(σ,τ) – (J,R) – DERIVATIONS ON JORDAN IDEALS

Ikram A. Saed*

Date of acceptance 16 /10 / 2008

Abstract:

Let R be an associative ring with center Z(R). A well known results proved by Bell and kappe concering derivations in prime rings have been extensively studied by many authors, several of these outhers extended these result for α - derivation like Yenigual and Argac and some of them extended these results for a (σ,τ) – derivations like M. Asharf.

The main purpose of this paper is to study the action of a (σ,τ) – (J,R) – derivation and a left (σ,τ) – (J,R) – derivation and (σ,τ) – (J,R) – derivation on Jordan ideals.

Keyword: Z(R): center of R, R: prime ring, d: derivation, δ: left derivation, F: generalized.

§ 1 Basic Concepts:

Definition 1.1: [3]

A ring R is called a prime if for any $a,b \in R$,

aRb = {0} , implies that either a = 0 or b = 0.

Definition 1.2: [4]

A ring R is called a semiprime ring if for any $a \in R$, aRa = {0} , implies that a = 0.

Definition 1.3: [3]

Let R be a ring. Define a Jordan product on R as follows

aob = ab + ba , for all $a,b \in R$.

Definition 1.4: [3]

An additive subgroup $A \subset R$ is called a Jordan subring of R if $a,b \in A$. implies that aob = ab + ba $\in$ A.

Definition 1.5: [3]

Let A be a Jordan subring of R and $J \subset A$ is an additive subgroup such that $a \in A$, $b \in J$ implies that ab + ba $\in$ J, then J is called a Jordan ideal of A.

Definition 1.6: [3]

A ring R is said to be n-torsion-free, where $n \neq 0$ is an integer such that whenever na = 0 with $a \in R$, then a = 0.

Definition 1.7: [3]

Let R be a ring. Define a lie product [,] on R as follows.

[x,y] = xy – yx , for all $x,y \in R$.

Properties 1.8: [8]

Let R be a ring, then for all $x,y,z \in R$, we have

(1) [x,yz] = y[x,z] + [x,y]z
(2) [xy,z] = x[y,z] + [x,z]y
(3) [x + y,z] =[x,z] + [y,z]
(4) [x,y + z] = [x,y] + [x,z]

Definition 1.9: [4]

Let R be a ring. An additive mapping d:R→R is called a derivation if d(xy) = d(x)y + xd(y), for all $x,y \in R$ and we say that d is a Jordan derivation if $d(x^2) = d(x)x + xd(x)$, for all $x \in R$.

*University of Technology

Definition 1.10: [8]

Let R be a ring. An additive mapping $\delta:R\rightarrow R$ is called a left derivation if

$\delta(xy) = x\delta(y) + y\delta(x)$, for all $x,y\in R$ and we say that δ is a Jordan left derivation if

$\delta(x^2) = 2x\delta(x)$ for all $x\in R$.

Definition 1.11: [8]

Let R be a ring. An additive mapping $d:R\rightarrow R$ is called a (σ,τ) – derivation where $\sigma,\tau:R\rightarrow R$ are two mappings of R, if $d(xy) = d(x)\,\sigma(y) + \tau(y)\,d(y)$, for all $x,y\in R$, and we say that d is a Jordan (σ,τ) – derivation if

$d(x^2) = d(x)\,\sigma(x) + \tau(x)\,d(x)$, for all $x\in R$.

Definition 1.12: [8]

Let R be a ring. An additive mapping $\delta:R\rightarrow R$ is called a left (σ,τ) – derivation where $\sigma,\tau:R\rightarrow R$ are two mapping of R,

if $\delta(xy) = \sigma(x)\,\delta(y) + \tau(y)\,\delta(x)$, for all $x,y\in R$ and we say that δ is a Jordan left (σ,τ) – derivation

if $\delta(x^2) = \sigma(x)\,\delta(x) + \tau(x)\,\delta(x)$, for all $x\in R$.

