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Solution of Wave Equation by Linear Regression Artificial Neural Network

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Abstract

Wave partial differential equations are one of the most important topics that model, for example, the wave motion of ground vibrations. Hence, finding approximate solutions to such equations with high accuracy and speed faster than difficult and complex analytical solutions has become possible through the use of artificial intelligence and machine learning methods. This research has three goals. The first is to transform the problem of the initial value of the wave equation into its legal form and find its accurate solution. The second is to write a linear-regression neural network algorithm. The third result is applying this algorithm to find a numerical solution to the initial value problem under the study. The last step is to compare the solution using a table and figures for certain parameters and initial conditions values to demonstrate the efficiency of the artificial neural network method. Approximate solutions were obtained with an error amount close to zero compared to the real solution of the wave differential equation by applying the artificial neural network that represents the linear regression equation. This gives the advantage of high speed in obtaining the solution to this type of differential equation.

Keywords: Artificial Neural Network, Linear Regression, LM Training, Partial Differential Equation, Wave Equation

Introduction

Wave partial differential equations (WPDE) are one of the foremost important topics that represent, for example, the wave motion of ground vibrations, the motion of a tsunami wave, or the propagation of heat waves. Therefore, finding approximate solutions to such equations with high accuracy faster than difficult and complex analytical solutions has become possible by using Artificial Neural Networks (ANN) and machine learning methods. Accordingly, the researchers could now use the ANN to find approximate solutions to many partial differential equations representing many physical, chemical, and life phenomena.

For example. In [1] they used the functional variable method to find approximate solutions of some nonlinear PDF, such as the Klein-Gordon equation, and the higher order nonlinear Schrödinger equation. The solutions resulting from this method represent single-wave solutions and periodic wave solutions.

In [2] an (ANN) has been adopted to solve high-order partial differential equations based on the

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initial condition and boundary conditions of these equations. In [3] a solution to a system of PDF has been suggested by proposing a new type of artificial neural network called Fourier network. Experiments were conducted on the Burger and Darcy equations and the Navier - Burger equations.

In [4] this work presents a neural network for solving integro PDE. In [5] it was able to interpret the solve as a superposition of k-wave solutions of quasi-linear systems of PDEs.

In [6] analytical and numerical methods are proposed for solutions of multidimensional linear and nonlinear wave equations. In [7] artificial intelligence, especially ANNs, is used to solve PDFs that represent physical problems, by building an artificial neural network based on some physical specifications of the differential equation that represents the wave, to improve the results from approximate solutions. Resulting. They studied the KdV–Burgers equation.

In [8] a solution to the heat wave differential equation is proposed using the fourth-order Runge-Kutta method. In [9] the heat flow was modeled using one- and two-dimensional differential equations. In [10] a new technique built on the principle of ANNs is proposed to study the problem of natural heat load generated in some mechanical engineering problems.

In [11] a comparison was made between three methods for creating artificial neural networks (ANNs) to simulate low- and high-order differential equations and the preference between them was determined.

In [12], they presented a review of three methods for creating ANNs which are used to solve higher-order PDEs. The philosophy of creating each method and its preference is explained. In [13], the researchers discussed the use of ANN networks to solve wave differential equations, such as the heat equation, and provided some approximate solutions to them.

In [14], the researchers proposed a new type of ANN to recognize the boundary conditions and initial values of PDEs, which helps shorten the time for learning and reduces the size of the data used in teaching the network while giving approximate solutions with a very small amount of error. In [15], they used the fractional series to solve fractional Burger's equation.

In [16] the researchers found the exact and approximate solution of fractional Burger's equation. In[17], the Haar wavelet is developed for the solution of non-linear and linear delay fractional order differential equations.

In[18], solution nonlinear fractional integro-differential Equation Theoretical and computational analysis. Also using the collocation method based on Haar wavelet to develop the combination of Caputo derivative.

In[19], the researchers have presented various forms of feedback artificial neural networks and have also provided solutions to some of the problems and obstacles facing the designer of such networks while maintaining the general structure of this type of network.

In[20], the shear strength of deep concrete blocks was simulated using artificial neural networks (ANNs) by using four methods to train these networks, relying on 70% of the input data as training data and 30% to match the obtained results, and comparing the results by using different statistical methods and thus determining the most efficient training method.

