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Modules Whose St-Closed Submodules are Fully Invariant

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Abstract

The duo module plays an important role in the module theory. Many researchers generalized this concept such as Ozcan AC, Hadi IMA and Ahmed MA. It is known that in a duo module, every submodule is fully invariant. This paper used the class of St-closed submodules to work out a module with the feature that all St-closed submodules are fully invariant. Such a module is called an Stc-duo module. This class of modules contains the duo module properly as well as the CL-duo module which was introduced by Ahmed MA. The behaviour of this new kind of module was considered and studied in detail, for instance, the hereditary property of the St-duo module was investigated, as the result; under certain conditions, every St-closed submodule of an St-duo module is also St-duo. Another characterization of the Stc-duo module was given. Additionally, the relationships of St-duo among some types of modules were investigated and discussed, for example; In the class of semi-extending modules, every weak duo module is an Stc-duo module. Also, the authors gave a case in which St-duo, duo, CL-duo and weak duo are equivalent. Furthermore, the St-duo module was used to make the concepts semi-extending and FI-extending equivalent.

Keywords: Closed submodule, Duo module, Fully Invariant submodule, St-closed submodule, Stc-duo module.

Introduction

Many researchers study generalizations of duo modules like Ozcan[1], Hadi[2] and Ahmed[3]. A submodule $\mathcal{N}$ of $\mathcal{M}$ is called fully invariant, for simply, it is indicated by $\mathcal{N} \leq_{fu} \mathcal{M}$, if for each f ∈ End($\mathcal{M}$), f($\mathcal{N}$) ⊆ $\mathcal{N}$, where End($\mathcal{M}$) is the endomorphism ring of $\mathcal{M}$[4]. An R-module $\mathcal{M}$ is said to be duo if every submodule of $\mathcal{M}$ is fully invariant[1]. A module $\mathcal{M}$ is called a weak duo if every direct summand of $\mathcal{M}$ is fully invariant[1]. A pure submodule $\mathcal{N}$ of $\mathcal{M}$ is defined as I$\mathcal{M} \cap \mathcal{N}$ = I$\mathcal{N}$ for each ideal I of R[5]. P-duo is a module whose every pure submodule is fully invariant[2]. A submodule $\mathcal{N}$ of $\mathcal{M}$ is termed essential (simply, $\leq_e \mathcal{M}$), if the intersection between $\mathcal{N}$ and any non-zero submodule of $\mathcal{M}$ is not zero[6]. A submodule $\mathcal{N}$ of $\mathcal{M}$ is called closed (briefly $\leq_c \mathcal{M}$), if $\mathcal{N}$ has no proper essential extension in $\mathcal{M}$, i.e., if $\mathcal{N} \leq_e \mathcal{H} \leq \mathcal{M}$, then $\mathcal{N} = \mathcal{H}$[4,7]. CL-duo is a module $\mathcal{M}$ such that each closed submodule of $\mathcal{M}$ is fully invariant[3].

This paper considers another generalization of the duo module named the Stc-duo module. A submodule $\mathcal{N}$ of a module $\mathcal{M}$ is called semi-essential (shortly, $\mathcal{N} \leq_{sem} \mathcal{M}$), if whenever $\mathcal{N} \cap P$ = (0), then P = (0) for each prime submodule P of $\mathcal{M}$[8,9] as well as[10-12]. $\mathcal{N}$ is St-closed in $\mathcal{M}$ (shortly, $\mathcal{N} \leq_{Stc} \mathcal{M}$), if $\mathcal{N}$ has no proper semi-essential extension in $\mathcal{M}$, i.e., if $\mathcal{N} \leq_{sem} \mathcal{H} \leq \mathcal{M}$, then

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$\mathcal{N}=\mathcal{H}$[13,14]. This article consists of two sections, in the first section, several properties of the Stc-duo modules are presented, for instance; in Proposition 2, the authors proved that if $\mathcal{M}$ is an Stc-duo module, then every St-closed submodule of $\mathcal{M}$ is Stc-duo provided that $\mathcal{M}$ be Stc-transitive and quasi Stc-injective module. Furthermore, For any Stc-duo and Stc-transitive module $\mathcal{M}$. If $\mathcal{N} \leq_{Stc} \mathcal{M}$ and $\mathcal{N}$ is Stc-summand submodule of $\mathcal{M}$, then the intersection of $\mathcal{N}$ with any direct summand of $\mathcal{M}$ is St-closed. Also, Let $\mathcal{M}$ be Stc-transitive, and each St-closed submodule of $\mathcal{M}$ is Stc-summand. If $\mathcal{M}$ is Stc-duo, then the sum of any St-closed submodule of $\mathcal{M}$ with a direct summand of $\mathcal{M}$ is fully invariant.