Definition 1.13: [6]

Let R be a ring. An additive mapping $F:R\rightarrow R$ is called a generalized (σ,τ) derivation associated with d, where $\sigma,\tau:R\rightarrow R$ are two mappings of R, if there exists a (σ,τ) – derivation $d:R\rightarrow R$ such that $F(xy) = F(x)\,\sigma(y) + \tau(x)\,d(y)$, for all $x,y\in R$.

§ 2 (σ,τ) – (J,R) – Derivations:

In this section first we will extend A.D. HAMDI, [5, Theorem 2.2.6] for a (σ,σ) - (J,R) derivation which acts as a homomorphism or as an anti-homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R, second we will generalize the above extension for a generalized (σ,σ) - (J,R) derivation. Finally we will extend the above result for (σ,τ) - (J,R) derivation which acts as a homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free orime ring R.

Now we introduce the following new definition which a generalize of definition 1.11.

Definition 2.1:

Let J be a Jordan ideal of a ring R. An additive mapping $d:R\longrightarrow R$ is called a (σ,τ) – (J,R) derivation where $\sigma,\tau:R\rightarrow R$ are two mappings of R, if

$d(xy) = d(x)\,\sigma(y) + \tau(x)\,d(y)$, for all $x\in J$, $y\in R$,

and we ay that d is a Jordan (σ,τ) – (J,R) derivation if

$d(a^2) = d(a)\,\sigma(a) + \tau(a)\,d(a)$ for all $a\in R$.

Example 2.2:

Let R be the ring of all 2x2 materices over commutative ring S of characteristic two.

Let $J = \left\{\begin{pmatrix} a & b \\ b & a\end{pmatrix} : a,b\in S\right\}$

It is clear that J is a Jordan ideal of R.

Define

$d:R\rightarrow R$, by $d\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0\end{pmatrix}$,

for all $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in R$

Let $\sigma,\tau:R\rightarrow R$ be two mappings, such that

$\sigma\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $\tau\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$,

For all $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in R$

Then d is (σ,τ) – (J,R) – derivation.

The following lemmas help us to prove the main theorems of this section:

Lemma 2.3: [8]

If R is a ring, J a nonzero Jordan ideal of R, then $2[R, R] J \subseteq J$ and $2J[R, R] \subseteq J$.

Lemma 2.4: [8]

Let R be a prime ring, J a nonzero Jordan ideal of R. if $a \in R$ and $aJ = \{0\}$ (or $Ja = \{0\}$) , then $a = 0$.

Lemma 2.5: [8]

Let R be a 2-trosion-free prime ring, J a nonzero Jordan ideal of R. if $aJb = \{0\}$ then $a = 0$ or $b = 0$.

Lemma 2.6: [8]

Let R be a 2-trosion-free prime ring, J a nonzero Jordan ideal of R. if J is a commutative Jordan ideal, then $J \subseteq Z(R)$.

Lemma 2.7:

Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R. Suppose that σ,τ are automorphisms of R. if R admits a (σ,τ) – (J,R) derivation d such that $d(J) = \{0\}$, then $d = 0$ or $J \subseteq Z(R)$.

Proof:

We have $d(u) = 0$, for all $u \in J$. This yields that $d(uor) = 0$, for all $u \in J$ and $r \in R$. Now using the fact that $d(u) = 0$, the above expression yields that

$\tau(u)\, d(r) + d(r)\, \sigma(u) = 0$, for all $u \in J$ and $r \in R$ …(1)

Replacing r by vr, $v \in J$ in (1) and using (1), we get

$\tau(u)\, d(vr) + d(vr)\, \sigma(u) = 0$

$\tau(u)\, [d(v)\, \sigma(r) + \tau(v)\, d(r)] + [d(v)\, \sigma(r) + \tau(v)\, d(r)]\, \sigma(u) = 0$

$\tau(u)\, \tau(v)\, d(r) + \tau(v)\, d(r)\, \sigma(u) = 0$

$\tau(u)\, \tau(v)\, d(r) + \tau(v)\, \tau(u)\, d(r) = 0$

$(\tau(u)\, \tau(v) + \tau(v)\, \tau(u))\, d(r) = 0$

$\tau(u) \circ \tau(v)\, d(r) = 0$, for all $u,v \in J$, $r \in R$.