Materials and Methods

In recent times, artificial neural networks (ANNs) have garnered significant interest from academics in several subjects. ANNs are being used to enhance conventional approaches for identifying optimum solutions. Additionally, ANNs are utilized to forecast efficient solutions with minimal costs and to prevent prospective issues. This study has emphasized the use of ANN in various scientific applications. Furthermore, the emphasis was placed on the advantages of ANNs, which enable their use in addressing diverse challenges across numerous domains in the next few years.

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Artificial Neural Network [19]

The architecture of ANN is derived from the organization and operation of the biological neural network. Artificial Neural Networks (ANN) has similarities with the neurons found in the brain since they are composed of neurons organized in different layers.
A feed-forward A common kind of neural network is one that consists of an input layer that receives outside data to recognize patterns, an output layer that offers a solution, and a hidden layer that acts as a middle layer and divides the other layers. Acyclic routes connect the neurons within the output layer to the neurons within the input layer. By altering the weights of its neurons in response to differences between the expected and actual outputs, the (ANN) uses a training technique to learn from the datasets. Fig 1 depicts the general structure of an (ANN).

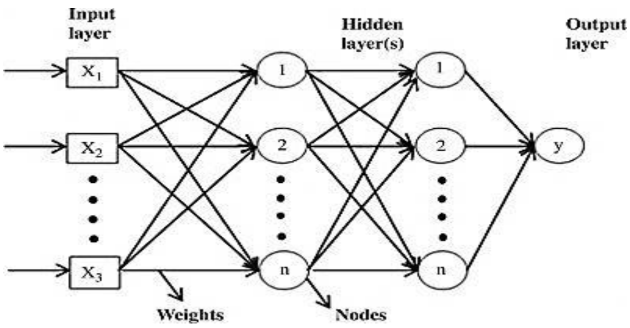

Figure 1. Structure of ANN.

Tanh-sigmoid function

The activation function that regularly achieves better performance than the sigmoid function is called the tangent hyperbolic function. The function in question is a sigmoid function that has been mathematically calibrated. Both are analogous and may be deduced from each other.

$$f(s) = \tanh(s) = \frac{2}{1+e^{-2s}} - 1.$$

Levenberg_Marquardt _Training (ANN-LM) [20]

The Levenberg_M (LM) training, otherwise known as the wet least squares approach, is used for the resolution of nonlinear least squares problems. This approach eschews the computation of the precise Hessian matrix and instead utilizes the Jacobian matrices and the slope vector. The damage function is represented mathematically as the sum of the squares errors.

$$f = \sum_{i=1}^{a} u_i^2 .$$

The variable "a" represents the digit of occurrences of data, whereas the variable "u" represents the array of errors. The Jacobian_M of the slop function is known as:

$$A_{i,j} = \frac{\partial u_i}{\partial w_i} .$$

Let i range from 1 to a and j range from 1 to b. Here, a represents the digit of cases at data, b represents the digit of parameters in the ANN, and A represents the Jacobian _M. The dimensions of the Jacobian_M are [a, b]. The calculation of the slope vector of the lost function is performed:

$$\nabla f = 2A^T u .$$

The approximation of the Hessian matrix is calculated:

$$Bf \cong 2A^T A + \beta Í .$$

The variable B represents the Hessian-M, β represents the factor that guarantees the positivity of the Hessian, and I is the identity_matrix. The first selection is made for the substantial parameter β. Furthermore, in the event of an error occurring during any iteration, the value of β will be augmented by a certain factor. In contrast, as the loss reduces, the value of β will be reduced to bring the LM algorithm closer to the Newton technique. The process of improving the parameters through the LM method is defined as follows:

$$\omega^{(k+1)} = \omega^{(k)} - \left(A^{(k)T} A^{j} + \beta^{(k)} I\right) (2A^{(k)T} u^{(k)} \quad ,$$

For $k = 0, 1, \ldots, n$.

In this section, the analytical solution was evaluated, and then the solution was found by using ANNs. Also, a comparison is given to ensure the efficiency of the ANNs using tables and graphs.

Analytic Solution for the problem

$$u_{\mathcal{TT}} - c^2 u_{xx} = 0 \quad , \qquad 1$$

With Ic_1: $u(x,0) = \varphi(x)$,

Ic_2: $u_t(x,0) = \psi(x)$.

The general solution is:

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$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$.

There for this example:

$A = 1\ ,\ B = 0\ ,\ C = -c^2$,

$B^2 - Ac = c^2 \ > 0$.

