On Semi-p-Compact Space[1]

*Nerjis Abdul Jabbar** *Suaad Gedaan**

Received 31, December, 2008
Accepted 28, October, 2009

Abstract:

The purpose of this paper is to introduce a new type of compact spaces, namely semi-p-compact spaces which are stronger than compact spaces; we give properties and characterizations of semi-p-compact spaces.

Key words: semi-p-open set, pre-open set and compact space.

Introduction:

Let (X,τ) be a topological space and let A be a subset of X. We denote the closure of A (the interior of A) by cl A (int A) respectively.

A subset A of (X,τ) is called pre-open set, see [1], [2] and [3], if $A \subseteq \text{int}(\text{cl } A)$. The complement of a pre-open set is called a pre-closed set; see [1], [2] and [3]. The intersection of all pre-closed sets containing A is called the pre-closure of A and is denoted by pre-clA, [2].

A subset A of (X,τ) is called semi-p-open, [1] if there exists a pre-open subset U of X such that $U \subseteq A \subseteq$ pre-clU. The complement of semi-p-open set is called semi p-closed set, see [3].

The family of all semi-p-open subsets of X is denoted by S-P-O(X). The intersection of all semi-p-closed sets containing A is called the semi-p-closure of A and is denoted by semi-p-cl A, see [1,3].

We study and define many concepts in this paper in order to give properties and characterizations of semi-p-compact spaces, like cluster and semi-p-cluster points, compact spaces, nets, filters, T_2 and semi-p-T_2 spaces, regular [redacted] spaces, almost [redacted] and semi-p-irrsolute functions. For more details of these concepts see [4], [2], [5], [6], [7] and [8].

Semi-p-Compact Spaces:

In this section, we define and study the concept of semi-p-compactness.

1 Definition

A family $\tilde{A}$ of semi-p-open subsets of a topological space (X,τ) which covers X is called semi-p-open cover of X.

2 Definition

A topological space (X,τ) is said to be semi-p-compact space if and only if every semi-p-open cover of X has a finite semi-p-open subcover.

Notice that every semi-p-compact space is compact, since every open subset of X is semi-p-open, but the converse is not true in general as the following example shows:

3 Example

Let $X = N \cup \{0\}$

$\tau = \{U \subseteq X \mid U \subseteq N \text{ or } (0 \in U \wedge U^c \text{ is finite})\}$

*Department of Mathematics-Ibn-Al-Haitham College of Education - University of Baghdad
[1]This paper is a part of a M.Sc. thesis by the second author supervised by the first author.

$\mathcal{F} = \{F \subseteq X \mid 0 \in F$ or $(F \subseteq N \wedge F$ is finite$)\}$
$S\text{-}P\text{-}O(X) = P(X) \setminus \{\{0\}\}$
Then (X,τ) is a compact space but not semi-p-compact space.

Semi-p-compactness is a weak hereditary property, as shown in the following proposition.

4 Proposition

A semi-p-closed subset of a semi-p-compact space is a semi-p-compact subspace.

Proof:

Let A be a semi-p-closed subset of a semi-p-compact space (X,τ) and let $\{G_\alpha : G_\alpha$ is semi-p-open subset of X, $\alpha \in \wedge\}$ be a semi-p-open cover of A. Since A^c is a semi-p-open set in X, so $\{G_\alpha : \alpha \in \wedge\} \cup \{A^c\}$ forms a semi-p-open cover of X which is a semi-p-compact space, then there exist finitely many members of index set $\wedge$ say $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = \bigcup_{i=1}^{n} G_{\alpha_i} \cup A^c$. But $A \subseteq X$ and $A \cap A^c = \phi$, therefore $A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$. Thus A is semi-p-compact.

In the following theorem we give a characterization of definition of semi-p-closure of a set.

5 Definition

Let A be a subset of a topological space (X,τ). The semi-p-closure of A (semi-p-cl A) is the intersection of all semi-p-closed subsets of X which contain A.
We shall call x, where $x \in X$, a semi-p-closure point of A if $x \in$ semi-p-cl A.

6 Theorem

Let (X,τ) be a topological space and let A be a subset of X. A point x in X is a semi-p-closure point of A if and only if every semi-p-open nbd (neighborhood) of x intersects A.

Proof:

The "only if"part

Assume that x is a semi-p-closure point of A, then $x \in K = \cap \{F \mid F$ is a semi-p-closed subset of X containing $A\}$. Suppose that there exists a semi-p-open nbd U of x such that $U \cap A = \phi$, therefore $A \subseteq U^c$ where U^c is a semi-p-closed subset of X with $x \notin U^c$, that is, $x \notin K$ which is a contradiction. Hence every semi-p-open nbd of x must intersects A.