Hence $[u \circ v]\tau^{-1} (d(r)) = 0$, for all $u,v \in J$ and $r \in R$.
This implies that $J\tau^{-1} (d(r)) = \{0\}$, for all $u,v \in J$ and $r \in R$.
Hence by lemma (2.4), we get

$d(r) = 0$, for all $r \in R$
Thus $d = 0$ on R

Now, we will prove the main theorems of this section.

Theorem 2.8:

Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R. If that σ is an automorphism of R and $d:R \rightarrow R$ is a (σ,σ) – (J,R) derivation then

(i) if d acts as a homomorphism on J, then either $d = 0$ on R or $J \subseteq Z(R)$.

(ii) if d acts as an anti-homomorphism on J, then either $d = 0$ on R or $J \subseteq Z(R)$.

Proof:

Suppose that $J \not\subseteq Z(R)$.

(i) if d acts as a homomorphism on J, then we have
$d(uv) = d(u)\, \sigma(v) + \sigma(u) d(v) = d(u)\, d(v)$, for all $u,v \in J$ …(1)
Replacing v by vw, $w \in J$ in (1), we get.
$d(u)\, \sigma(v)\, \sigma(w) + \sigma(u)\, (d(v)\, \sigma(w) + \sigma(v)\, d(w))$
$= d(u)\, (d(v)\, \sigma(w) + \sigma(v)\, d(w))$

Using (1), the above relation yields that
$(d(u) - \sigma(u))\, \sigma(v)\, d(w) = 0$, for all $u,v,w \in J$.

That is , σ^{-1} (d(u) – σ(u)) vσ^{-1} (d(w)) = 0 , for all u,v,w∈J.

and hence σ^{-1} (d(u) – σ(u)) Jσ^{-1} (d(w)) = {0},
For all u,v,w∈J. By Lemma (2.5), we get

either d(u) – σ(u) = 0 or d(w) = 0 , for all u,w∈J.
if d(w) = 0 , for all w∈J , then by using lemma (2.7) ,
we get d = 0 on R.
if d(u) – σ(u) = 0 , for all u∈J, then relation (1) implies that σ(u) d(v) = 0 , for all u,v∈J.
Now replace u by uw, to get σ(u) σ(w) d(v) = 0 ,
for all u,v,w∈J.
that is, uwσ^{-1} (d(v)) = 0 , for all u,w,v∈J.
and hence uJσ^{-1}(d(v)) = {0} , for all u,v∈J.

Thus by lemma (2.5), we get either u = 0 or d(v) = 0 ,
for all u,v∈J, But since J is a nonzero Jordan ideal of R, we find that d(v) = 0 , for all v∈J and hence by Lemma (2.7), we get the required result.

(ii) If d acts as an anti-homomorphism on J, then we have
d(uv) = d(u) σ(v) + σ(u) d(v) = d(v) d(u), for all u,v∈J..(2)
Replacing u by uv in (2), we get
(d(u) σ(v) + σ(u) d(v)) σ(v) + σ(u) σ(v) d(v)
= d(v) (d(u) σ(v) + σ(u) d(v))
Using (2), the above relation yields that.
σ(u) σ(v) d(v) = d(v) σ(u) d(v), for all u,v∈J ………(3)

Again replace u by wu, w∈J, in (3), we get
σ(w) σ(u) σ(v) d(v) = d(v) σ(w) σ(u) d(v), for all u,v,w∈J..(4)
in view of (3), the relation (4) yields that
[d(v), σ(w)] σ(u) d(v) = 0 , for all u,v,w∈J,
That is, σ^{-1}([d(v), σ(w)]) uσ^{-1} (d(v)) = 0 , for all u,v,w∈J
and hence σ^{-1} ([d(v), σ(w)]) Jσ^{-1} (d(v)) = {0} , for all v,w∈J.

