Quasi-posinormal operators

Sadiq Naji Nassir*

Received 17, March, 2009
Acceptance 26, July, 2009

Abstract:

In this paper, we introduce a class of operators on a Hilbert space namely quasi-posinormal operators that contain properly the classes of normal operator, hyponormal operators, M–hyponormal operators, dominant operators and posinormal operators . We study some basic properties of these operators .Also we are looking at the relationship between invertibility operator and quasi-posinormal operator .

Key words: posinormal operators , Hyponormal operators ,M- hyponormal operators, dominant operators.

Introduction:

Let B(H) denote the set of all bounded linear operators on a Hilbert space H., an operator T is said to be posinormal operator if there exists a positive operator P $\in$B(H), such that $TT^* = T^*PT$ .Also, T is posinormal operator if and only if $Range(T) \subseteq Range(T^*)$, [1,2].An operator T is called hyponormal operator if $T^*T - TT^* \geq 0$,or equivalently $\|T^*x\| \leq \|Tx\|$ for all x in H [3] ,and T is called dominant operator if for each $\lambda \in$ ¢ there exists a number $M_\lambda > 0$ such that $\|(T-\lambda)^* x\| \leq M_\lambda \|(T-\lambda)x\|$ for all x$\in$ H. Furthermore , if the set of constants M_λ are bounded by a positive number M then T is called M-hyponormal operator [4,5,6,p480]. Let $\sigma(T)$, $\sigma_p(T), \sigma_{ap}(T)$ and $r(T) = \sup\{|\lambda|, \lambda \in \sigma(T)\}$ denote the spectrum, the point spectrum , the approximate point spectrum of T and the spectral radius of T, [6,p196,502] . An operator is said to be normaloid if $\|T\| = r(T)$,[7,8 ,p267].In this paper , we give some types of operators namely quasi-posinormal operators .

1- Some basic properties of quasi- posinormal operator .

We start this section by giving the definition of quasi-posinormal operator ,and we give some basic properties of these operators

Definition 1.1

Let T$\in$B(H). We call T is a quasi-posinormal operator if $Range(T^2) \subseteq Range(T^*)$.

Example 1.2

Let H= ℓ_2(¢) ={x:x=(x$_1$,x$_2$,x$_3$,…,.x$_n$,..): $\sum_{i=1}^{\infty} |x_i|^2 < \infty$} , the Unilateral shift operator on H is defined by U(x$_1$, x$_2$, x$_3$,) =(0,x$_1$, x$_2$, x$_3$,…).It is known that U^*(x$_1$, x$_2$, x$_3$,…) =(x$_2$,x$_3$, x$_4$,……) and

*Department of Mathematics-College of Science-University of Baghdad

U^2 (x$_1$, x$_2$, x$_3$,…) =U(0,x$_1$, x$_2$, x$_3$,…) =(0,0,x$_1$, x$_2$, x$_3$,…).

Now let y∈Range(U^2) then y=(0,0,x$_1$, x$_2$, x$_3$,…) for some x in H .If we assume x=(0,0,0,x$_1$, x$_2$, x$_3$,…) then U^*(x)= (0,0,x$_1$, x$_2$, x$_3$,…) = y, and y ∈Range(U *) . Hence U is quasi-posinormal operator .

Now we give an operator that is not quasi-posinormal operator.

Example 1.3

Let H= ℓ$_2$(¢) ={x:x=(x$_1$,x$_2$,x$_3$..,x$_n$ ..): $\sum_{i=l}^{\infty}|x_i|^2 < \infty$} , the Bilateral shift operator on H is defined by B(x$_1$, x$_2$, x$_3$,…) =(x$_2$,x$_3$, x$_4$,……). It is known that B^*(x$_1$, x$_2$, x$_3$,…) =(0,x$_1$, x$_2$, x$_3$,…). Now let y= (1,0,0,0,…,0,…) then y∈Range(B^2) and $B^*(x) \neq$ y for all x in H .Hence y∉ $Range(B^*)$ and therefore B is not quasi-posinormal operator.

The above example also shows that if T is quasi-posinormal operator then T^* is not quasi -posinormal operator .

In [9]. Douglas proved the following theorem

Theorem 1.4 [9]

For A,B ∈ B(H) the following statements are equivalent :

1-Range (A)⊆ Range (B)

2-AA*≤ λ^2 BB* for some λ≥ 0

3-there exists a T∈ B(H) such that A=BT.

Moreover if one of 1, 2, and 3 holds then there is a unique operator T such that

a-$\|T\|^2$ = inf { μ : $\mu \geq 0$ and AA*≤ μ BB*}

b-KerA = KerT; and

c-Rnage(T) ⊆ $\overline{Range(B^*)}$.