In Theorem 1, a characterization of Stc-duo modules is given. In the second section, the relationships of Stc-duo modules with some other classes of modules are investigated, for instance;

- $\mathcal{M}$ is Stc-duo if and only if $\mathcal{M}$ is a CL-duo module, provided that every semi-essential extension of any submodule of a module $\mathcal{M}$ is a fully essential module.
- In the class of semi-extending modules, every weak duo module is an Stc-duo module.
- Let $\mathcal{M}$ be a fully prime over a principal ideal ring (simply, PIR). Consider these assertions:

1. $\mathcal{M}$ is a duo module.
2. $\mathcal{M}$ is an Stc-duo module.
3. $\mathcal{M}$ is a CL-duo module.
4. $\mathcal{M}$ is a P-duo module.
5. $\mathcal{M}$ is a weak duo module.

Then $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4$, and if $\mathcal{M}$ is semisimple then $5 \Longrightarrow 1$.

- Take an almost semi-extending module $\mathcal{M}$ over PIR. Consider the conditions below:

1. $\mathcal{M}$ is a duo module.
2. $\mathcal{M}$ is a CL-duo module.
3. $\mathcal{M}$ is a P-duo module.
4. $\mathcal{M}$ is an Stc-duo module.

Then $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4$. Moreover, if $\mathcal{M}$ is an St-semisimple module, then 4 implies 1.

Finally, it should be noted that $\mathcal{R}$ is indicated to a commutative ring with identity and $\mathcal{M}$ is a unitary $\mathcal{R}$-module.

Stc-duo Modules

This section introduced and studied the concept of Stc-duo modules as a proper generalization of duo modules.

Definition 1: If every St-closed submodule of an $\mathcal{R}$-module $\mathcal{M}$ is fully invariant then $\mathcal{M}$ said to be Stc-duo. Any ring $\mathcal{R}$ is Stc-duo if $\mathcal{R}$ as an $\mathcal{R}$-module is Stc-duo.

Remark 1: It is clear that every duo module is Stc-duo. The converse is not always true such as the Z-module Q is Stc-duo since the only St-closed submodule in Q is itself which is fully invariant. Note that <0> $\nleq_{Stc}$ Q since Q has only prime submodule which is <0>, so that <0> $\leq_{Sem}$Q. In contrast, Q is not duo module, since the submodule Z of Q is not fully invariant in Q because there is a homomorphism f: Q⟶Q defined by:

$$f(s) = \frac{1}{4}s \quad \text{for all } s \in Z$$

It is obvious that $f(Z) \nsubseteq Z$.

Examples and Properties:

1. $\mathcal{M} = Z_2 \oplus Z_2$ is not Stc-duo, since the St-closed submodule $<\bar{0},\bar{1}>$ of $\mathcal{M}$ is not fully invariant. Indeed, ∃ homo f: $\mathcal{M} \rightarrow \mathcal{M}$ defined by:

$$f(\bar{a},\bar{b}) = (\bar{b},\bar{a}) \quad \text{for all } (\bar{a},\bar{b}) \in Z_2 \oplus Z_2$$

Therefore:

$$f(<\bar{0},\bar{1}>) = <\bar{1},\bar{0}> \nleq <\bar{0},\bar{1}>$$

An $\mathcal{R}$-module $\mathcal{M}$ is multiplication if every submodule $\mathcal{N}$ of $\mathcal{M}$ is of form $\mathcal{N} = I\mathcal{M}$ for some ideal I of $\mathcal{M}$[15]. It is known that any multiplication module is a duo module.

2. Any multiplication module is Stc-duo.
3. Each commutative ring is an Stc-duo ring.

Proof: This follows directly by point 2.

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A non-zero module $\mathcal{M}$ is termed uniform (semi-uniform) if every non-zero submodule of $\mathcal{M}$ is essential[4] (semi-essential)[8]

4. Obviously, each uniform module is an Stc-duo module.

5. Every semi-uniform module is Stc-duo.

Proof: Take a semi-uniform module $\mathcal{M}$. Either $\mathcal{M}$ has only proper St-closed submodule (0) which is fully invariant in $\mathcal{M}$. Or $\mathcal{M}$ hasn't a proper St-closed submodule, in fact, if $\mathcal{M}$ has only a prime submodule, then (0) is not St-closed[13].

6. Z_{P^∞} as Z-module is an Stc-duo module, because Z_{P^∞} is uniform and by point 4 it is Stc-duo.

A module $\mathcal{M}$ is named St-semisimple if every submodule of $\mathcal{M}$ is St-closed[16].

7. Let $\mathcal{M}$ be an St-semisimple module. If $\mathcal{M}$ is Stc-duo then it is a duo module.

8. Every simple module is an Stc-duo module.

Proof: The only prime submodule in any simple module $\mathcal{M}$ is (0), so (0) $\nleq_{Stc} \mathcal{M}$. Hence $\mathcal{M}$ is the only St-closed submodule in $\mathcal{M}$. Thus, $\mathcal{M}$ is Stc-duo.

Remark 2: For any simple $\mathcal{R}$-module U, the direct sum U $\oplus$ U is not Stc-duo.