By using lemma (2.5), we get either [d(v), σ(w)] = 0 or d(v), for all v,w∈J.
if d(v) = 0 , for all v∈J, then by using lemma (2.7),
we get d = 0 on R .
if [d(v), σ(w)] = 0, for all v,w∈J.
Replacing v by vw in the above relation, we get
0 = [d(vw), σ(w)]
= [d(v) σ(w) + σ(v) d(w), σ(w)]
= [d(v) σ(w), σ(w)] + [σ(v) d(w), σ(w)]
= σ(v) [d(w), σ(w)] + [σ(v), σ(w)] d(w)
for all v,w∈J, this implies that
σ(v) [d(w), σ(w)] + [σ(v), σ(w)] d(w) = 0 for all v,w∈J.. (5)

Replace v by v_1v, v_1∈J in (5), and using (5), to get
[σ(v_1), σ(w)] σ(v) d(w) = 0 , for all v,v_1,w∈J.
That is [v_1,w] vσ^{-1}(d(w)) = 0 , for all v,v_1,w∈J.
and hence [v_1,w] Jσ^{-1} (d(w)) = {0}, for all v_1,w∈J.
By lemma (2.5), we get either [v_1,w] = 0
or d(w) = 0 , for all v_1,w∈J.

Now let
J_1 = {w∈J/[v_1,w] = 0 , for all v_1∈J} and
J_2 = {w∈J/d(w) = 0 }
Clearly, J_1 and J_2 are additive proper subgroups of J whose union is J.
Since a group can not be the set theoretic union of two proper subgroups.
hence J=J_1 or J=J_2

if $J=J_1$, that is, $[v_1,w] = 0$, for all $v_1,w\in J$.
if follows that J is commutative, then by Lemma (2.6),
we get $J\subseteq Z(R)$, which is a contradiction.
On the other hand if $J=J_2$, we get then by lemma (2.7),
the required result.

Now we introduce the following new definition which a generalize of definition 2.1

Definition 2.9:

Let J be a Jordan ideal of a ring R, An additive mapping $F:R\rightarrow R$ is called a generalized (σ,τ) – (J,R) associated with d, where $\sigma,\tau:R\rightarrow R$ are two mappings of R, if there exists a (σ,τ) – (J,R) derivation $d:R\rightarrow R$ such that
$F(xy) = F(x)\,\sigma(y) + \tau(x)\,d(y)$, for all $x\in J$, $y\in R$.

Example 2.10:

Let
R= $\left\{\begin{pmatrix} x & y \\ z & w \end{pmatrix} : x, y, z, w \in N, \text{ where N is the ring of integers}\right\}$

be a ring of 2x2 matrices with respect to the usual addition and multiplication.
Let J= $\left\{\begin{pmatrix} x & y \\ y & x \end{pmatrix} : x, y \in N\right\}$
it is clear that J is a Jordan ideal of R.
Let $F:R\rightarrow R$, defined by $F\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 0 \\ z & 0 \end{pmatrix}$,
For all $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in R$, and let $d:R\rightarrow R$ defined by $d\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 0 \\ z & 0 \end{pmatrix}$, for all $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in R$.
Suppose that $\sigma,\tau:R\rightarrow R$ are two mappings such that
$\sigma\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$,
$\tau\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & w \end{pmatrix}$, for all $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in R$.
it is clear that d is a (σ,τ) – (J,R) derivation .
Then F is a generalized (σ,τ) – (J,R) derivation associated with d.

