An Approximate Solution of Fredhom Integral Equation Using Bernstein Polynomials

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Abstract

In this paper, Bernstein polynomials method has used to find an approximate solution for Fredholm integral equation of the second kind. These polynomials are incredibly useful mathematical tools, because they are simply defined, can be calculated quickly on computer systems and represent a tremendous variety of functions. They can be differentiated and integrated easily.

1. Introduction

The integral equation is an equation in which the unknown function y(x) appears under the integral sign.

The general form of integral equation is given by [1] :

$$h(x)y(x) = f(x) + \int k(x,t)y(t)dt \ldots(1)$$

where $h(x)$, $f(x)$ and the kernel $k(x,t)$ are known functions ; $y(x)$ is the function to be determined.

The general integral equations which are linear involve the integral operator:

$$L = \int k(x,t)dt \ldots(2)$$

which satisfy the linearity condition:

$$L(a_1 f_1(t) + a_2 f_2(t)) = a_1 L(f_1(t)) + a_2 L(f_2(t)) \ldots(3)$$

Where a_1, a_2 are constants and $L(f(t)) = \int k(x,t) f(t)dt$.

The integral equation is called homogenous If $f(x) = 0$, otherwise it is called non homogenous [2].

We can distinguish between two types of integral equations which are:

1. Integral equation of the first kind when $h(x) = 0$ in equation (1).

$$f(x) = \int k(x,t)y(t)dt \ \ldots(4)$$

2. Integral equation of the second kind when $h(x) = 1$ in equation (1).

$$y(x) = f(x) + \int k(x,t)y(t)dt \ \ldots(5)$$

Integral equations can be classified into different kinds according to the limits of integral:

1. If the upper limit of the integral in equation (1) is variable then equation (1) is called Volterra integral equation.

Volterra integral equation of the first kind is[3]:-

$$f(x) = \int_a^x k\{x,t)y(t)dt \ \ldots(6)$$

where a is constant and x is variable

Volterra integral equation of the second kind is:-

$$y(x) = f(x) + \int_a^x k(x,t)y(t)dt \ \ldots(7)$$

2. If the limits of equation (1) are constants then the equation is called Fredholm integral equation.

The Fredholm integral equation of the first kind is:-

$$f(x) = \int_a^b k(x,t)y(t)dt \ldots(8)$$

where a, b are constants.

Fredholm integral equation of the second kind is[3]:-

$$y(x) = f(x) + \int_a^b k(x,t)y(t)dt \ \ldots(9)$$

3. The integral equation in equation (1) is said to be singular if the range of integration is infinite, e.g., $0 < x < \infty$

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or $-\infty < x < \infty$, or if the kernel k(x, t) is unbounded.

4. If the kernel k(x, t) in equation (1) depends only on the difference x-t, such a kernel is called a difference kernel unbounded. and the equation:

$$h(x)y(x) = f(x) + \int k(x-t)y(t)dt \quad \ldots(10)$$

is called integral equation of convolution type.

2. Bernstein polynomials

The Bernstein polynomials of degree n are defined by [4], [5].

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

for $i = 0,1,2,...,n \ldots(11)$

where

$\binom{n}{i} = \frac{n!}{i!(n-i)!}$, n is the degree of polynomials, I is the index of polynomials and t is the variable.

The exponents on the t term increase by one as i increases, and the exponents on the (1-t) term decrease by one as i increases.

2.1 Properties of Bernstein polynomials

Bernstein polynomials properties are:[4],[5]

1. $B_i^n(0) = B_i^n(1) = 0$ for $i = 0,1,2,...,n-1$

$B_0^n(0) = B_n^n(1) = 1$

$B_i^n(t) = 0$ if $i < 0$ or $i > n$

$B_i^n(t) \geq 0$ in [0,1]

$B_i^n(t) = B_{n-i}^n(1-t)$

$\sum_{i=0}^{n} B_i^n(t) = 1$

2. A Matrix Representation for Bernstein Polynomials

In many applications, a matrix formulation for the Bernstein polynomials is useful. These are straight forward to develop if only looking at a linear combination in terms of dot products. Given a polynomial written as a linear combination of the Bernstein basis functions [4].

$$B(t) = c_0 B_0^n(t) + c_1 B_1^n(t) + c_2 B_2^n(t) + ... + c_n B_n^n(t)$$

$\ldots(12)$

It is easy to write this as a dot product of two vectors

$$B(t) = \left[[B_0^n(t) \;\; B_1^n(t) \;\; B_2^n(t) \ldots B_n^n(t) \right] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \ldots(13)$$

which can be converted to the following form:

$$B(t) = \left[1 \;\; t \;\; t^2 \cdots t^n\right] \begin{bmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ b_{20} & b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \ldots (14)$$

where b_{nn} are the coefficients of the power basis that are used to determine the respective Bernstein polynomials,