Proof: Take a submodule U $\oplus$ (0) in U$\oplus$U. Clearly, U $\oplus$ (0) $\nleq_{sem} U \oplus U$ and there is no proper (hence semi-essential) submodule in U $\oplus$ U contains U $\oplus$ (0) properly. Thus, U $\oplus$ (0) $\leq_{Stc}$ U $\oplus$ U. In contrast, U $\oplus$ (0) $\nleq_{fu}$ U $\oplus$ U. In fact, $\exists$ f: U $\oplus$ U $\rightarrow$ U $\oplus$ U defined by:

$$f(x,y) = (y,x) \quad \forall (x,y) \in U \oplus U$$

Note that:

$$f(<a,0>) = <0,a> \nsubseteq <a,0>$$

Next, Stc-duo module is not closed under arbitrary submodules, to verify that consider the following example.

Example 1: Consider a field F and a vector space U over F. Let T= F $\oplus$ U. Define a multiplication on T by:

$$(a,u)(b,v) = (ab, av+bu) \quad \forall (a,u), (b,v) \in T$$

T is a commutative ring implying that T is an Stc-duo T-module. In contrast, the submodule $\mathcal{H}$ = (0) $\oplus$ U of T is not Stc-duo. In fact, $\mathcal{H}$ is a direct sum of two simple modules, so by Remark 2, $\mathcal{H}$ is not Stc-duo module.

The following definitions are needed to be introduced.

Definition 2: If every direct summand of a module $\mathcal{M}$ is an St-closed submodule, then $\mathcal{M}$ is said to be Stc-summand

Definition 3: If all St-closed submodules of a module $\mathcal{M}$ satisfy the transitive property, then $\mathcal{M}$ is named Stc-transitive. Thus means if A $\leq_{Stc}$ B $\leq_{Stc}$ $\mathcal{M}$ then A $\leq_{Stc}$ $\mathcal{M}$.

Remark 3: If $\mathcal{M}$ is a chain module, then $\mathcal{M}$ is Stc-transitive, where an R-module $\mathcal{M}$ is called chain, if every pair of submodules A and B in $\mathcal{M}$ are comparable. i.e., either A$\leq$ B or B$\leq$A[4].

Under certain conditions, a submodule of Stc-duo module is Stc-duo as shown in the result below.

Proposition 1: Take $\mathcal{M}$ to be an Stc-summand and Stc-transitive module. Then every direct summand of $\mathcal{M}$ is Stc-duo whenever $\mathcal{M}$ is Stc-duo.

Proof: Assume $\mathcal{H} \leq^{\oplus} \mathcal{M}$, and $\mathcal{N} \leq_{Stc} \mathcal{H}$. Consider the projection homomorphism ρ: $\mathcal{M} \rightarrow \mathcal{H}$, and f $\in$ End($\mathcal{H}$), where End($\mathcal{H}$) indicates to the endomorphism ring of $\mathcal{H}$. So, h = jfρ $\in$ End($\mathcal{M}$), where j: $\mathcal{H} \rightarrow \mathcal{M}$. Since $\mathcal{H} \leq^{\oplus} \mathcal{M}$ and $\mathcal{M}$ is Stc-summand, then $\mathcal{H} \leq_{Stc} \mathcal{M}$. By transitivity of St-closed, this yield $\mathcal{N} \leq_{Stc} \mathcal{M}$. But $\mathcal{M}$ is Stc-duo, thus, h($\mathcal{N}$) belong to $\mathcal{N}$, hence (jfρ)($\mathcal{N}$) = f($\mathcal{N}$)$\subseteq$ $\mathcal{N}$. That is $\mathcal{H}$ is Stc-duo.

It is now possible to deduce the following using Remark 3 and Proposition 1.

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Corollary 1: Let $\mathcal{M}$ be a chain Stc-summand module. If $\mathcal{M}$ is Stc-duo then every direct summand of $\mathcal{M}$ is Stc-duo.

Another condition can be applied to make Stc-duo modules closed under submodules as follows. Before that, an $\mathcal{R}$-module $\mathcal{M}$ is called Stc-self-injective, if every homomorphism from any St-closed submodule of $\mathcal{M}$ to $\mathcal{M}$ can be extended to a homomorphism θ: $\mathcal{M} \longrightarrow \mathcal{M}$ [17].

Proposition 2: Let $\mathcal{M}$ be an Stc-transitive and Stc-self-injective module. If $\mathcal{M}$ is an Stc-duo module, then every St-closed submodule of $\mathcal{M}$ is Stc-duo.

Proof: Take $\mathcal{N} \leq_{Stc} \mathcal{M}$ and $\leq_{Stc} \mathcal{N}$. Assume that f ∈ End($\mathcal{N}$), so Fig 1 is considered below:

$$\begin{tikzcd} \mathcal{N} \arrow[r, "i"] \arrow[d, "f"'] & \mathcal{M} \arrow[ddl, dashed, "h"] \\ \mathcal{N} \arrow[d, "i"'] & \\ \mathcal{M} & \end{tikzcd}$$