We generalize the theorem 2.8 as follows:

Theorem 2.11:

Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R. Suppose that σ is an automorphism of R and $F:R\rightarrow R$ is a generalized (σ,σ) – (J,R) derivation associated with a derivation d.
(i) if F acts as a homomorphism on J, then either d = 0 on R $J\subseteq Z(R)$.
(ii) if F acts as anti-homomorphism on J, then either d = 0 on R or $J\subseteq Z(R)$.

Proof:

Suppose that $J\not\subseteq Z(R)$
(i) if F acts as a homomorphism on J, then we have
$F(uv) = F(u)\,\sigma(v) + \sigma(u)d(v) = F(u)\,F(v)$, for all $u,v\in J$ …(1)
Replacing v by vw, $w\in J$ in (1), we get.
$F(u)\,\sigma(v)\,\sigma(w) + \sigma(u)\,(d(v)\,\sigma(w) + \sigma(v)\,d(w))$

$= F(u)\,(F(v)\,\sigma(w) + \sigma(v)\,d(w))$

using (1), the above relation yields that $(F(u) - \sigma(u))\,\sigma(v)\,d(w) = 0$, for all $u,v,w \in J$.
that is , $\sigma^{-1}(F(u) - \sigma(u))\,v\sigma^{-1}(d(w)) = 0$, for all $u,v,w \in J$.

and hence $\sigma^{-1}(F(u) - \sigma(u))\,J\sigma^{-1}(d(w)) = \{0\}$, for all $u,v,w \in J$.
hence by Lemma (2.5), we get either $F(u) - \sigma(u) = 0$ or $d(w) = 0$, for all $u,w \in J$.
if $d(w) = 0$, for all $w \in J$, then by using lemma (2.7) ,
we get $d = 0$ on R.
if $F(u) - \sigma(u) = 0$, for all $u \in J$, then relation (1) implies that $\sigma(u)\,d(v) = 0$, for all $u,v \in J$.
Now replace u by uw, to get
$\sigma(u)\,\sigma(w)\,d(v) = 0$, for all $u,v,w \in J$.
This implies that $uw\sigma^{-1}(d(v)) = 0$ and hence
$uJ\sigma^{-1}(d(v)) = \{0\}$, for all $u,v \in J$.

Thus by lemma (2.5), we get either $u = 0$
or $d(v) = 0$, for all $u,v \in J$.
But since J is a nonzero Jordan ideal of R, we find that
$d(v) = 0$, for all $v \in J$ and hence by Lemma (2.7), we get the required result.

(ii) If F acts as an anti-homomorphism on J, then we have
$F(uv) = F(u)\,\sigma(v) + \sigma(u)\,d(v) = F(v)\,F(u)$, for all $u,v \in J$..(2)
Replacing u by uv in (2), we get
$(F(u)\,\sigma(v) + \sigma(u)\,d(v))\,\sigma(v) + \sigma(u)\,\sigma(v)\,d(v)$
$= F(v)\,(F(u)\,\sigma(v) + \sigma(u)\,d(v))$
Using (2), the above relation yields that.
$\sigma(u)\,\sigma(v)\,d(v) = F(v)\,\sigma(u)\,d(v)$, for all $u,v \in J$ ………(3)

Again replace u by wu, $w \in J$, in (3), to obtain
$\sigma(w)\,\sigma(u)\,\sigma(v)\,d(v) = F(v)\,\sigma(w)\,\sigma(u)\,d(v)$,
for all $u,v,w \in J$..……… ……(4)

in view of (3), the relation (4) yields that
$[F(v), \sigma(w)]\,\sigma(u)\,d(v) = 0$, for all $u,v,w \in J$,
This implies that is, $\sigma^{-1}([F(v), \sigma(w)])\,u\sigma^{-1}(d(v)) = 0$,
for all $u,v,w \in J$ and hence
$\sigma^{-1}([F(v), \sigma(w)])\,J\sigma^{-1}(d(v)) = \{0\}$, for all $v,w \in J$.