We note that the matrix in this case is lower triangular. In the quadratic case n=2, the matrix representation is:

$$B(t) = \left[1 \;\; t \;\; t^2\right] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

The cubic case n=3, the matrix representation is

$$B(t) = \left[1 \;\; t \;\; t^2 \;\; t^3\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

3. A recursive Definition of the Bernstein Polynomials.

The Bernstein polynomial of degree n can be defined by blending together two Bernstein polynomials of degree (n-1). That is, the n[th] degree Bernstein polynomial can be written as, [4].

$$B_k^n(t) = (1-t)B_k^{n-1}(t) + tB_{k-1}^{n-1}(t) \ldots(15)$$

The (n=1)Bernstein polynomial of degree n form a partition of unity in that they all sum to one,[4].

$$\sum_{i=0}^{n} B_i^n(t) = \sum_{i=0}^{n-1} B_i^{n-1}(t) = \sum_{i=0}^{n-2} B_i^{n-2}(t) = \cdots = \sum_{i=0}^{1} B_i^1(t) = 1 \cdots(16)$$

Bernstein polynomials of degree n can be written in terms of the power basis. This can be directly calculated using the equation (11) and the binomial theorem as follows, [4].

$$B_k^n(t)=\binom{n}{k}t^k(1-t)^{n-k}=\sum_{i=k}^{n}(-1)^{i-k}\binom{n}{i}\binom{i}{k}t^i$$

Where the binomial theorem is used to Expand $(1-t)^{n-k}$.

3. Solution of Fredholm integral equation with Bernstein polynomials

In this section Bernstein polynomials to find the approximate solution for Fredholm integral equation, will be introduced.

Let us reconsider the Fredholm integral equation of the second kind.

$$y(x)=f(x)+\int_a^b k(x,t)y(t)dt \quad \ldots (17)$$

A applying the Bernstein polynomials method for equation (17) by using equation (9),(12), we get the following formula.

$$y(t)=\left[B_0^n(t) \quad B_1^n(t) \quad B_2^n(t)\ldots B_n^n(t)\right]\begin{bmatrix}c_0\\c_1\\\vdots\\c_n\end{bmatrix}$$

If n=2

$$B(t)=\left[c_0B_0^2(t) \quad c_1B_1^2(t) \quad c_2B_2^2(t)\right]=y(t)$$

$$y(t)=\left[B_0^2(t) \quad B_1^2(t) \quad B_2^2(t)\right]\begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}$$

$$y(t)=\left[\binom{2}{0}t^0(1-t)^{2-0} \quad \binom{2}{1}t^1(1-t)^{2-1} \quad \binom{2}{2}t^2(1-t)^{2-2}\right]\begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}$$

$$y(t)=\left[(1-t)^2 \quad 2t(1-t) \quad t^2\right]\begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}$$

$$y(t)=\left[1-2t+t^2 \quad 2t-2t^2 \quad t^2\right]\begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}$$

$$y(t)=\left[1 \quad t \quad t^2\right]\begin{bmatrix}1&0&0\\-2&2&0\\1&-2&1\end{bmatrix}.\begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}\ldots(18)$$

If we substitute equation(18) in to equation(17), we obtain,

$$\left[1 \quad x \quad x^2\right]\begin{bmatrix}1&0&0\\-2&2&0\\1&-2&1\end{bmatrix}.\begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}=f(x)+\int_a^b k(x,t)\left[1 \quad t \quad t^2\right]\begin{bmatrix}1&0&0\\-2&2&0\\1&-2&1\end{bmatrix}\begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}dt$$

$$c_0(1-x)^2+2c_1x(1-x)+c_2x^2=f(x)+\lambda\int_a^b k(x,t)\left[c_0(1-t)^2+2c_1t(1-t)+c_2t^2\right]dt$$

And after performing the integration.

$$c_0(1-x)^2+2c_1x(1-x)+c_2x^2=f(x)+\lambda\left(\int_a^b k(x,t)c_0(1-t)^2dt+2\int_a^b k(x,t)c_1t(1-t)dt+\int_a^b k(x,t)c_2t^2dt\right)$$

$$=f(x)+c_0\int_a^b k(x,t)(1-2t+1^2)dt+2c_1\int_a^b k(x,t)(t-t^2)dt+c_2\int_a^b k(x,t)t^2dt \ldots(19)$$

now to find all integration in equation(19).
Then in order to determine c_0, c_1 *and* c_2, we need three equation;
Now Choice $x_i, i = 1,2,3$ in the interval [a, b], which gives three equations.
Solve the three equation by Gauss elimination to find the values c_0, c_1 *and* c_2.
The following algorithm summarizes the steps for finding the approximate solution for the second kind of Fredholm integral equation.