Figure 1. Stc-self-injective module

where i: $\mathcal{N} \longrightarrow \mathcal{M}$ is the inclusion. Because $\mathcal{N} \leq_{Stc} \mathcal{M}$ and $\mathcal{M}$ is Stc-self-injective then there is a homomorphism h ∈ End($\mathcal{M}$) with h∘i = i∘f. Now, (h∘i) ($\mathcal{K}$)= h($\mathcal{K}$), but $\mathcal{K} \leq_{Stc} \mathcal{N}$ and $\mathcal{N} \leq_{Stc} \mathcal{M}$ and since $\mathcal{M}$ is Stc-transitive, then $\mathcal{K} \leq_{Stc} \mathcal{M}$. Moreover, $\mathcal{M}$ is Stc-duo, therefore, h($\mathcal{K}$) ⊆ $\mathcal{K}$. Beside that:

(h∘i) ($\mathcal{K}$) = (i∘f) ($\mathcal{K}$) = f($\mathcal{K}$), thus h($\mathcal{K}$) = f($\mathcal{K}$) ⊆ $\mathcal{K}$

Thus, $\mathcal{N}$ is Stc-duo.

According to [13], the intersection of any two St-closed submodules may not be St-closed. Nonetheless, the class of the Stc-duo module plays an important role in satisfying that. Before that, the following lemma is needed.

Lemma 1[1] :

Let $\mathcal{M} = \bigoplus_{i\in I} \mathcal{M}_i$ be a direct sum of submodules $\mathcal{M}_i$, i∈I, and $\mathcal{N} \leq_{fu} \mathcal{M}$, then $\mathcal{N} = \bigoplus_{i\in I} (\mathcal{N} \cap \mathcal{M}_i)$.

Proposition 3: For any Stc-duo and Stc-transitive module $\mathcal{M}$. If $\mathcal{N} \leq_{Stc} \mathcal{M}$ and $\mathcal{N}$ is Stc-summand submodule of $\mathcal{M}$, then the intersection of $\mathcal{N}$ with any direct summand of $\mathcal{M}$ is St-closed.

Proof: Take $\mathcal{N}_1 \leq^{\oplus} \mathcal{M}$, so $\mathcal{M}=\mathcal{N}_1 \oplus \mathcal{K}$ for some $\mathcal{K} \leq \mathcal{M}$. Since $\mathcal{M}$ is Stc-duo then $\leq_{fu} \mathcal{M}$. By Lemma 1, $\mathcal{N} = (\mathcal{N} \cap \mathcal{N}_1) \oplus (\mathcal{N} \cap \mathcal{K})$. That is $\mathcal{N} \cap \mathcal{N}_1 \leq^{\oplus} \mathcal{N}$. But $\mathcal{N}$ is Stc-summand, therefore $\mathcal{N} \cap \mathcal{N}_1 \leq_{Stc} \mathcal{N}$. Since $\mathcal{N} \leq_{Stc} \mathcal{M}$ and $\mathcal{M}$ is Stc-transitive, then $(\mathcal{N} \cap \mathcal{N}_1) \leq_{Stc} \mathcal{M}$.

Proposition 4: Let $\mathcal{M}$ be Stc-transitive, and each St-closed submodule of $\mathcal{M}$ is Stc-summand. If $\mathcal{M}$ is Stc-duo, then the sum of any St-closed submodule of $\mathcal{M}$ with a direct summand of $\mathcal{M}$ is fully invariant.

Proof: Take an St-closed submodule $\mathcal{N}_1$ of $\mathcal{M}$ and a direct summand $\mathcal{N}_2$ of $\mathcal{M}$, so $\mathcal{M} = \mathcal{N}_2 \oplus \mathcal{K}$ for some $\mathcal{K} \leq \mathcal{M}$. Now, $\mathcal{N}_1 \leq_{Stc} \mathcal{M}$, by assumption $\mathcal{N}_1 \leq_{fu} \mathcal{M}$. By Lemma 1, $\mathcal{N}_1 = (\mathcal{N}_2 \cap \mathcal{N}_1) \oplus (\mathcal{K} \cap \mathcal{N}_1)$. Thus, $\mathcal{N}_1 + \mathcal{N}_2 = (\mathcal{N}_2 \cap \mathcal{N}_1) \oplus (\mathcal{K} \cap \mathcal{N}_1) + \mathcal{N}_2$. Now, $(\mathcal{K} \cap \mathcal{N}_1)$ is a direct summand of $\mathcal{N}_1$ and $\mathcal{N}_1 \leq_{Stc} \mathcal{M}$, Also, since $\mathcal{M}$ is Stc-transitive, then $(\mathcal{K} \cap \mathcal{N}_1) \leq_{Stc} \mathcal{M}$. Thus, $\cap \mathcal{N}_1 \leq_{fu} \mathcal{M}$. On the other hand, $\mathcal{N}_2 \leq^{\oplus} \mathcal{M}$ and $\mathcal{M}$ is Stc-summand, thus, $\mathcal{N}_2 \leq_{fu} \mathcal{M}$. This implies that $(\mathcal{K} \cap \mathcal{N}_1) + \mathcal{N}_2 \leq_{fu} \mathcal{M}$, hence $\mathcal{N}_1 + \mathcal{N}_2 \leq_{fu} \mathcal{M}$.

The following theorem gives a characterization of the definition of Stc-duo module.