It is of hyperbolic type:

$\frac{dx}{dt} = \mp C\left[\frac{B\mp\sqrt{B^2-AC}}{A}\right]$,

$\xi(x, \mathcal{T}) = x + c\,\mathcal{T}$,

$\eta(x, \mathcal{T}) = x - c\,\mathcal{T}$,

$\xi_x = \eta_x = 1\ , \xi_t = c\ , \eta_t = -c\ ,\ \xi_{xx} = \xi_{xt} = \xi_{tt} = \eta_{xx} = \eta_{xt} = \eta_{tt} = 0$,

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$,

$u_t = u_\xi \xi_t + u_\eta \eta_t = c\,u_\xi - c\,u_\eta$,

$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$,

$u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}$.

Substitution in Eq 1 gets:

$\Rightarrow\ c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta} - c^2\left[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right] = 0$,

$\Rightarrow\ u_{\xi\eta} = 0$.

Integrate w.r.t ξ to get:

$u_\eta = f_1(\eta)$, $f_1 is\ arbitary\ function$.

Again Integrate w.r.t η to get:

$u(\xi, \eta) = f(\xi) + \int f_1(\eta) d\eta$,

$= f(\xi) + g(\eta)$,

$u(x, \mathcal{T}) = f(x + c\,\mathcal{T}) + g(x - c\,\mathcal{T})$. 2

Is the condition:

$IC_1 \Rightarrow f(x) + g(x) = \varphi(x)$, 3

$IC_2:\ u_t(x, 0) = \psi(x)$,

$\Rightarrow c f'(x) - c\, g'(x) = \psi(x)$. 4

Integrate Eq. (4) from 0 to x to get:

$f(x) - g(x) = \frac{1}{c}\int_0^x \psi(r) dr + f(0) - g(0)$. 5

Eq 3 + Eq 5 $\Rightarrow$ $f(x) = \frac{1}{2}\varphi(x) + \frac{1}{2}\left[f(0) - g(0) + \frac{1}{c}\int_0^x \psi(r) dr\right]$. 6

Eq 3 - Eq. 5 $\Rightarrow$ $g(x) = \frac{1}{2}\varphi(x) - \frac{1}{2}\left[g(0) - f(0) - \frac{1}{c}\int_0^x \psi(r) dr\right]$. 7

Substations Eq 6 and Eq 7 in Eq 2 we get:

$u(x, \mathcal{T}) = f(x + c\,\mathcal{T}) + g(x - c\,\mathcal{T})$,

$= \frac{1}{2}\varphi(x + c\,\mathcal{T}) + \frac{1}{2c}\int_0^{x+ct} \psi(r) dr + \frac{1}{2}\varphi(x - c\,\mathcal{T}) - \frac{1}{2c}\int_0^{x-ct} \psi(r) dr$,

$= \frac{1}{2}\varphi(x + c\,\mathcal{T}) + \varphi(x - c\,\mathcal{T}) + \frac{1}{2c}\int_0^x \psi(r) dr$.

As a Special case let:

$\varphi(x, \mathcal{T}) = Ke^{-(x+c\,\mathcal{T})}\sin(x + c\,\mathcal{T})$,

$\psi(x, \mathcal{T}) = \cos(x + c\,\mathcal{T})$,

$$u(x, \mathcal{T}) = \frac{1}{2}\left[e^{-(x+c\,\mathcal{T})}\sin(x + c\,\mathcal{T}) + e^{-(x-c.\mathcal{T})}\sin(x - c\,\mathcal{T})\right] + \frac{1}{2}\left[\sin(x + c\,\mathcal{T}) - \sin(x - c\,\mathcal{T})\right].$$

Results and Discussion

The Matlab ANNs tools and the "feed-forward neural net" (FFNN) module were used in this study to construct an FFNN. The selection of the degree of layers and neurons, as well as the training procedure ("TANSING"—LM t training approach), aimed to achieve a harmonious balance between simplicity and accuracy. The output value of the neural network was evaluated by comparing it with the theoretical value to validate the correctness and efficiency of this technology, as seen in Fig 2.

