The Relationships between Relatively Cancellation Modules and Certain Types of Modules

*Hatem Yahya Khalaf **

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Abstract:

Let R be a commutative ring with identity and M be unitary (left) R-module. The principal aim of this paper is to study the relationships between relatively cancellation module and multiplication modules, pure submodules and Noetherian (Artinian) modules.

Key words: relatively cancellation module, multiplication module, Noetherian module, Artinian module, pure submodule.

Introduction:

Gilmer [1] has been defined the concept of cancellation ideal to be the ideal I of R which satisfies the following: whenever AI = BI with A and B are ideals of R implies A = B.

Mijbass in [2] has been generalized this concept to modules. He has been defined the cancellation module as follows: An R-module M is called a cancellation module, whenever AM=BM with A and B are ideals of R implies A=B.

Rasheed in [3] has been introduced the concept of relatively cancellation module as follows: An R-module M is called relatively cancellation, whenever AM=BM with A is a prime ideal of R and B is any ideal of R implies A = B.

Clearly, the class of cancellation modules contains the class of relatively cancellation modules.

This paper contains three sections. In section one; we study the relationships between relatively cancellation R-module and multiplication modules. We shall prove that under certain condition, a multiplication module is a relatively cancellation R-module, see theorem (1.1) and we shall study more another of properties.

The relation between relatively cancellation modules and pure ideals are studied in section two, see theorem (2.1). The last section is devoted to study the relation between the class of relatively cancellation modules and the class of Noetherian (Artinian) modules, see theorem (3.1) and theorem (3.2).

Finally, we remark that R in this paper stands for a commutative ring with identity and all modules are unitary.

1- Multiplication Modules and Relatively Cancellation Modules

In this section, we establish some relationships between relatively cancellation R-modules and multiplication modules.

An R-module M is said to be multiplication module if for every submodule N of M, there exists an ideal I of R such that N=I M, [4].

The following theorem gives a sufficient condition under which the module M is relatively cancellation module.

*Department of Mathematics-Ibn-Al-Haitham College of Education - University of Baghdad

1.1 Theorem:

Let M be a multiplication R-module and $ann_M(ann_R(M)) \not\subseteq P$ for all maximal ideals P of R. Then M is relatively cancellation module.

proof: Suppose that AM=BM where A is prime ideal of R and B is any ideal of R. Let $a \in A$, $a \neq 1$ (since A is prime ideal of R) and let $K=\{r \in R: ra \in B\}$.

Now, suppose that $K \neq R$. Then there exists maximal ideal P such that $K \subseteq P$. It is clear that $PM \neq M$. Therefore M is P-cyclic by [4,lemma (1)]. Thus there exists $m \in M$ and $p \in P$ such that $(1 - p)M \subseteq Rm$. In special case $(1 - p)am \in (1 - p)BM = B(1 - p)M \subseteq BM$. Then there exists $b \in B$ such that $(1 - p)am = bm$, which implies that $[(1 - p)a - b]m = 0$, but $(1 - p)ann_R(m) \subseteq ann_R(M)$. Then $(1 - p)[(1 - p)a - b] \in ann_R M$, but $ann_M(ann_R(M)) \not\subseteq P$, then there exists $s \in ann_M(ann_R(M))$ and $s \notin P$, which implies that $s(1 - p)^2 a - s(1 - p)b = 0$. Therefore $s(1 - p)^2 a = s(1 - p)b \in B$. Then $s(1 - p)^2 \in K \subseteq P$, implies that $(1 - p)^2 \in P$ since $s \notin P$. Therefore P=R. This is a contradiction (by definition of maximal ideal). Then K=R and $a \in B$, which implies that $A \subseteq B$.

Similarly, we can prove that $B \subseteq A$. Therefore A=B, which completes the proof.

Recall that an R-submodule N of an R-module M is called prime submodule if and only if $N \neq M$ and whenever $rx \in N$ for $r \in R$ and $x \in M$ we have either $r \in [N \underset{R}{:} M]$ or $x \in N$, [5].