Theorem 1: A module $\mathcal{M}$ is Stc-duo if and only if for every f ∈ End($\mathcal{M}$) and every cyclic St-closed <m> of $\mathcal{M}$, ∃r ∈ $\mathcal{R}$ with f(m) = rm.

Proof: ⇒) Let f ∈ End($\mathcal{M}$) and $\mathcal{R}m \leq_{Stc} \mathcal{M}$. Since $\mathcal{M}$ is Stc-duo then $\mathcal{R}m \leq_{fu} \mathcal{M}$, so f(m)∈ $\mathcal{R}$m. Thus, ∃ r ∈ $\mathcal{R}$ such that f(m) = rm.

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⇐) Take $\mathcal{N} \leq_{Stc} \mathcal{M}$, and f∈End($\mathcal{M}$). Note that ∀n ∈ $\mathcal{N}$, f(n)∈ $\mathcal{M}$. By the hypothesis, ∃r ∈ $\mathcal{R}$ with f(n) = r n ∈ $\mathcal{N}$. Therefore, f(n)∈$\mathcal{N}$. This means $\mathcal{M}$ is Stc-duo module.

As an application of Theorem 1, one can show the following, before that, a module $\mathcal{M}$ is said to be countably generated if it can be generated by a countable set[3].

Proposition 5: A module $\mathcal{M}$ is Stc-duo if each countably generated submodule of $\mathcal{M}$ is Stc-duo.

Proof: Let $\mathcal{R}x \leq_{Stc} \mathcal{M}$ and $f: \mathcal{M} \longrightarrow \mathcal{M}$. Consider the following sum of cyclic submodules:

$$\mathcal{H} = \mathcal{R}x + \mathcal{R}f(x) + \mathcal{R}f^2(x) + \mathcal{R}f^3(x) + \ldots\ldots$$

$\mathcal{H}$ is a countably generated submodule of $\mathcal{M}$. Restrict f on $\mathcal{H}$, this obtains $f_{|H}$∈ End($\mathcal{H}$), hence $\mathcal{R}$x $\leq_{Stc} \mathcal{H}$[13]. But $\mathcal{H}$ is Stc-duo, so by Theorem 1, $\exists t \in \mathcal{R}$ such that $f(x) = tx$. By Theorem 1 $\mathcal{M}$ is an Stc-duo module.

In general, the direct sum of two Stc-duo modules need not be Stc-duo, for example, Z_2 is Stc-duo Z-module, but $Z_2 \oplus Z_2$ is not Stc-duo as shown in Examples and Properties, point (1).

Lemma 2[1]:

Consider the direct sum $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ of submodules $\mathcal{M}_1$ and $\mathcal{M}_2$. Then $\mathcal{M}_1 \leq_{fu} \mathcal{M}$ if and only if Hom ($\mathcal{M}_1$, $\mathcal{M}_2$) = 0.

Proposition 6: Assume that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is an Stc-summand module where $\mathcal{M}_1$ and $\mathcal{M}_2$ are submodules of $\mathcal{M}$. If $\mathcal{M}_1$ is Stc-duo then Hom ($\mathcal{M}_1, \mathcal{M}_2$) = 0.

Proof: Since $\mathcal{M}$ is Stc-summand then $\mathcal{M}_1 \leq_{Stc} \mathcal{M}$. But $\mathcal{M}$ is Stc-duo, therefore $\mathcal{M}_1 \leq_{fu} \mathcal{M}$. By Lemma 2, Hom ($\mathcal{M}_1, \mathcal{M}_2$) = 0.

Example 2: The contrapositive of Proposition 6 can be used to verify that the Z-module Z ⊕ Z is not St-duo even though Z is Stc-duo.

The Relationships of Stc-duo with Other Modules

Some types of modules are related to Stc-duo modules for instance CL-duo, weak duo and P-duo modules. This section studies these relationships.

Remark 4: Every CL-duo module is Stc-duo.

Proof: Take a CL-duo module $\mathcal{M}$ and $\mathcal{N} \leq_{Stc} \mathcal{M}$, then $\leq_c \mathcal{M}$. But $\mathcal{M}$ is CL-duo, therefore $\mathcal{N} \leq_{fu} \mathcal{M}$.

The converse of Remark 4 seems not true, since closed need not be St-closed[13], but the authors haven't an example for that.

Now, the relationship between Stc-duo and CL-duo modules has been discussed. Remember that, if every proper submodule of a module $\mathcal{M}$ is a prime submodule then $\mathcal{M}$ is called fully prime[18].

Proposition 7: Let $\mathcal{M}$ be a fully prime module, then $\mathcal{M}$ is Stc-duo if and only if CL-duo.

Proof: ⟹) Assume that $\mathcal{M}$ is Stc-duo and $\mathcal{N} \leq_c \mathcal{M}$. There are two cases: if $\mathcal{N}$ = 0, then clearly $\mathcal{N} \leq_{fu} \mathcal{M}$ Otherwise, $\mathcal{M}$ is fully prime implies that $\mathcal{N} \leq_{Stc} \mathcal{M}$ [Remark 1.8][13]. But $\mathcal{M}$ is Stc-duo, thus $\mathcal{N} \leq_{fu} \mathcal{M}$.

