ON M- Hollow modules

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Abstract:

Let R be associative ring with identity and M is a non- zero unitary left module over R. M is called M- hollow if every maximal submodule of M is small submodule of M. In this paper we study the properties of this kind of modules.

Key words: Maximal Submodule, Small Submodule, Hallow Module, Projective Module and Lifting Module.

Introduction:

Let R be an associative ring with identity and M be a non- zero unitary left module over R. A submodule N of a module M is called small submodule of M denoted by $N << M$, if $N+L \neq M$ for any proper submodule L of M [1] . M is called hollow module if every proper submodule of M is small submodule [2] . A proper submodule N of a module M is called a maximal submodule in M if whenever K is a submodule of M with $N< K$ then $K= M$.

A module P is called projective R-module if for every epimorphism $ß : B\rightarrow C$ and every homomorphism $\psi :P\rightarrow C$ there is a homomorphism $\lambda :P\rightarrow B$ with $\psi =ß \lambda$ [1] .

Note that if P is local projective module then every maximal submodule in P is a small submodule of P [3] .

In this paper we introduce the notation of M- hollow module that is a module in which every maximal submodule is small submodule. And we discuss some basic properties of this concept

Further more we introduce in section 3 the notation of M-lifting module and study the main properties of this modules.

1- M - hollow module

In this Section we introduce the concept of M-hollow modules and study the basic Properties of this type of modules

Definition 1.1

A non –zero module M is called M-hollow module, if every maximal submodule of M is small submodule of M.

It is clear that every hollow module is M-hollow .

In the following proposition we give some of the basic properties of M-hollow modules

Proposition 1.2

Let M be a finitely generated module, then M is M-hollow iff M is hollow .

Proposition 1.3

Epimorphic image of M-hollow module is M-hollow .

Proof : Let M be M- hollow and let $f : M\rightarrow M^{\backslash}$ an epimorphism with $M^{\backslash}$. Suppose $N^{\backslash}$ be a maximal Submodule of $M^{\backslash}$,Now $f^{-1}(N^{\backslash})$ is maximal

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Submodule of M Since other wise $f^{-1}(N^\backslash) = M$ and hence $f(f^{-1}(N^\backslash)) = M^\backslash$ then $N^\backslash = M^\backslash$ which is contradiction with $N^\backslash < M^\backslash$ thus $f^{-1}(N^\backslash)$ is Proper submodule therefore $f^{-1}(N^\backslash) << M$ and hence $f(f^{-1}(N^\backslash)) << f(M)$ this means $N^\backslash << M^\backslash$.

Corollary 1.4

Let M be a module . If M is M-hollow module then M/N is M-hollow for every proper submodule N of M .

Proof : Let N be a proper submodule of M-hollow module M . Let $\pi : M \rightarrow M/N$ be a natural epimorphism then M/N is M-hollow module .

Proposition 1.5

Let K be a small submodule of a module M. If M /K is M-hollow module then M is M-hollow.

Proof : Suppose M / K is M-hollow with K << M and Let N a maximal submodule of M with $M=N+L$ where $L \leq M$ then $M/K = (N+K)/K$ implies $M/K=((N+K)/K)+(L+K/K)$, N+K/K is proper submodule of M/K to show N+K/K is maximal in M/K. Suppose $N+K/K < J/K \leq M/K$ thus J/K = M/K(since N is maximal in M which is means J=M).Then N+K/K is small in M/K and hence L+K/K =M/K then L+K=M but K << M then L=M .

Let M be a module . If M is M-hollow module then M/N is M-hollow for every proper Submodule N of M .

Proof : Let N be a proper submodule of M-hollow module M . Let $\pi : M \rightarrow M/N$ be a natural epimorphism then M/N is M-hollow module .

Proposition 1.6

Let M be a module then M is M-hollow and finitely generated module If and only if M is cyclic and has unique maximal submodule.

Proof : Let M be finitely generated M-hollow then $M = Rx_1+Rx_2+\text{-------}+Rx_n$, $x_i \in M$, i=1,2,-----,n.

If $M \neq Rx_1$ then Rx_1 is proper submodule of M thus by [1,prop.2.3.11,p.28] $\exists$ N maximal submodule of M s.t $Rx_1 < N$ but M is hollow so N<<M then Rx_1<<M then $M=Rx_2+Rx_3+\text{------}+Rx_n$

So we delete the Summand one by one until we have $M=Rx_i$ for Some i , then M is cyclic module.

Suppose M_1, M_2 are two distinct maximal submodules then $M=M_1+M_2$ but M is M-hollow Thus $M=M_1$ or $M=M_2$ which is contradiction . The Converse is clear .

