Jordan θ-Centralizers of Prime and Semiprime Rings

*Abdulrahman H. Majeed** *Mushreq I. Meften**

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Abstract:

The purpose of this paper is to prove the following result: Let R be a 2-torsion free ring and T: R→R an additive mapping such that T is left (right) Jordan θ-centralizers on R. Then T is a left (right) θ-centralizer of R, if one of the following conditions hold (i) R is a semiprime ring has a commutator which is not a zero divisor . (ii) R is a non commutative prime ring . (iii) R is a commutative semiprime ring, where θ be surjective endomorphism of R . It is also proved that if T(xoy)=T(x)oθ(y)=θ(x)oT(y) for all x, y ∈ R and θ-centralizers of R coincide under same condition and θ(Z(R)) = Z(R) .

Key words: prime ring, semiprime ring, left (right) centralizer, centralizer, Jordan centralizer, left (right) θ -centralizer, θ-centralizer , Jordan θ-centralizer.

Introduction:

Throughout this paper, R will represent an associative ring with the center Z. R is called prime if aRb = (0) implies a = 0 or b = 0 and semiprime if aRa = (0) implies a = 0. A mapping D: R→R is called derivation if D(xy) = D(x)y+xD(y) holds for all x, y ∈ R. A left (right) centralizer of R is an additive mapping T: R→ R which satisfies T(xy) = T(x)y (T(xy) = xT(y)) for all x, y ∈ R. A centralizer of R is an additive mapping which is both left and right centralizer . If a ∈ R, then $L_a(x)$ = ax is a left centralizer and $R_a(x)$ = xa is a right centralizer.

A mapping D: R→R is called (θ,θ) derivation if D(xy) = D(x)θ(y)+ θ(x)D(y) holds for all x, y ∈ R[1]. A left (right) θ-centralizer of R is an additive mapping T: R→R which satisfies T(xy) = T(x)θ(y) (T(xy) = θ(x)T(y)) for all x, y ∈ R. A θ-centralizer of R is an additive mapping which is both left and right θ-centralizer. If a ∈ R, then $L_a(x)$ = aθ(x) is a left θ-centralizer and $R_a(x)$ = θ(x)a is a right θ-centralizer[2][3].

A mapping D: R→R is called Jordan (θ,θ) derivation if $D(x^2)$ = D(x)θ(x)+ θ(x)D(x) holds for all x ∈ R[7]. A Jordan left (right) θ-centralizer of R is an additive mapping T: R→R which satisfies $T(x^2)$ = T(x)θ(x) ($T(x^2)$ = θ(x)T(x)) for all x ∈ R. A Jordan θ-centralizer of R is an additive mapping which is Jordan both left and right θ-centralizer[2,3] .

If R is a ring with involution *, then every additive mapping E: R→R which satisfies $E(x^2)$ = E(x)x* + xE(x) for all x ∈ R is called Jordan *-derivation. These mappings are closely connected with a question of representability of quadratic forms by bilinear forms. Some algebraic properties of Jordan *-derivations are

*Department of Mathematics, College of Science, Baghdad University, Iraq

considered in [4], where further references can be found. For quadratic forms see [5].

Brešar and Zalar obtained a representation of Jordan *-derivations in terms of left and right centralizers on the algebra of compact operators on a Hilbert space. They arrived at a problem whether an additive mapping T which satisfies a weaker condition $T(x^2) = T(x)x$ is automatically a left centralizer. They proved that this is in fact so if R is a prime ring (generally without involution). In [6] Zalar generalize this result on semiprime rings. In [7] A. H. Majeed and H. A. Shaker extended the results of Zalar [6].

An easy computation shows that every centralizer T is satisfies $T(xoy)=T(x)oy= xoT(y)$. B. Zalar in [6] prove that every additive mapping $T: R\rightarrow R$ which satisfies $T(xoy)=T(x)oy= xoT(y)$ of a semiprime ring is a centralizer.

An easy computation shows that every θ-centralizer T satisfies $T(xoy)=T(x)o\theta(y)=\theta(x)oT(y)$.

In the present paper we generalize results of Zalar[6] to θ-centralizer .

1. The first result.

To prove our first result, we need two lemmas which we now state.