⟸) It is followed by Remark 4.

An R-module $\mathcal{M}$ is called fully essential if every semi-essential submodule of $\mathcal{M}$ is essential[13].

Proposition 8: $\mathcal{M}$ is Stc-duo if and only if $\mathcal{M}$ is a CL-duo module, provided that every semi-essential extension of any submodule of a module $\mathcal{M}$ is a fully essential module.

Proof: Suppose that $\mathcal{M}$ is Stc-duo and $\mathcal{N} \leq_c \mathcal{M}$. There are two cases: if $\mathcal{N}$ = 0, then clearly $\mathcal{N} \leq_{fu} \mathcal{M}$ Otherwise, $\mathcal{M}$ is fully prime implies that $\mathcal{N} \leq_{Stc} \mathcal{M}$ [Proposition 1.7][13]. Since $\mathcal{M}$ is Stc-duo, therefore, $\mathcal{N} \leq_{fu} \mathcal{M}$.

Recall that an $\mathcal{R}$-module is termed semi-extending if every submodule of $\mathcal{M}$ is semi-essential in a direct summand of $\mathcal{M}$. Equivalently, $\mathcal{M}$ is semi-

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extending if every St-closed submodule of $\mathcal{M}$ is a summand of $\mathcal{M}$[14, 16].

Stc-duo and week duo are thought to be independent, unfortunately, no example could be found to demonstrate that. Under certain conditions, the weak duo module can be Stc-duo as shown below.

Proposition 9: In the class of semi-extending modules, every weak duo module is an Stc-duo module.

Proof: Assume that $\mathcal{M}$ is semi-extending and $\mathcal{N} \leq_{Stc} \mathcal{M}$. Since $\mathcal{M}$ is semi-extending then $\mathcal{N} \leq^{\oplus} \mathcal{M}$. On the other hand, $\mathcal{M}$ is a weak duo so $\mathcal{N} \leq_{fu} \mathcal{M}$.

It is thought that the two types of modules, Stc-duo and P-duo, are independent, however, no example was identified. Under certain conditions P-duo modules can be Stc-duo as shown in the following before, remember that an R-module $\mathcal{M}$ is named almost semi-extending if every submodule of $\mathcal{M}$ is semi-essential in a pure submodule of $\mathcal{M}$. Equivalently, every St-closed submodule of $\mathcal{M}$ is pure[16].

Proposition 10: Take an almost semi-extending module $\mathcal{M}$, If $\mathcal{M}$ is a P-duo module then it is Stc-duo.

Proof: Let $\mathcal{N} \leq_{Stc} \mathcal{M}$. Since $\mathcal{M}$ is almost semi-extending then $\mathcal{N}$ is pure in $\mathcal{M}$. Also, $\mathcal{M}$ is P-duo, therefore $\mathcal{N} \leq_{fu} \mathcal{M}$. Thus, $\mathcal{M}$ is Stc-duo.

Remember that a module $\mathcal{M}$ is semisimple if every submodule of $\mathcal{M}$ is a direct summand [6].

Theorem 2: Let $\mathcal{M}$ be a fully prime over a principal ideal ring (simply, PIR). Consider these assertions:

6. $\mathcal{M}$ is a duo module.
7. $\mathcal{M}$ is an Stc-duo module.
8. $\mathcal{M}$ is a CL-duo module.
9. $\mathcal{M}$ is a P-duo module.
10. $\mathcal{M}$ is a weak duo module.

Then $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4$. Furthermore, if $\mathcal{M}$ is semisimple then $5 \Longrightarrow 1$.

Proof:

1⇒ 2: It is clear.

2 ⟹ 3: Since $\mathcal{M}$ is fully prime, then by Proposition 7, the result follows.

3 ⟹ 4: Since $\mathcal{M}$ is defined over PIR, so by [Proposition 5.4][3], $\mathcal{M}$ is P-duo.

4 ⟹ 5: It is followed by [2]

5 ⟹ 1: Assume that $\mathcal{N} \leq \mathcal{M}$. Because $\mathcal{M}$ is semisimple then $\leq^{\oplus} \mathcal{M}$. Besides that $\mathcal{M}$ is a weak duo, thus, $\mathcal{N} \leq_{fu} \mathcal{M}$.

A module $\mathcal{M}$ is named purely extending if each submodule of $\mathcal{M}$ is essential in a pure submodule of $\mathcal{M}$[19].

Theorem 3: For a semisimple fully prime module $\mathcal{M}$, the following are equivalent:

1. $\mathcal{M}$ is a duo module.
2. $\mathcal{M}$ is an Stc-duo module.
3. $\mathcal{M}$ is a CL-duo module.
4. $\mathcal{M}$ is a weak duo module.

Proof:

1 ⇒ 2: it is straightforward.

2 ⇒ 3: It follows from Proposition 7.

3 ⇒ 4: Take $\leq^{\oplus} \mathcal{M}$, so $\mathcal{N} \leq_{c} \mathcal{M}$. Since $\mathcal{M}$ is CL-duo, then $\mathcal{N} \leq_{fu} \mathcal{M}$.