If we put A= T^2 and B= T* we get a special case from Douglas theorem

Which gives a characterization of totally quasi-posinormal operator.

Theorem 1.5

Let T∈B(H), the following statement are equivalent ;

1- $Range(T^2) \subseteq Range(T^*)$, i.e. T is quasi-posinormal operator.

2- $T^2 T^{*2} \leq \lambda^2 T^* T$ for some λ≥0 ; and

3- there exists an operator C_T ∈B(H) , such that $T^2 = T^* C_T$

Moreover if 1,2,and 3 hold then there is a unique operator C_T ∈B(H) such that

a- $\|C_T\|^2 = \inf\{\mu, T^2 T^{*2} \leq \mu T^* T\}$.

b-KerT^2 =ker C_T ; and

c- $Range(C_T) \subseteq \overline{Range(T)}$.

Let [T]={AT : A∈B(H) } the left ideals in B(H) generated by T. We have the following corollary.

Corollary 1.6

T is quasi-posinormal operator if and only if T^{*2} ∈[T].

Proof :

Let T be a quasi-posinormal operator then $T^2 = T^*$C for some bounded operator C∈B(H) and $T^{*2} = C^* T$ implies T^{*2} ∈[T].Conversely, if T^{*2} ∈ [T] then T^{*2} =KT for some K∈B(H), and hence $T^2 = T^* K^*$ so T is quasi-posinormal operator .

Proposition 1.7

Let T∈B(H) ,then T is quasi-posinormal operator if and only if for each x in H , there exists a constant M ≥ 0 such that $\|T^{*2}x\| \leq M\,\|Tx\|$.

Proof :

Let T be a quasi-posinormal then

$$\|T^{*2}x\|^2 = \langle T^* T^* x, T^* T^* x\rangle = \langle T^2 T^{*2} x, x\rangle$$

$\leq M <T^{*}Tx, x> = M <Tx, Tx> = M \|Tx\|^2$

.for some M$\geq$ 0.

Conversely ,let $\|T^{*2}x\| \leq M \|Tx\|$

$<T^2T^{*2}x, x> = <T^{*2}x, T^{*2}x> = \|T^{*2}x\|^2$

$\leq M^2 \|Tx\|^2 = M^2 <Tx, Tx> = M^2 <T^{*}Tx, x>$

,this implies for each x in H , then $T^2T^{*2} \leq M^2T^{*}T$, hence T is quasi-posinormal operator .

Proposition 1.8

Let T$\in$B(H),if T is posinormal operator then T is quasi-posinormal.

Proof:

Since

$Range(T^2) \subseteq Range(T) \subseteq Range(T^{*})$

then T is quasi-posinormal .

Corollary 1.9

Every Dominant operator in particular every M-hyponormal operator, hyponormal operator , normal operator are quasi-posinormal operators.

The converse of the above Proposition is not true, see the following example .

Example 1.10

Let H=ℓ_2(¢)={x:x=($x_1,x_2,x_3,\ldots x_n$,..): $\sum_{i=l}^{\infty}|x_i|^2 < \infty$} ,we define T by

T(x_1, x_2, x_3,…) =(x_2,0,0,0,..).

It is easy to check that

T^{*}(x_1, x_2, x_3,…) =(0,x_1,0,0,0,0…) but T^2(x_1, x_2, x_3,…) =T(x_2,0,0,0,……) =(0,0,0,0,0,0…)and $Range(T^2) \subseteq Range(T^{*})$,hence T is quasi-posinormal operator.

Easily we see that $Range(T) \not\subset Range(T^{*})$.

Therefore T is not posinormal operator.

2- Invertibility, translates and quasi-posinormal operator

In this section we are looking at the relationship between invertibilty operators and quasi -posinormal operator .A quasi- posinormal operator need not be an invertible operator (see example 1.2) ,we start this section by the following theorem

Theorem 2.1

Let T$\in$B(H),be an invertible operator then

1- T is quasi-posinormal operator .

2- T^{-1} is quasi-posinormal operator.

Proof :

1- $\|T^{*2}x\| \leq \|T^{*2}\|\|x\| \leq \|T^{*2}\|\|T^{-1}\|\|Tx\|$

for all x in H ,we take M=$\|T^{*2}\|\|T^{-1}\|$,

hence T is quasi-posinormal operator

2-

$\|(T^{-1})^{*2}x\| \leq \|(T^{-1})^{*2}\|\|x\| \leq \|(T^{-1})^{*2}\|\|T\|\|T^{-1}x\|$

for all x in H, we take M=$\|(T^{-1})^{*2}\|\|T\|$,

hence T^{-1} is quasi-posinormal operator.