The "if" part

Assume that every semi-p-open nbd of x intersects A, and suppose that X is not a semi-p-closure point of A, therefore $x \notin K$, that is there exists a semi-p-closed subset F of X with $A \subseteq F$ such that $x \notin F$, it follows that $x \in F^c$ which is a semi-p-open set in X and $A \cap F^c = \phi$. That implies a contradiction with our assumption. Hence x must be a semi-p-closure point of A. ■

7 Definition

Let (X,τ) be a topological space and let $(f, X, A, \geq)$ be a net in X. A point x_0 in X is called a "semi-p-cluster point of f " if for each $a \in A$ and for each semi-p-open nbd U of x_0 there exists $b \in A$ such that $b \geq a$ and $f(b) \in U$.

8 Definition

Let (X,τ) be a topological space and let $(f, X, A, \geq)$ be a net in X, then f is said to be "semi-p-convergent" to a point x_0 in X if for each semi-p-open nbd. N of x_0 there exists an element $a_0 \in A$ such that $f_a \in N$ for each $a \geq a_0$.

9 Theorem

Let (X,τ) be a topological space and let $(f, X, A, \geq)$ be a net in X. For each $a \in A$, let $M_a = \{ f(x) : x \geq a$ in $A\}$ then a point p of X is a semi-p-cluster point of f if and only if $p \in$ semi-p-cl M_a for each $a \in A$.

Proof:

The "only if"part

Assume that p is a semi-p-cluster point of f and let N be a semi-p-open nbd. of p, then for each $a \in A$, there exists an element $x \geq a$ in A such that $f(x) \in N$.

Hence $M_a \cap N \neq \phi$ for each $a \in A$. Since N is an arbitrary nbd., so by theorem 2.6 $p \in$ semi-p- cl M_a for each $a \in A$.

The "if" part

Assume that $p \in$ semi-p-cl M_a for each $a \in A$ and suppose, if possible, p is not a semi-p-cluster point of f, then there exists a semi-p-open nbd. N of p and an element $a \in A$ such that $f(x) \notin N$ for every $x \geq a$ in A. This implies that $N \cap M_a = \phi$, it follows that $p \notin$ semi-p-cl M_a for this a which is a contradiction. Hence p must be a semi-p-cluster point of the net f. ■

10 Definition

Let (X,τ) be a topological space and let $\mathcal{F}$ be a filter on X. A point x in X is called a "semi-p-cluster point of $\mathcal{F}$" if each semi-p-open nbd. of x intersects every member of $\mathcal{F}$.

Notice that, every semi-p-cluster point of a filter is a cluster point.

11 Theorem

Let (X,τ) be a topological space and let $\mathcal{F}$ be a filter on X. A point p in X is a semi-p-cluster point of $\mathcal{F}$ if and only if $p \in$ semi-p-cl F for each $F \in \mathcal{F}$.

Proof:

The "only if"part

Let p be a semi-p-cluster point of $\mathcal{F}$, then each semi-p-open nbd. of p intersects every member of $\mathcal{F}$, that is, for each semi-p-open nbd. U of p, $U \cap F \neq \phi$ for each $F \in \mathcal{F}$. It follows that, $p \in$ semi-p-cl F for each $F \in \mathcal{F}$, by theorem 6.

The "if" part

Assume that $p \in$ semi-p-cl F for each $F \in \mathcal{F}$, then by theorem 6 every semi-p-open nbd. of p intersects F for each $F \in \mathcal{F}$, that is every semi-p-open nbd. of p intersects every member of $\mathcal{F}$. Hence p is a semi-p-cluster point of $\mathcal{F}$. ■

In the next theorem we give two characterizations of semi-p-compact spaces.

12 Theorem

Let (X,τ) be a topological space then the following statements are equivalent:

1. X is a semi-p-compact space,
2. Every collection of semi-p-closed subsets of X with the FIP (finite intersection property) has a non-empty intersection,
3. Every filter on X has a semi-p-cluster point.

Proof:

$(1 \Rightarrow 2)$ Assume that X is a semi-p-compact space and let $\{F_\alpha : \alpha \in \wedge\}$ be a collection of semi-p-closed subsets of X with FIP. Suppose that $\bigcap_{\alpha \in \wedge} F_\alpha = \phi$, then by De-Morgan Laws $X = \bigcup_{\alpha \in \wedge} F_\alpha^c$ where F_α^c is a semi-p-open set for each $\alpha \in \wedge$. Therefore $\{F_\alpha^c : \alpha \in \wedge\}$ is a semi-p-open cover of X which is a semi-p-compact space, then there exist finitely many members $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = \bigcup_{i=1}^{n} F_{\alpha_i}^c$, it follows by De-Morgan Laws that $\bigcap_{i=1}^{n} F_{\alpha_i} = \phi$ which is a contradiction with our assumption that $\{F_\alpha : \alpha \in \wedge\}$ has a FIP. Hence $\bigcap_{\alpha \in \wedge} F_\alpha \neq \phi$.

