاردي المحتث من الدالة φ(z)=sz+t عندما $|s| \le 1, \; |t| < 1$ و $|s|+|t| \le 1$. لقد أعطينا وصفا جيدا للمؤثرات الاعتيادية مع بعض الأنواع الأخرى من المؤثرات المرتبطة بها. بالإضافة إلى ذلك فقد درسنا المؤثر الاعتيادي الجوهري وأعطينا بعض النتائج الأخرى التي هي حسب علمنا جديدة.