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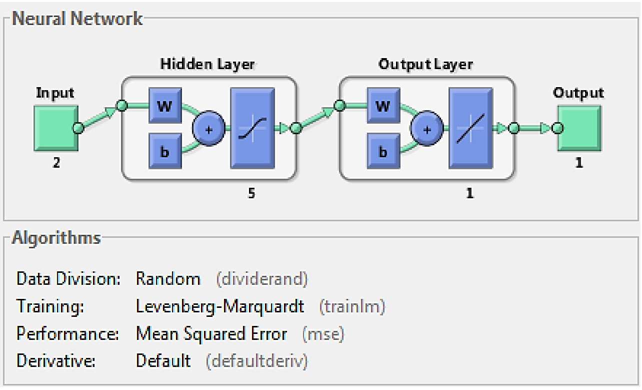


Figure 2. Structure of ANN in Matlab

The approximate function of the structure has the form:

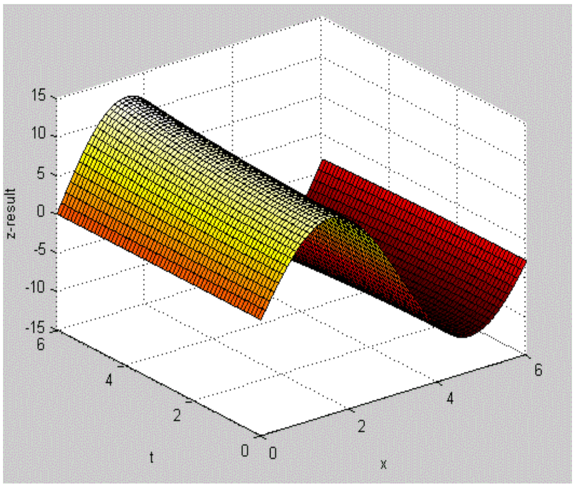
$$Result = output = (w_2 * \tanh (w_1 * [x, y] + b_1) + b_2)$$

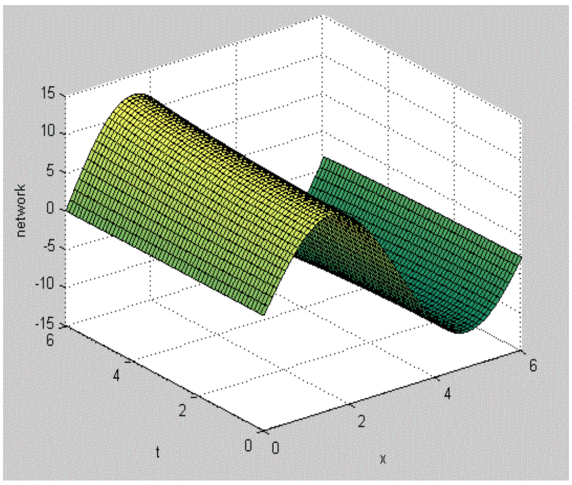
After applying the neural network saturation to the problem, the obtained results are shown in Table 1 when C = 0.1

Table 1. Theoretical, neural value, and Accuracy of example.

X↓	t	Theoretical $\mathbf{u_a(x,y)}$	Trainlm $\mathbf{u_t(x,y)}$	Error = $\lvert\mathbf{u_a - u_t}\rvert$
0	0	0.000	1.33129721365e-002	1.33129721365e-002
0	0.5	4.74791694442e-002	5.71473017104e-002	9.66813226611e-003
1	1	3.61507282645e-001	3.61154991472e-001	3.52291172383e-004
1	1.5	3.61019452487e-001	3.61965892074e-001	9.46439587427e-004
2	2	4.26043293452e-002	4.42288874693e-002	1.62455812402e-003
2	2.5	2.35433399532e-002	2.48871056180e-002	1.34376566481e-003
3	3	-2.8111072561e-001	-2.8185862312e-001	7.47897508035e-004
3	3.5	-3.2642100232e-001	-3.2660257033e-001	1.81568003578e-004
4	4	-2.6642803923e-001	-2.6595291707e-001	4.75122167868e-004
4	4.5	-2.9565583834e-001	-2.9560974878e-001	4.60895628375e-005
5	5	1.29123515675e-001	1.28268993223e-001	8.54522451544e-004
6	6	5.40619704312e-001	5.27156592342e-001	1.34631119701e-002
		MSE	6.16e-006	

After applying the neural network saturation to the problem, the result shown in Fig 3 when C = 0.5 with MSE = 6.91e-003