By using this concept, we have the following:

1.2 Proposition:

Let M be a multiplication relatively cancellation module, N be a submodule of M such that $[N \underset{R}{:} M]$ is a prime ideal of R. Then N is a prime submodule of M.

proof: Let N be a submodule of an R-module M and $rx \in N$ for $r \in R$, $m \in M$. We want to prove that N is prime submodule of M, i.e. either $m \in N$ or $r \in [N \underset{R}{:} M]$ suppose that $m \notin N$, to show that $r \in [N \underset{R}{:} M]$ since $rm \in N$ and M is multiplication R-module, then $N = [N \underset{R}{:} M]M$, which implies that $rm \in [N \underset{R}{:} M]$, we have $m \in M$, $m \notin N$. Thus $rM \subseteq [N \underset{R}{:} M]M$ by [3,theorem (1.1.51)p.7], we get $r \in [N \underset{R}{:} M]$.

The following result is an immediate consequence of proposition (1.2), but first the following definition is needed.

A submodule N of an R-module M is said to be quasi-prime submodule if whenever $r_1 r_2 m \in N$ for $r_1, r_2 \in R$ and $m \in M$, then either $r_1 m \in N$ or $r_2 m \in N$. Equivalently, a proper submodule N of an R-module M is quasi-prime if and only if $[M \underset{R}{:} (m)]$ is a prime ideal of R for each $m \in M$, [6].

1.3 Corollary:

Let M be a multiplication relatively cancellation module. N be a submodule of M such that $[N \underset{R}{:} M]$ is prime ideal of R. Then N is a quasi-prime submodule of M.

proof: The result follows directly by proposition (1.2) and [5,proposition (2.1.3)].

Recall that a proper submodule N of an R-module M is called irreducible if for every submodules L_1 and L_2 of M such that $L_1 \cap L_2 = N$, then either $L_1 = N$ or $L_2 = N$, [7].

By using this concept we can give the following proposition.

1.4 Proposition:

Let M be a multiplication relatively cancellation module, N is an irreducible submodule of an R-module M, then the following statements are equivalent:

1. N is a prime submodule.
2. N is a quasi-prime submodule.
3. $[N \underset{R}{:} M]$ is prime ideal of R.

proof: (1)⇒(2), by [6,proposition (2.1.3),p.40].

(2)⇒(3), by [6,corollary (2.1.5),p.41].

(3)⇒(1), by proposition (1.2).

Next, we give the following proposition

1.5 Proposition:

Let M be a multiplication R-module and N be a submodule of M such that N is relatively cancellation module. Then M is relatively cancellation module.

proof: M is multiplication module and N is a submodule of M. Then N = JM where J is an ideal of R. Let AM=BM where A is prime ideal of R and B is any ideal of R. Now, AJM = BJM. Then AN = BN. Then A = B (since N is relatively cancellation module) and have M is relatively cancellation module.

Now, we have the following proposition.

1.6 Proposition:

Let M be a multiplication relatively cancellation R-module and N is a proper submodule of M. Then the following statements are equivalent.

(1) N is relatively cancellation submodule.

(2) $(N \underset{R}{:} M)$ is relatively cancellation ideal of R.

(3) N=AM where A is relatively cancellation ideal of R.

proof: (1) ⇒ (2)

Suppose that N is relatively submodule and $A(N \underset{R}{:} M) = B(N \underset{R}{:} M)$ where A is prime ideal of R and B is any ideal of R. Then $A(N \underset{R}{:} M)M = B(N \underset{R}{:} M)M$, implies that AN=BN. Therefore A=B (since N is relatively cancellation submodule) and hence $(N \underset{R}{:} M)$ is relatively cancellation ideal of R.

(2) ⇒ (3)

Only put $A=(N \underset{R}{:} M)$ we get the result.

(3) ⇒ (1)

Let CN = DN where C is prime ideal of R and D is any ideal of R. Let N=AM where A is relatively cancellation ideal of R. Then CAM = DAM and implies that CA = DA and hence N is relatively cancellation submodule.