Lemma 1.7

Let M be M-hollow module which has a maximal submodule K then RadM=K.

Proof : Let L be a nother maximal submodule in M ,then K+L =M, But M is M-hollow then K=M which is contradiction with maximality of K , therefore RadM = K.

An R- module M is called local module if M has a unique maximal submodule N which contains all proper submodule of M [2] .

Proposition 1.8

Let M be a local module then M is M-hollow and cyclic .

Proof: Suppose that M is a local module, then it has a unique maximal N which contain all other submodule of M. Let $w \in M$ with $w \notin N$ then Rw submodule of M . If $M \neq Rw$ then $Rw \leq N$ then $w \in N$ this is a contradiction , then Rw=M and hence M is cyclic , Now if N+K =M for some K< M then $K \leq N$ then $M=N+K \leq N$ then M=N which is a contradiction , then K=M, then N<<M hence M is M-hollow.

Proposition 1.9

Let M be a module, M is M-hollow and $RadM \neq M$ if and only if M is M-hollow and cyclic

Proof : Let M be a M-hollow module with Rad M≠ M , then M has a maximal submodule and by (Lemma 1.7) RadM is the unique maximal of M and M is M-hollow therefor RadM<<M and M\RadM is a simple module thus cyclic , then M\RadM=(m+RadM) for some m∈M (we claim that M=Rm). Let w∈M then w +RadM ∈ M \ RadM hence there is r∈R such that w+RadM= r (m+RadM) = r m+RadM i.e w-r m ∈RadM thus w-r m = y for some y∈Rad M thus w= y + rm ∈ Rm + Rad M , hence M=Rm+RadM . But RadM<<M then M=Rm .

Conversely, since M is cyclic then M is finitely generated and thus Rad M ≠ M.

Proposition 1.10

Let M be a module, M is M-hollow if and only if RadM is a small and maximal in M.

Proof : Let RadM be a small and maximal submodule of in M .To proof M is M-hollow , let L be a maximal submodule of M , therefore M=L+RadM. But RadM is small thus L=M which is contradiction, this imply Rad M is the unique maximal submodule of M & small thus M is M-hollow module. The converse is clear by (1.6)

Definition 1.11 [3]

A pair (p,f) is a projection cover of a module M in case P is a Projective module f:P→M where f is an epimorphism and ker f << P. (we call P itself a projective cover of M)

Proposition 1.12

Let f:P→M be aprojective cover of M , if M is a M-hollow module then P is a M-hollow.

Proof : Let M be a M-hollow module and since f:P→M is epimorphism then P/kerf is isomorphic to M and hence it is M-hollow and kerf << P, thus P is a M-hollow module(by prop. 1.3 & 1.4).

We need the following Lemmas.

Lemma 1.13 [4]

If P is a projective module, then P is a local module if and only if End(P) is a local ring .

Lemma 1.14 [4]

Let M be a module, M is a local module if and only if RadM is a small and maximall in M.

Now we can prove the following proposition .

Proposition 1.15

Let P be a projective module then the following is equivalent:

(1) P has a small and maximal submodule.

(2) Rad P is a small and maximal submodule in P.

(3) P is a local module.

(4) End (P) is a local ring.

(5) P is M-hollow

(6) P is a projective cover for a simple module.

Proof:

(1)→(2)

Let N be a maximal and small submodule in P, then Rad P≤ N. Moreover N<< P then N≤ P and hence N=Rad P.

(2)→(3)

P is a local module (1.14)

(3)→(4)

Since P is a local projective module then End (P) is a local ring

(4)→(5)

Let N be a maximal submodule in P. We must show that N<<P.

Now, since P is a projective module and End (P) is a local ring then P is a local module (1.12) and hence P is a hollow module. Thus N<<P.

(5)→(6)

Since P is a projective module then Rad P ≠ P, i.e., P has a maximal submodule, say N. Now, P/N is a simple module.

Let π:P→P/N be the natural epimorphism. We have ker π =N and

$N \ll P$ by (4) then π is a projective cover for P/N.

(6)→(1)

Let P be a projective cover for a simple module , say M. So there exists an epimorphism $g:P\to M$ such that $\ker g \ll P$. We only have to show that kerg is a maximal submodule in P. By first isomorphism theorem $P/\ker g \cong M$ and M is a simple module then P/kerg is also a simple module and this implies that kerg is a maximal submodule in P.

2- M-lifting module:

Recall that a module M is called lifting if for any submodule N of M , there exist submodules A,B of M such that $M=A\oplus B$, $A\le N$ and $N\cap B \ll B$ [5]

In the following we introduce M-lifting modules and give some properties of this kind of modules.