Corollary 2.2

Let T$\in$B(H), and $\lambda \notin \sigma(T)$ then T-λI is quasi-posinormal operator.

Before we state the next theorem we need the following lemma which appeared in [10].

Lemma 2.3

Let $\{a_n\}$ be a sequence of positive numbers , which satisfy the relation $a_1^2 \leq a_2$ and $a_n^2 \leq a_{n-1}a_{n+1}$ for n=2,3,… then $a_1^n \leq a_n$ for n=1,2,3,4,5,… .

Theorem 2.4

Let T be an invertible operator and $\|T^{-1}\| \leq 1$ then

1- $\|T^2x\|^{n+1} \leq M^{n(n+1)/2}\|T^{n+2}x\|$ for $\|x\| = 1$ and n=1,2,…., there exists a constant M>0 such that

2- if T^{n+1}x =0 then T^2x =0 for all x in H .

Proof :

1-Let k=n+1. We want to show that $\|T^2x\|^k \le M^{k(k-1)/2}\|T^{k+1}x\|$

Let $a_1 = \|T^2x\|$,and $a_k = M^{k(k-1)/2}\|T^{k+1}x\|$ k=2,3,…

Since

$\|T^2x\|^2 = <T^2x, T^2x> = <x, T^{*2}T^2x>$

$\le \|T^{*2}T^2x\|\|x\| \le M\|T^3x\|$ then $a_1^2 \le a_2$.

Now

$a_k^2 = M^{k(k-1)}\|T^{k+1}x\|^2 = M^{k(k-1)} <T^{k+1}x, T^{k+1}x>$

$= M^{k(k-1)} <T^{*2}T^{k+1}x, T^{k-1}x>$

$\le M^{k(k-1)}\|T^{*2}T^{k+1}x\|\|T^{k-1}x\|$

$\le M^{k(k-1)}M\|T^{k+2}x\|\|T^{k-1}x\|$

$\le M^{k^2-k+1}\|T^{-1}\|\|T^{k+2}x\|\|T^kx\|$

$\le a_{k+1}a_{k-1}$ then by Lemma 2.3

$a_1^k \le a_k$ and $\|T^2x\|^k \le M^{k(k-1)/2}\|T^{k+1}x\|$

2- $\|T^2x\|^n = \|x\|^n \left\|T^2\frac{x}{\|x\|}\right\|^n$

$\le \|x\|^n M^{n(n-1)/2}\left\|T^{n+1}\frac{x}{\|x\|}\right\| \le 0$, hence T^2x=0 for all x in H .

Theorem 2.5

Let T be a quasi-posinormal operator and then

1-λT is a quasi-posinormal operator for λ ∈¢

2- the translate T+λI need not be a quasi-posinormal operator

Proof :

1- $\|(\lambda T)^{*2}x\| = |\lambda|^2\|T^{*2}x\| \le M|\lambda|^2\|Tx\| \le M|\lambda|\|\lambda Tx\|$ for all x in H .

2- consider the case T=B-5I (where B is the operator defined in example1.3). Since $5 \notin \sigma(B)$, then T is an invertible operator by theorem 2.1 T is quasi-posinormal operator. But T+5I=B is not quasi posinormal operator

Definition 2.6

Let T∈B(H), the quasi-spectrum of T , denoted Q(T) is the set {λ: T -λI is not quasi-posinormal operator }

Proposition 2.7

let T∈B(H),be a quasi- posinormal operator then

1- Q(T) ⊆ $\sigma(T)$.

2- If λ∈ $\sigma_p(T)$ and $(T-\lambda)^{*2}x \ne 0$ for all x≠0 ∈ H then λ ∈Q(T).

3- If λ∈ $\sigma_{ap}(T)$ and $(T-\lambda)^{*2}x \ne 0$ for all x≠0 ∈ H then λ ∈Q(T).

Proof :

(1) By corollary 2.2 makes that Q(T) is a subset of $\sigma(T)$.

(2) Suppose λ∉Q(T) then T-λI is quasi-posinormal operator and $\|(T-\lambda I)^{*2}x\| \le M\|(T-\lambda)x\|$ for all x in H. Now λ∈ $\sigma_{ap}(T)$ then there exists x≠0such that $(T-\lambda)x = 0$ and $(T-\lambda)^{*2}x = 0$ contradiction , hence λ ∈Q(T)

(3) by the same way we can prove it .