Lemma 1.1.[6]

Let R be a semiprime ring. If $a,b \in R$ such that $axb=0$ for all $x \in R$, then $ab=ba=0$.

Lemma 1.2.[6]

Let R be a semiprime ring and $A,B: R\times R\rightarrow R$ biadditive mappings. If $A(x,y)\, w\, B(x,y) = 0$ for all $x, y,w \in R$, then $A(x,y)\, w\, B(u,v) = 0$ for all $x, y, u, v,w \in R$.

Theorem 1.3

Let R be a 2-torsion free ring, Then every Jordan left (right) θ-centralizer is a left (right) θ-centralizer, if one of the following statements hold:-

(i) R is a semiprime ring has a commutator which is not a zero divisor.

(ii) R is a non commutative prime ring .

(iii) R is a commutative semiprime ring .

Where θ be surjective endomorphism of R

Proof:

$T(x^2) = T(x)\,\theta(x)$ for all $x \in R$ …(1)

If we replace x by x + y, we get

$T(xy + yx) = T(x)\,\theta(y) + T(y)\,\theta(x)$...(2)

By replacing y with xy + yx and using (2), we arrive at

$T(x(xy+yx)+(xy+yx)x)=T(x)\theta(xy)+2T(x)\theta(yx) +T(y)\theta(x2)$ ...(3)

But this can also be calculated in a different way.

$T(x^2y + yx^2) + 2T(xyx) = T(x)\theta(xy) + T(y)\theta(x^2) + 2T(xyx)$ ...(4)

Comparing (3) and (4), we obtain

$T(xyx) = T(x)\theta(yx)$ for all $x,y \in R$ …(5)

If we linearize (5), we get

$T(xyz + zyx) = T(x)\,\theta(yz) + T(z)\,\theta(yx)$ ...(6)

Now we shall compute $j = T(xyzyx + yxzxy)$ for all $x,y,z \in R$ in two different ways. Using (5), we have

$j = T(x)\,\theta(yzyx) + T(y)\,\theta(xzxy)$…(7)

Using (6), we have

j = T(xy) θ(zyx) + T(yx) θ(zxy)...(8)

Comparing (7) and (8) and introducing a biadditive mapping B(x,y) = T(xy) - T(x)θ(y), we arrive at

B(x,y) θ(zyx) + B(y, x) θ(zxy) = 0

for all x,y,z ∈ R…(9)

Equality (2) can be rewritten in this notation as B(x,y) = -B(y,x) for all x,y ∈ R. Using this fact and equality (9), we obtain

B(x,y) θ(z) [θ(x), θ(y)] = 0

for all x,y,z ∈ R…(10)

Using Lemma 1.2, we have

B(x, y) θ(z) [θ(u), θ(v)] = 0

for all x,y,z,u,v∈R …(11)

Using Lemma 1.1, we have

B(x, y)[θ(u), θ(v)] = 0 for all x,y,,u,v ∈ R …(12)

If R has a commutator which is not a zero divisor

Using (12) and θ is onto, we have

B(x,y) = 0 for all x,y ∈ R

If R is a non commutative prime ring

Using (11) and θ is onto, we have

B(x,y) = 0 for all x,y ∈ R

If R is a commutative semiprime ring

Now we shall compute j = T(xyzyx) in two different ways. Using (5) we have

j = T(x) θ(yzyx)…(13)

j = T(xy) θ(zyx)…(14)

Comparing (13) and (14) , we arrive at

B(x, y) θ(z) θ(yx) = 0 for all x,y,z ∈ R … (15)

Let Ψ(x,y)=θ(x)θ(y), it's clear that Ψ is a biadditive mapping, therefore

B(x, y) θ(z) Ψ(y,x) = 0 for all x,y,z ∈ R

Using Lemma 1.2, we have

B(x, y) θ(z) Ψ(u,v) = 0 for all x,y,z,u,v ∈ R

Implies that

B(x, y) θ(z) θ(uv) = 0for all x,y,z,u,v ∈ R ….(16)

By replacing θ(v) with B(x, y) θ(z), θ is onto , and R is a semiprime ring, we have

B(x,y) = 0 for all x,y ∈ R

If T(x^2)= θ(x)T(x), we obtain the assertion of the theorem with similar approach as above, the proof is complete.□

Corollary 1.4

Let R be a 2-torsion free prime ring .Then every Jordan left (right) θ-centralizer is a left (right) θ-centralizer, where θ be surjective endomorphism of R .