4 ⇒ 1: Let $\mathcal{N} \leq \mathcal{M}$. Since $\mathcal{M}$ is semisimple, then $\mathcal{N} \leq^{\oplus} \mathcal{M}$. But $\mathcal{M}$ is a weak duo, therefore, $\mathcal{N} \leq_{fu} \mathcal{M}$.

Theorem 4: Take an almost semi-extending module $\mathcal{M}$ over PIR. Consider the conditions below:

5. $\mathcal{M}$ is a duo module.
6. $\mathcal{M}$ is a CL-duo module.
7. $\mathcal{M}$ is a P-duo module.
8. $\mathcal{M}$ is an Stc-duo module.

Then $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4$. Moreover, if $\mathcal{M}$ is an St-semisimple module, then 4 implies 1.

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Proof:

1 ⟹ 2: Clear.

2 ⟹ 3: Let $\mathcal{N}$ be pure in $\mathcal{M}$. Since R is PIR, then $\mathcal{N} \leq_{c} \mathcal{M}$ [Exersies 15, P.242][20]. On the other hand, $\mathcal{M}$ is CL-duo, therefore $\mathcal{N} \leq_{fu} \mathcal{M}$.

3 ⟹ 4: Let $\leq_{Stc} \mathcal{M}$. Since $\mathcal{M}$ is almost semi-extending then $\mathcal{N}$ is pure. On contrast, $\mathcal{M}$ is P-duo, so $\mathcal{N} \leq_{fu} \mathcal{M}$.

4 ⟹ 1: Let $\mathcal{N} \leq \mathcal{M}$. Since $\mathcal{M}$ is St-semisimple, then $\mathcal{N} \leq_{Stc} \mathcal{M}$. Additionally, $\mathcal{M}$ is Stc-duo, therefore $\mathcal{N} \leq_{fu} \mathcal{M}$.

Remember that an R-module $\mathcal{M}$ is named FI-extending if every fully invariant submodule is essential in a direct summand of $\mathcal{M}$[21,22]. It is known that every extending module is FI-extending but not conversely. In the following proposition, a duo module is used as a condition under which the converse becomes true.

Proposition 11: Take a duo module $\mathcal{M}$, then $\mathcal{M}$ is extending. If it is an FI-extending module.

Proof: Suppose that $\mathcal{N} \leq \mathcal{M}$. Since $\mathcal{M}$ is a duo then $\mathcal{N} \leq_{fu} \mathcal{M}$. On the other hand, $\mathcal{M}$ is FI-extending, so $\mathcal{N}$ is essential in a direct summand of $\mathcal{M}$. Thus, $\mathcal{M}$ is extending.

A module $\mathcal{M}$ is called semi-extending if every St-closed submodule of $\mathcal{M}$ is a direct summand[16]. The last proposition led us to introduce the following.

Definition 4: A module $\mathcal{M}$ is said to be FI-semi-extending if every fully invariant submodule of $\mathcal{M}$ is essential in a direct summand of $\mathcal{M}$.

Obviously, every semi-extending module is FI-semi-extending.

Proposition 12: In the class of Stc-duo modules, every FI-extending module is semi-extending.

Proof: Assume that $\mathcal{M}$ is an Stc-duo and FI-extending module, take $\mathcal{N} \leq_{Stc} \mathcal{M}$. Since $\mathcal{M}$ is an Stc-duo module, then $\mathcal{N} \leq_{fu} \mathcal{M}$. As well as $\mathcal{M}$ is FI-semi-extending, therefore, $\mathcal{N}$ is essential in a direct summand of $\mathcal{M}$. Thus, $\mathcal{M}$ is a semi-extending module.

At the end of this section, the relationships of the Stc-duo module among other related modules can be summarized in Table 1. Moreover, the role of the Stc-duo module in the relationship between FI-extending and semi-extending modules is considered.

Table 1. The relationships of the Stc-duo module among other related modules

Results	If an $\mathcal{R}$-module $\mathcal{M}$ is	With Conditions	Then $\mathcal{M}$ is
Remark 1	duo	-	
Remark 4	CL-duo	-	
Proposition 9	weak duo	$\mathcal{M}$ is a semi-extending module	**Stc-duo module**
Proposition 10	P-duo	$\mathcal{M}$ is an almost semi-extending module	
Proposition 7		$\mathcal{M}$ is a fully prime module	
Proposition 8	Stc-duo	Every semi-essential extension of any submodule of $\mathcal{M}$ is fully essential module	**CL-duo module**
Example and Properties 7		$\mathcal{M}$ is an St-semisimple module	**Duo module**
Proposition 12	FI-extending	$\mathcal{M}$ is an Stc-duo module	**semi-extending module**

Results and Discussion

The main results in this paper are the following:

1. For any Stc-duo and Stc-transitive module $\mathcal{M}$. If $\mathcal{N} \leq_{Stc} \mathcal{M}$ and Stc-summand submodule of $\mathcal{M}$, then the intersection of $\mathcal{N}$ with any direct summand of $\mathcal{M}$ is St-closed.