2- Pure Submodules and Relatively Cancellation Modules

In this section we introduce a condition under which every module containing pure submodule satisfies a property of relatively cancellation is a relatively cancellation module. A submodule N of M is called pure if I M ∩ N = I N for every ideal I of R, [8, prop. (1.8), p.10].

We start by the following result.

2.1 Proposition:

Let M be an R-module and N is a pure submodule of M. If N is relatively cancellation submodule. Then M is relatively cancellation module.

proof: Let AM = BM where A is prime ideal of R and B is any ideal of R. Now, N is pure submodule of M. Then N∩AM = AM and N∩BM = BM. Therefore AN = BN, but N is relatively cancellation submodule. Hence A = B, which completes the proof.

The following results are consequences of proposition (2.1).

2.2 Corollary:

Let M be a cyclic module and N be a submodule of M such that $rM \cap N = rN$, $\forall r \in R$. If N is relatively cancellation submodule. Then M is relatively cancellation module.

proof: We have M is cyclic module and $rM \cap N = rN$, $\forall r \in R$. Then N is pure by [9] and according to the proposition (2.1), we obtain the result.

Fieldhouse has been defined the regular module as follows: An R-module M is called regular module if every submodule of M is pure [8].

2.3 Corollary:

Let M be a regular module and N be a relatively cancellation submodule of M. Then M is relatively cancellation module.

proof: From definition a regular module and proposition (2.1)

Recall that an ideal I of a ring R is called semimaximal if I is an intersection of finitely many maximal ideal of R, [10].

2.4 Corollary:

Let N be a relatively cancellation submodule of an R-module M and ann(M) is semimaximal ideal of R. Then M is relatively cancellation module.

proof: According to [10, prop. (1.3.5), p.27] and proposition (2.1).

3- Noetherian (Artinian) Relatively Cancellation Modules

In this section, we give the relation between the relatively cancellation module and stationary chain.

We show that every acceding (decending) chain of ideals in R is stationary if an R-module M is Noetherian (Artinian) relatively cancellation module see theorem (3.1) and theorem (3.2).

Now, we state and prove the following proposition.

3.1 Proposition:

Let M be a Noetherian relatively cancellation module and let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be acceding chain from ideals in R such that A_n is prime ideals of R $\forall$ n. Then the above chain is stationary.

proof: Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be an ascending chain. Then $A_1M \subseteq A_2M \subseteq A_3M \subseteq \ldots$ but M is Noetherian module. Then there exists $n \in$ N such that $A_nM = A_kM$ $\forall\, n \leq k$. Now, A_n is prime ideal of R and M is relatively cancellation module. Then $A_n = A_k$ $\forall\, n \leq k$. Therefore $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ is stationary.

We end this section by the following proposition.

3.2 Proposition:

If M be an Artinian relatively cancellation R-module and $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ decreasing chain from ideals in R such that A_n is prime ideal $\forall n$. Then the above chain is stationary.

proof: Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ be decreacing chain form ideals in R. Then $A_1M \supseteq A_2M \supseteq A_3M \supseteq \ldots$. But M is an Artinian module, then there exists $n \in$ ¥ such that $A_nM = A_kM$ $\forall\, n \leq k$. Therefore $A_n = A_k$ (since M is relatively cancellation module) and hence the chain $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ is stationary.

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العلاقة بين موديولات الحذف نسبيا وبعض انواع اخرى من الموديولات

حاتم يحيى خلف*

*قسم الرياضيات - كلية التربية- ابن الهيثم - جامعة بغداد

الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايد وليكن M موديلاً احادياً أيسراً على الحلقة R.

ان هدفنا الرئيس في بحثنا هذا هو دراسة العلاقة بين موديولات الحذف نسبياً وبين انواع اخرى من الموديولات مثل الموديولات الجدائية، الموديولات الجزئية النقية والموديولات النويثرية (الآرتينية).