2. The second result.

We again divide the proof in few lemmas.

Lemma 2.1.

Let R be a semiprime ring, D a θ-derivation of R and a ∈ R some fixed element. Where θ be surjective endomorphism of R

(i) $D(x)D(y) = 0$ for all $x, y \in R$ implies $D = 0$.

(ii) $a\theta(x) - \theta(x)a \in Z$ for all $x \in R$ implies $a \in Z(R)$.

Proof :

(i) $D(x)\,\theta(y)D(x) = D(x)D(yx) - D(x)D(y)\,\theta(x) = 0$ for all $x,y \in R$

But θ is onto, and R is a semiprime ring, we have D=0

(ii) Define $D(x) = a\theta(x) - \theta(x)a$. It is easy to see that D is a (θ,θ)-derivation. Since $D(x)\in Z(R)$ for all $x\in R$, we have $D(y)\theta(x)= \theta(x)D(y)$ and also $D(yz)\theta(x) = \theta(x)D(yz)$.

Hence

$$D(y)\theta(zx) + \theta(y)D(z)\theta(x) = \theta(x)D(y)\theta(z) + \theta(xy)D(z)$$

$$D(y)[\theta(z),\theta(x)] = D(z)[\theta(x),\theta(y)]$$

Since θ is surjective take $a=\theta(z)$.

Obviously $D(z) = 0$, so we obtain

$$0 = D(y)[a,\theta(x)] = D(y)D(x) \text{ for all } x,y \in R$$

From (i) we get $D = 0$ and hence $a \in Z(R)$.□

Lemma 2.2.

Let R be a semiprime ring and $a \in R$ some fixed element. If $T(x) = a\theta(x)+\theta(x)a$,and $T(xoy) = T(x)o\theta(y) = \theta(x)oT(y)$ for all $x, y \in R$ then $a\in Z$. Where θ be surjective endomorphism of R

Proof :

$$T(xy + yx) = T(x)\theta(y) + \theta(y)T(x) \quad \text{for all } x,y \in R$$

gives us

$$a\theta(xy) + a\theta(yx) + \theta(xy)a + \theta(yx)a = (a\theta(x) + \theta(x)a)\,\theta(y) + \theta(y)(a\theta(x) + \theta(x)a)$$

Implies that

$$a\theta(yx) + \theta(xy)a - \theta(x)a\theta(y) - \theta(y)a\theta(x) = 0 = (a\theta(y) - \theta(y)a)\theta(x) - \theta(x)(a\theta(y) - \theta(y)a) \quad \text{for all } x,y \in R$$

The second part of Lemma 2.1 now gives us $a \in Z(R)$.□

Lemma 2.3.

Let R be a semiprime ring, and $T: R\to R$ an additive mapping which satisfies $T(xoy) = T(x)o\theta(y) = \theta(x)oT(y)$ for all $x, y \in R$. Then T maps from Z(R) into Z(R). Where θ be surjective endomorphism of R

Proof :

Take any $c \in Z$ and denote $a = T(c)$.

$$2T(cx) = T(cx + xc) = T(c)\theta(x) + \theta(x)T(c) = a\theta(x) + \theta(x)a$$

A straightforward verification shows that $S(x) = 2T(cx)$ is satisfies $S(xoy) = S(x)o\theta(y) = \theta(x)oS(y)$ for all $x, y \in R$

By Lemma 2.2, we have $T(c) \in Z(R)$.□

Theorem 2.4.

Let R be a 2-torsion free ring and $T: R\to R$ an additive mapping which satisfies $T(xoy) = T(x)o\theta(y) = \theta(x)oT(y)$ for all $x, y \in R$. Then T is a θ-centralizer of R, if one of the following statements hold :-

(i) R is a semiprime ring has a commutator which is not a zero divisor .

(ii) R is a non commutative prime ring .

(iii) R is a commutative semiprime ring .