$(2 \Rightarrow 3)$

Let F be a filter on X, then F has a FIP. In particular the collection {semi-p-cl F:F∈ F } of semi-p-closed subset of X has the FIP, so by 2 there exists at least one point x∈∩{semi-p-cl F: F∈ F }, that is, x∈semi-p-cl F for each F∈ F . Hence by theorem 11 x is a semi-p-cluster point of F .

(3 ⇒ 1)

Assume that every filter on X has a semi-p-cluster point. To prove X is a semi-p-compact space. Let $\Im$ be a semi-p-open cover of X and suppose, if possible, $\Im$ has no finite subcover. The collection $\wp$ = {X – G : G∈$\Im$} has the FIP. For if there exists a finite subcollection $\{X – G_i \mid 1 \le i \le n\}$ of $\wp$ such that $\cap \{X – G_i \mid 1 \le i \le n\} = \phi$. This implies that $\cup \{G_i \mid 1 \le i \le n\}$ = X which contradicts our supposition that $\Im$ has no finite subcover. Thus $\wp$ must have the FIP. It follows that there exists an ultra filter F on X containing $\wp$. By 3 F has a semi-p-cluster point x ∈ X, then by theorem 11 x ∈ semi-p-cl F for each F ∈ F . In particular x ∈ semi-p-cl (X – G) for each G ∈$\Im$. But X – G is a semi-p-closed subset of X for each G ∈ $\Im$, then semi-p- cl (X – G) = X – G for each G ∈ $\Im$. This implies x ∈ ∩ { X – G: G ∈ $\Im$} = X – ∪ {G∣ G ∈ $\Im$}. Hence x ∉ ∪ {G∣ G ∈ $\Im$} which contradicts the fact that $\Im$ is a semi-p-open cover of X. Thus $\Im$ must have a finite subcover and consequently X is semi-p-compact space. ■

13 Proposition

Let (X,τ) be a topological space. If X is a semi-p-compact space then every net in X has a semi-p-cluster point.

Proof:

Let (f, X,A,≥) be a net in X. For each a ∈ A, let $M_a=\{f(x):x \ge a\}$ since A is directed by ≥, so the collection { M_a: a ∈ A} has the FIP, in particular the collection {semi-p-cl M_a: a ∈ A} of semi-p-closed subsets of X is also has the FIP. It follows by theorem 12 that ∩ {semi-p-cl M_a: a∈A} ≠ϕ, let p∈∩{semi-p-cl M_a: a ∈ A}, then p∈semi-p-cl M_a for each a ∈ A, thus by theorem 9 p is a semi-p-cluster point of f. ■

It seems that the converse of proposition 13 is not true in general, but we could not get a counter example.

14 Definition [3]

Let f :(X,τ) ⟶ (Y,τ′) be any function, then f is said to be "semi-p-irresolute function" if the inverse image of any semi-p-open subset of Y is a semi-p-open subset of X.

15 Proposition

The semi-p-irresolute image of a semi-p-compact space is a semi-p-compact.

Proof:

Let f be a semi-p-irresolute function from a semi-p-compact space (X,τ) onto a topological space (Y,τ′). To prove Y is a semi-p-compact space let $\{G_\alpha:\alpha\in\wedge\}$ be a semi-p-open cover of Y, then $\{f^{-1}(G_\alpha):\alpha\in\wedge\}$ is a semi-p-open cover of X which is semi-p-compact space, then there exist finitely many members of ∧ say $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = \bigcup_{i=1}^{n} f^{-1}(G_{\alpha_i})$, it follows that $Y = \bigcup_{i=1}^{n} G_{\alpha_i}$. Thus Y is a semi-p-compact space. ■

16 Corollaries

1. The semi-p-irresolute image of a semi-p-compact space is a compact space.

2. Semi-p-compactness is a topological property.

17 Definition

A topological space (X,τ) is said to be "semi-p-T_2-space" if for each two distant points x and y in X, there exists two semi-p-open subsets U and V of X, such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

18 Proposition

A semi-p-compact subset of a T_2-space is semi-p-closed.