Remark 2.8

The sum and the product of two quasi-posinormal operators need not be quasi-posinormal operator. We can see that by the following examples

1- Let H= ℓ_2(¢), Let T_1 =U the unilateral shift operator and T_2 is the operator defined on H by T_2 ($x_1,x_2,x_3,\ldots$)= $(0,0,0,-x_3,-x_4,-x_5,\ldots\ldots.)$ it is clear that T_2 is hyponormal operator hence T_2 quasi-posinormal operator .Now (T_1+T_2) ($x_1, x_2, x_3,\ldots$)= T_1($x_1, x_2, x_3,\ldots$)+T_2($x_1, x_2, x_3,\ldots$)= (0, x_1, x_2,0,0,0,0,0….),and $(T_1+T_2)^*$(

$x_1, x_2, x_3, \ldots) = (x_2, x_3, 0,0,0,\ldots..)$.If we take $x = (0,0, x_3, x_4, x_5 \ldots)$ such that $x_3 \neq 0$,then $\|(T_1+T_2)x\|^2 = \|0\|^2$ which implies $\|(T_1+T_2)x\| = 0$, but $\|(T_1+T_2)^{*2}x\|^2 = \|(T_1+T_2)^*(0, x_3, 0,0,0,)\|^2 =$ $\|(x_3, 0,0,0,0,\ldots)\|^2 = |x_3|^2$ then for all M >0 that $\|(T_1+T_2)^{*2}x\| \geq M\,\|(T_1+T_2)x\|$ and (T_1+T_2) is not quasi-posinormal operator.

2- Let H= ℓ_2(¢), T_1 =U the unilateral shift operator and T_2 be the operator defined on H by $T_2(x_1, x_2, x_3,\ldots) = (x_1, x_2, 0,0,0,\ldots)$ then T_2 is self-adjoint operator hence is quasi-posinormal operator but $T_1T_2(x_1, x_2, x_3,\ldots) = T_1(x_1, x_2, 0,0,0,\ldots) = (0, x_1, x_2, 0,0,0,0,\ldots)$ and T_1T_2 is not quasi-posinormal operator by above example (1) .

Remark 2.9

Let T∈B(H) be a quasi-posinormal operator then T is not normaloid operator. i.e. the spectral radius of T is not necessarily equal to $\|T\|$,for example let $\{e_n\}_{n=1}^{\infty}$ be an orthogonal basis of a Hilbert space H and T be the a weighted shift defined by $Te_1 = e_2$, $Te_2 = 2e_3$ and $Te_i = e_{i+1}$ for i≥3 ,in [11]. Wadhwa.B.L proved that T is M-hyponormal operator, and not normaloid operator but by Corollary 1.9 T is quasi posinormal operator and not normaloid operator.

References:

1- Amelia Bucur 2003." Posinormality versus hyponormality for cesaro oerators" General math. Vol11, no1-2,pp33-46.

2-B.P. Duggal and C.Kubrusly . 2006." Weyls theorem for posinormal operators ",J Korean Math.Soc.No.3,pp.529-541.

3- S.Mecheri 2005 " ON normality of operators " Revista Colomiona de Math. Vol.39, pp.1-9.

4-Ahmed Bachir 2004 " Generalized Derivation" SUT Journal of Math. Vol.40,No.2 , pp111-116.

5- Stampfli J.G., and Wadhwa B.L..1976. " *An asymmetric Putnam-Fuglede theorem for* d*ominant operators*" . Indiana Univ.math.,J 25.359-365.

6- Balmohan V.Limaye, 2006. "functional analysis" second E ditor , New age international LTD.Delhi

7- Raul.E. Curto and Young Min Han 2003 " Weyls theorem for algebraically paranormal operators" Integral Equations and operator theory vol.47,pp.307-314.

8- Istratesca,V. 1981. "introduction to linear operator theory" Marcel Dekke,ing, New York

9- Douglas, R.G. 1966. "On majorization,Factorization and range Inclusion of Operators on Hilbert Space" , Proc Amer.Soc. 17,pp.413-416.

10- Stampfli, JG .1976. "Alocal spectral Theory for operators V:spectral subspaces for hyponormal operators ,Trans. Amer.Math.Soc.Vol.217:pp.285-298.

11-Wadhwa. B.L. 1974 . "M-hyponormal operators ".Duke, Math. J vol.41:pp.655-660.

المؤثرات الشبه السوية الموجبة

صادق ناجي ناصر*

*قسم الرياضيات –كلية العلوم- جامعة بغداد

الخلاصة :

في هذا البحث سندرس صنفاً من المؤثرات المعرفة على فضاء هلبرت سوف نطلق على عناصره اسم المؤثر شبه السوي الموجب ويضم كلا من صنف المؤثرات السوية, المؤثرات فوق السوية و المؤثرات فوق السوية من النمط M ,المؤثرات المهيمنة والمؤثرات السوية الموجبة و سوف ندرس بعض الصفات الاساسية لهذا الصنف من المؤثرات وكذلك البحث عن العلاقة التي تربط هذا الصنف بالمؤثرات التي لها نظير .