A

Figure 3. A- The exact solution, B- The neural network result

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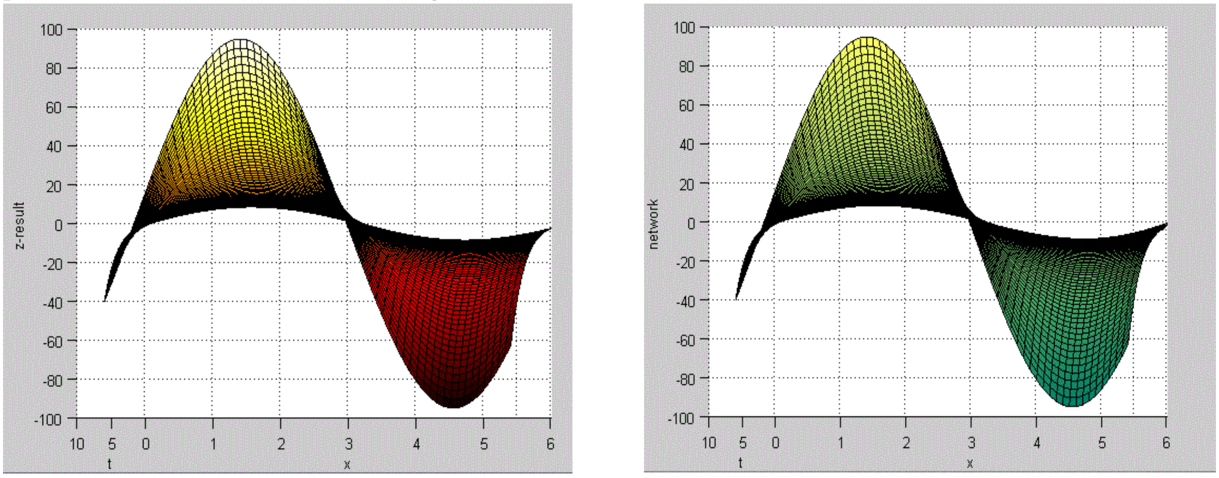
After applying the neural network saturation to the problem, the result is shown in Fig 4 when C = 1 with MSE= 6.29e-003



A B

Figure 4. A- The exact solution, B- The neural network result

Conclusion

The main goal of modeling is to give a reasonable interpretation of the situation (physical, chemical, biological, medical, etc.), and this interpretation is given by finding the solution of the model. Solutions are in closed form, approximate, or numerical. Surely the best one if possible, is the closed form, but there are difficulties in finding it. Therefore, they go to find the solution approximately or numerically by a certain method, such as Sumuda Adomain Decomposition, the Runge Kutte method, the Artificial Neural Networks method, etc.

This paper solves the model theoretically to find the closed-form solution by classification of the partial differential equation and also by the artificial neural network method, which is efficient in finding the solution and takes a very short time to find the solution from the complex analytical method.

Authors' Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Anbar.

Authors' Contribution Statement

A.N., S. M.A., S.M.A., and A.S.N., they contributed to the design and implementation of the research, the analysis of the results, and the writing of the manuscript.

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حل المعادلة الموجية بواسطة الشبكة العصبية الاصطناعية للانحدار الخطي

ناهض سليم السيف[1]، محمود عريبي شمران[2]، سماح محمد علي [2], سعد ناجي العزاوي[2]

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الخلاصة

تعتبر المعادلات التفاضلية الموجية من اهم المواضيع التي تمثل على سبيل المثال الحركة الموجية للاهتزازات الأرضية . ومن هنا فان ايجاد حلول تقريبيه لمثل هذه المعادلات بدقة وسرعه عالية وبشكل اسرع من الحلول التحليلية والمعقدة , اصبح ممكنا من خلال استخدام الذكاء الاصطناعي واساليب التعلم الالي. في هذا البحث هناك ثلاثة أهداف الأول هو تحويل مشكلة القيمة الأولية للمعادلة الموجية إلى شكلها القانوني وإيجاد حلها الدقيق. والثاني هو كتابة خوارزمية الشبكة العصبية الاصطناعية للانحدار الخطي. النتيجة الثالثة هي تطبيق هذه الخوارزمية لإيجاد حل عددي لمسألة القيمة الأولية قيد الدراسة. وأخيرا هو مقارنة الحل بواسطة جدول وأشكال لقيم معينة من المعلمات والشروط الأولية لبيان كفاءة طريقة الشبكة العصبية الاصطناعية. تم الحصول على نتائج الحلول التقريبية ذات الخطأ البسيط جداً مقارنة بالحل الحقيقي للمعادلة التفاضلية الموجية من خلال تطبيق الشبكة العصبية الاصطناعية التي تمثل معادلة الانحدار الخطي والتي تعطي ميزة السرعة العالية في الحصول على حل هذا النوع من التفاضلية .

الكلمات المفتاحية: الشبكات العصبية الاصطناعية، الانحدار الخطي، تدريب (L.M)، المعادلات التفاضلية الجزئية، معادلة الموجة.