Where θ be surjective endomorphism of R , and θ(Z(R)) = Z(R)

Proof :

T(xy + yx) = T(x)θ(y) + θ(y)T(x) = θ(x)T(y) + T(y)θ(x) for all x,y ∈ R

If we replace y by xy + yx, we get

T(x)θ(xy + yx) + θ(xy + yx)T(x) = T(xy + yx)θ(x) + θ(x)T(xy + yx) = (T(x)θ(y) + θ(y)T(x))θ(x) + θ(x)(T(x)θ(y) + θ(y)T(x)) for all x,y∈R

Now it follows that [T(x),θ(x)]θ(y) = θ(y)[T(x),θ(x)] holds for all x,y ∈R, but θ is surjective, then we get [T(x), θ(x)] ∈ Z(R)

The next goal is to show that [T(x),θ(x)] = 0 holds. Take any c ∈ Z(R).

2T(cx) = T(cx + xc) = T(c)θ(x) + θ(x)T(c) = 2T(x) θ(c), for all x∈R

Using Lemma 2.3, we get

T(cx) = T(x)θ(c)= T(c)θ(x) for all x ∈ R,

[T(x),θ(x)]θ(c)=T(x)θ(x)θ(c)-θ(x)T(x)θ(c) = T(c)θ(x2) - θ(x)T(c)θ(x) = 0

Since R is semiprime, θ(Z(R))=Z(R) , and [T(x),θ(x)] itself is central element , our goal is achieved.

$2T(x^2)$=T(xx + xx)= (x)θ(x)+θ(x)T(x)

2T(x)θ(x)=2θ(x)T(x) for all x∈R

Theorem 1.3 now concludes the proof.□

Corollary 2.5.

Let R be a 2-torsion free prime ring and T: R→R an additive mapping which satisfies T(xoy) = T(x)oθ(y) = θ(x)oT(y) for all x, y ∈ R. Then T is a θ-centralizer of R, where θ be surjective endomorphism of R , and θ(Z(R)) = Z(R) or θ(Z(R))≠0 .

References:

1. Zaidi, S. M. A. Ashraf, M. and Ali S. 2004. On Jordan ideals and left (θ,θ)-derivation in prime rings, IJMMS. 37: 1957-1964 .
2. Daif, M. N., Tammam El-Sayiad, M. S. and Muthana N. M.2008. An Equation Related to θ-Centralizers in Semiprime Rings, Int. math. forum 3(24): 1165 – 1177.
3. Daif, M. N., Tammam El-Sayiad, M. S. and Muthana N. M.2008. An Identity on θ-Centralizers of Semiprime Rings, Int. math. Forum 3(19): 937 – 944.
4. Majeed, A.H. and Shaker, H.A. 2008. Some Results on Centralizers, Dirasat, Pure Sciences 35(1): 23-26.
5. Šemrl, P. 1991. Quadratic functionals and Jordan *-derivations, Studia Math. 97:157-165.
6. Zalar, B. 1991. On centralizers of semiprime rings **.**Comment. Math. Univ.Carolinae 32(4): 609-614.
7. Brešar, M. and Vukman, J. 1989. On some additive mapping in rings with involution, Aequationes Math. 38 : 178-185.

تمركزات جوردنθ في الحلقات الأولية وشبه الأولية

عبد الرحمن حميد مجيد*   مشرق إبراهيم مفتن*

*جامعة بغداد / كلية العلوم / قسم الرياضيات

الخلاصة:

الهدف من البحث هو برهان النتيجة الآتية : لتكون R حلقة طليقة الالتواء من الدرجة الثانية و T: R→R دالة جمعية بحيث إن T تكون تمركز جوردن θ يساري (يميني) على R ، فإن T تكون تمركز θ يساري (يميني) على R إذا تحقق أحد الشروط الآتية:- (i) R تكون حلقة شبه أولية تحتوي على مبادل غير قاسم للعناصر غير الصفري . (ii) R تكون حلقة أولية غير أبدالية . (iii) R تكون حلقة شبه أولية أبدالية . وإن θ دالة تشاكلية شاملة على R. وأيضا نبرهن إذا كان T(xoy)=T(x)oθ(y)=θ(x)oT(y) لكل x,y في R فإن T تكون تمركز θ تحت الشروط نفسها أعلاه و θ(Z(R))=Z(R) .