2. Let $\mathcal{M}$ be Stc-transitive, and every St-closed submodule of $\mathcal{M}$ is Stc-summand. If $\mathcal{M}$ is Stc-duo, then the sum of any St-closed submodule of $\mathcal{M}$ with any direct summand of $\mathcal{M}$ is fully invariant.

3. A module $\mathcal{M}$ is Stc-duo if and only if $\forall$ f $\in$ End($\mathcal{M}$) and for each cyclic <m> $\leq_{Stc} \mathcal{M}$, $\exists$t $\in$ R $\ni f(m)$ = tm.

4. Given $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is an Stc-summand module where $\mathcal{M}_1$ and $\mathcal{M}_2$ are submodules of $\mathcal{M}$. If $\mathcal{M}_1$ is Stc-duo then Hom $(\mathcal{M}_1, \mathcal{M}_2) = 0$.

5. In the class of semi-extending modules, every weak duo module is an Stc-duo module.

6. For almost semi-extending modules, every P-duo module is Stc-duo.

7. Let $\mathcal{M}$ be a fully prime module, then $\mathcal{M}$ is Stc-duo if and only if CL-duo.

8. $\mathcal{M}$ is Stc-duo if and only if $\mathcal{M}$ is a CL-duo module, provided that every semi-essential extension of any submodule of a module $\mathcal{M}$ is a fully essential module.

9. Let $\mathcal{M}$ be a fully prime over a principal ideal ring (simply, PIR). These assertions are considered:

1. $\mathcal{M}$ is a duo module. 2. $\mathcal{M}$ is Stc-duo. 3. $\mathcal{M}$ is CL-duo. 4. $\mathcal{M}$ is P-duo. 5. $\mathcal{M}$ is a weak duo.

Then 1$\Longrightarrow$ 2 $\Longrightarrow$ 3 $\Longrightarrow$ 4, and when $\mathcal{M}$ is semisimple then 5 $\Longrightarrow$ 1

10. The following statements are equivalent for a semisimple fully prime module $\mathcal{M}$:

1. $\mathcal{M}$ is a duo module. 2. $\mathcal{M}$ is Stc-duo. 3. $\mathcal{M}$ is CL-duo. 4. $\mathcal{M}$ is a weak duo.

11. Let $\mathcal{M}$ be almost semi-extending over PIR. The following are considered:

1. $\mathcal{M}$ is a duo module. 2. $\mathcal{M}$ is CL-duo. 3. $\mathcal{M}$ is P-duo. 4. $\mathcal{M}$ is Stc-duo. Then 1$\Rightarrow$ 2 $\Longrightarrow$ 3 $\Longrightarrow$ 4. And if $\mathcal{M}$ is St-semisimple, then 4 implies 1.

12. Each FI-extending is a semi-extending module in the class of Stc-duo modules.

The above results created a new class of modules and named Stc-duo modules. Many properties of this kind of module were given and discussed in detail. All relationships between Stc-duo and related modules were studied using certain conditions.

Conclusion

By using the concept of an St-closed submodule, the authors introduced a new kind of module named the Stc-duo module in which all St-closed submodules are fully invariant. It contains a duo and CL-duo module but it is independent of the weak duo and P-duo. The main features of the Stc-duo modulexamplee as analogues of those in the duo module are investigated. Some of these properties need certain conditions added to Stc-duo. Conditions under which (submodule of the Stc-duo module is also Stc-duo) are given. The intersection of the St-closed submodule with other submodules is considered. Another characterization of the definition of the Stc-duo module is given. The relationships of the Stc-duo module with CL-duo, weak duo and P-duo modules are established. The author discovered that Stc-duo contains CL-duo and is independent of the concepts of weak duo and P-duo modules**.**

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Authors' Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

Authors' Contribution Statement

This work was carried out in collaboration between the two authors. A.M.A. introduced all the results of this paper. A.M.R. contributed to the printing, revision, and proofreading ofthe research. Both authors read and approved the final manuscript.

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المقاسات التي كل مقاساتها الجزئية المغلقة من النمط-St تكون ثابتة تماماً

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الخلاصة

في المقاس الثنائي يكون كل مقاس جزئي فيه ثابت تماماً. هذا البحث يتناول دراسة المقاس الذي كل مقاس مغلق من النمط -Stc فيه يكون ثابت تماماً، ويسمى هذا المقاس بالمقاس الثنائي من النمط -Stc. ان المقاس الثنائي من النمط -Stc يحوي المقاس الثنائي بشكل فعلي فضلاً عن المقاس الثنائي من النمط -CL الذي قُدم من قبل منى عباس احمد. ان سلوك هذا النوع من المقاسات سيتم دراستهُ بشكل مفصل مع اعطاء تشخيصاً اخر له. كما ان العلاقة بين المقاس الثنائي من النمط -Stc بالمقاسات الثنائية من الانماط P, CL, weak قد دُرست بالتفصيل.

الكلمات المفتاحية: المقاس الجزئي المغلق، المقاس الثنائي، المقاس الجزئي الثابت تماماً، المقاس المغلق من النمط -Stc، المقاس الثنائي من النمط Stc.