4. Algorithm (BPFI)

Step1:
Choice n the degree of Bernstein polynomials

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \text{ for } i = 0,1,2,...,n$$

Step2:
Put the Bernstein polynomials in Fredholm integral equation for second kind.

$$B_i^n(x) = f(x) + \int_a^b k(x,t) B_i^n(t) dt$$

Step3:
Compute $\int_a^b k(x,t) B_i^n(t) dt$

Step4:
Compute $c_0, c_1, \cdots, c_n$, where $x_i, i = 1,2,3,\cdots,n$, $x_i \in [a,b]$

Example(1):
Consider the following Fredholm integral equation of the second kind:

$$u(x) = e^{-x} - \int_0^1 x e^t u(t) dt$$

Which has the exact solution

$$u(x) = e^{-x} - \frac{x}{2}$$

Here
$f(x) = e^{-x}$, $\lambda = -1$, $k(x,t) = xe^t$
When Bernstein polynomials algorithm is applied, Table(1) and Fig(1) presents the comparison between the approximate solutions using Bernstein polynomials method and exact values $u(x) = e^{-x} - \frac{x}{2}$ depending on least square error (L.S.E).

4. Conclusion

In this work, Bernstein polynomials method has been proved effectiveness in solving Fredholm integral equation of the second kind. From solving some numerical examples the following points have been identified:

1. This method can be used to solve the all kinds of linear Fredholm integral equation.
2. It is clear that using the Bernstein polynomial basis function to approximate when the n^{th} degree of Bernstein polynomial is increases the error is decreases.

Table (1) The results of Example(1) using **(BPFI)** algorithm.

x	Exact	BPFI Of degree n=2
0	1	1
0.1	0.8548	0.8590
0.2	0.7187	0.7241
0.3	0.5908	0.5955
0.4	0.4703	0.4730
0.5	0.3565	0.3568
0.6	0.2488	0.2467
0.7	0.1466	0.1428
0.8	0.0493	0.0452
0.9	-0.0434	-0.0463
1	-0.1321	-0.1316
L.S.E		1.0e-3

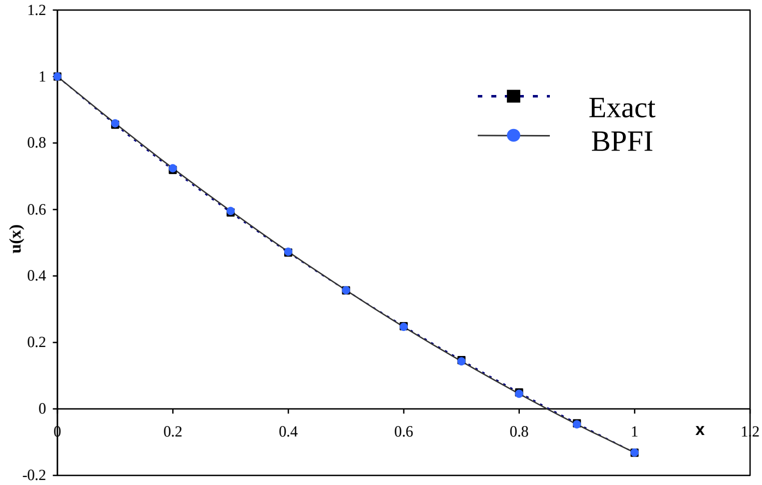

Fig.1 Approximation and Exact solution of Fredholm integral equation of Example1

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حل تقريبي لمعادلة فريدهولم التكاملية باستخدام طريقة متعددة حدود برنشتن

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الخلاصة:

في هذا البحث استخدمت طريقة متعددة حدود برنشتن التقريبية لإيجاد الحل التقريبي لمعادلة فريدهولم التكاملية من النوع الثاني. كما أن متعددة الحدود تستعمل بشكل كبير في الطرق الرياضية وذلك لبساطة تعريفها ولسهولة وسرعة الحسابات فيها. وتنوع دوالها.