Definition 2.1

An R-module M is called M-lifting if for any maximal submodule N of M , there exist submodules A , B such that $M=A\oplus B$ with $A \le N$ and $N\cap B \ll B$.

We easily prove the following

Remark 2.2

An R-module M is M-lifting If and only if for any maximal $N\le M$ there exist A, $B\le M$ such that $M = A\oplus B$ with $A\le N$ and $N \cap B \ll M$.

It is clear that lifting module is M-lifting. The following proposition is give characterization of M- lifting modules.

Proposition 2.3

Let M be an R-module the following statements are equivalent.

1- M is M-Lifting

2- Every maximal Submodule N of M , N can be written as $N=A\oplus B$ and A is a direct summand of M and $B \ll M$.

3- For every maximal submodule N of M there exists a direct summand K of M such that $K\le N$ and $N/K \ll M/K$.

<u>Proof:</u> (1)⇒(2) Let N be maximal submodule of M . By condition (1) there exist submodules K , H of M such that $M= K\oplus H$ with $K\le N$ and $N \cap H \ll M$. Since $N=N\cap M$ So, $N=N\cap(K\oplus H) = K\oplus(N\cap H)$. Assume $A=K$, $B=N\cap H$ then $N =A\oplus B$ where A is direct summand of M and $B\ll M$.

(2) ⇒(3) Let N be a maximal submodule of M , By condition (2) , $N=A\oplus B$ with A is a direct summand of M and $B\ll M$.

Assume $K=A$, so K is a direct summand of M .

To prove $N / K \ll M / K$ Let $\Pi: M\to M/K$ be the natural Projection . Since $B\ll M$ then $\Pi(B) \ll M / K$ [1]

We claim that $\Pi(B) =N/K$. To show that let $x \in\Pi(B)$. so $x=\Pi(b)$ for some $b\in B$, Hence $x = b + k \in N / K$ because $B\subseteq N$, thus $\Pi (B) \subseteq N / K$.

Now if $x\in N/ K$, then $x= a+ b+ k$,where $a\in A$, $b\in B$. But $A = K$, hence $x =b + k\in\Pi(B)$, then $N / K \subseteq \Pi (B)$, thus $N/ K= \Pi (B)$ and hence $N/K\ll M/K$

(3)⇒(1) Let N be a maximal submodule of M, by (3) There exists a direct summand K of M such that $K\subseteq N$ and $N/K\ll M/K$. This implies that $M = K\oplus H$ for some submodule H of M . To show $N\cap H \ll M$, since $N=N\cap M$, then $N = N\cap(K\oplus H) = K \oplus (N\cap H)$ (moduler Low) . But $M= K \oplus H$ then $M / K \cong H$. Let g be an isomorphism, $g: M / K \to H$ which is defined by $g(m + K)= h$, if $m =k+h$ where $k\in K$, $h\in H$. We claim that $g(N/K) = N\cap H$, let $x\in N / K$ then

$x = n+k$ where $n \in N$,since $n \in N \subseteq M = K \oplus H$, $n = k_1 + h_1$ where $k_1 \in K$, $h_1 \in H$ and so $g(n+K) = g(k_1 + h_1 + K) = h_1$ but $h_1 = n - k_1$ and $k_1 \in K \subseteq N$ hence $h_1 \in N \cap H$, then $g(N/K) \subseteq N \cap H$. Now , Let $d \in N \cap H$, then $d \in H$ and $g(d+K) = g(0+d+K) = d$ then $d = g(d+K) \in g(N/K)$ then $N \cap H \subseteq g(N/K)$ thus $g(N/K) = N \cap H$, but $N/K < M/K$ therefore $g(N/K) << H$ i.e $N \cap H << H$ hence $N \cap H << M$.

It is known that every hollow module is lifting module [6]. To generalize this statement we give the following proposition.

Proposition 2.4

Every M-hollow module is M-lifting

Proof

Let $N \leq M$ be maximal, if $N \neq M$, then $N << M$ and since $N = \{0\} \oplus N$ thus by definition 3.1, We get the result.

The converse of proposition 2.4 is not true in general as in the following example .

Example

Let M be Z-module, $M = Z_2 \oplus Q$, $N = \{0\} \oplus Q$ is a unique maximal submodule of M, then it clear that M is M-lifting but not M-hollow.

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حول المقاسات المجوفة من النوع M

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الخلاصة:

لتكن R حلقة تجميعية ذات عنصر محايد وليكن M مقاسا احاديا غير صفري ايسر معرف على R . يقال ان المقاس M مجوف من النوع M اذا كان كل مقاس جزئي اعظم من M يكون مقاسا جزئيا صغيرا في M. في هذا البحث سندرس خواص هذا النوع من المقاسات.