Proof:

Let A be a semi-p-compact subset of the T_2-space (X,τ), so A is compact since every semi-p-compact is compact, but X is a T_2-space (given) so A is closed in X [5,p.156,prop.11] but every closed subset of A is semi-p-closed, so A is semi-p-closed. ■

Notice that, a semi-p-compact subset of semi-p-T_2-space need not be semi-p-closed as the following example shows:

19 Example

Let $X=\{1,2,3\}$, $\tau = \{X,\phi,\{2,3\}\}$, $F = \{x,\phi,\{1\}\}$.

S-P-O(X) = $\{X,\phi,\{2,3\},\{2\},\{3\},\{1,3\},\{1,2\}\}$

S-P-C(X) = $\{X,\phi,\{1\},\{1,3\},\{1,2\},\{2\},\{3\}\}$

Clear that X is semi-p-T_2 space. If $A = \{2,3\}$ then A is semi-p-compact subset of X, but not semi-p-closed.

20 Definition [3]

A topological space (X,τ) is said to be:

1. 'semi-p-regular space" if and only if for each point $x \in X$ and for each closed subset F of X such that $x \notin F$, there exist two disjoint semi-p-open subsets U and V of X such that $x \in U$ and $F \subseteq V$.

2. "Almost semi-p-regular space" if and only if for each point x in X and for each semi-p-closed subset F of X such that $x \notin F$, there exist two semi-p-open disjoint subsets U and V of X such that $x \in U$ and $F \subseteq V$.

3. "semi-p-normal space" if and only if for each two disjoint closed subsets F_1 and F_2 of X, there exist two disjoint semi-p-open subsets U and V of X such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

Notice that, every regular space is a semi-p-regular and every normal space is a semi-p-normal.

21 Proposition

A compact T_2 – space is a semi-p-regular space.

Proof:

Clear.

22 Corollary

A semi-p-compact T_2-space is a semi-p-regular.

Proof:

Clear.

23 Proposition

A semi-p-compact T_2-space is an almost semi-p-regular space.

Proof:

Let (X,τ) be a semi-p-compact T_2-space and let F be a semi-p-closed subset of X and x be any point in X with $x \notin F$, then $x \neq y$ for each $y \in F$. Since X is a T_2-space, so there exist two disjoint open subsets U_y and V_y of X such that $x \in U_y$ and $y \in V_y$. Then the family $\{ V_y : y \in F\}$ forms an open cover of F, but it is compact set, since every semi-p-compact set is compact and F is semi-p-compact by proposition 4 therefore, we get finitly many elements $y_1, \ldots, y_n$ of F such that $F \subseteq \bigcup_{i=1}^{n} V_{y_i}$. Now, let $V = \bigcup_{i=1}^{n} V_{y_i}$ and $U = \bigcap_{i=1}^{n} U_{y_i}$, then U and V are two disjoint open subset of X such that $x \in U$ and $F \subseteq V$. But every open set is semi-p-open, so X is an almost semi-p-regular space.

24 Proposition

A compact T_2 – space is a semi-p-normal space.

Proof:

Clear.

25 Corollary

A semi-p-compact T_2-space is a semi-p-normal (normal) space.

Proof:

Clear.

References

1. Navalagi, G.B., 1991, Definition Bank in General Topology, Mathematics Subject Classification, 54 G, pp.50.
2. Veera Kumar M.K.R.S., 2002, pre-semi-closed sets, Indian J. Math. 44 (2): 165-181.
3. Al-Khazaraji, R.B., 2004, On Semi-p-open Sets, M. Sc. Thesis, College of Education- Ibn Al-Haitham, University of Baghdad.
4. Bourbaki, N., 1989, Elements of Mathematics, General Springer-Verlage Berlin, Heidelberg, New York, London, Paris, Tokyo, 2nd Edition, pp.437.
5. Gemignani, M.C., 1972, Elementary Topology, University of New Yourk, Addison-Wesley Publishing Company, Inc. pp.270.
6. Hofman, K.H., 2003, Introduction To General Topology, An Introductory Course for the Fourth Semester in Wahlpflichtbereich, pp.52.
7. Reilly, I.L., 2005, Hacettepe J. Math. St. 345, 27-34.
8. Miguel Galdas, 2001, Some Properties of Contra-B-Continuous Functions, Mem. Fac. Sci. (Math.), 22, 19-28.

فضاءات الرص شبه - p

*نرجس عبد الجبار** *سعاد جدعان**

*قسم الرياضيات - كلية التربية- ابن الهيثم - جامعة بغداد

الخلاصة:

الغرض من هذا البحث تقديم نوع جديد من فضاءات الرص وهو فضاء الرص شبه-p وهو اقوى من فضاءات الرص، وكذلك اعطينا خواصاً ومميزات لفضاء الرص شبه – p.