By using Lemma (2.5), we get either $[F(v), \sigma(w)] = 0$ or $d(v)$, for all $v,w \in J$.
if $d(v) = 0$, for all $v \in J$, then by using Lemma (2.7),
we get $d = 0$ on R .
if $[F(v), \sigma(w)] = 0$, for all $v,w \in J$.
Replacing v by vw in the above relation, we get
$0 = [F(vw), \sigma(w)]$
$= [F(v)\,\sigma(w) + \sigma(v)\,d(w), \sigma(w)]$
$= [F(v)\,\sigma(w), \sigma(w)] + [\sigma(v)\,d(w), \sigma(w)]$
$= \sigma(v)\,[d(w), \sigma(w)] + [\sigma(v), \sigma(w)]\,d(w)$
for all $v,w \in J$, this implies that
$\sigma(v)\,[d(w), \sigma(w)] + [\sigma(v), \sigma(w)]\,d(w) = 0$
for all $v,w \in J$..………… (5)

Now, replace v by v_1v, $v_1 \in J$ in (5), and using (5), to get
$[\sigma(v_1), \sigma(w)]\,\sigma(v)\,d(w) = 0$, for all $v,v_1,w \in J$.
That is, $[v_1,w]\,v\sigma^{-1}(d(w)) = 0$, for all $v,v_1,w \in J$.
and hence $[v_1,w]\,J\sigma^{-1}(d(w)) = \{0\}$, for all $v_1,w \in J$.
By lemma (2.5), we get either
$[v_1,w] = 0$ or $d(w) = 0$, for all $v_1,w \in J$.
Now let
$J_1 = \{w \in J/[v_1,w] = 0$, for all $v_1 \in J\}$

and

$J_2 = \{w \in J / d(w) = 0\}$

Clearly, J_1 and J_2 are additive proper subgroups of J whose union is J.

Since a group can not be the set theoretic union of two proper subgroups, hence $J=J_1$ or $J=J_2$.

if $J=J_1$, that is, $[v_1,w] = 0$, for all $v_1,w \in J$.

if follows that J is commutative, so by Lemma (2.6), we get $J \subseteq Z(R)$, which is a contradiction on the other hand if $J=J_2$,

then by lemma (2.7), we get the required result.

In the following theorem our objective is to extend theorem 2.8 to a (σ,τ)–(J,R) derivation of a 2-torsion-free prime ring R which acts as a homomorphism on a Jordan ideal J of R.

Theorem 2.12:

Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R. Suppose that σ,τ are automorphism of R and $d:R \rightarrow R$ is a (σ,τ) – (J,R) derivation. If d acts as a homomorphism on J, then d = 0 on R.

Proof:

Since d acts as a homomorphism on J, then

we have

$d(uv) = d(u)\,\sigma(v) + \tau(u)\,d(v) = d(u)\,d(v)$,

for all $u,v \in J$ ……………………(1)

Replacing v by vw, $w \in J$ in (1), we get

$d(u)\,\sigma(v)\,\sigma(w) + \tau(u)\,(d(v)\,\sigma(w) + \tau(v)\,d(w))$

$= d(u)\,(d(v)\,\sigma(w) + \tau(v)\,d(w))$

using (1), the above relation yields that

$(d(u) - \tau(u))\,\tau(v)\,d(w) = 0$, for all $u,v,w \in J$,

This implies that $\tau^{-1}(d(u) - \tau(u))v\,\tau^{-1}(d(w)) = 0$,

for all $u,v,w \in$ and hence

$\tau^{-1}(d(u) - \tau(u))\,J\tau^{-1}(d(w)) = \{0\}$, for all $u,w \in J$.

By using Lemma (2.5), we get either $d(u) - \tau(u) = 0$ or $d(w) = 0$, for all $u,w \in J$.

if $d(w) = 0$, for all $w \in J$, then by lemma (2.7), we get d = 0 on R.

if $d(u) - \tau(u) = 0$, for all $u \in J$, we get $d(u) = \tau(u)$, for all $u \in J$.

Then the relation (1) implies that

$d(u)\,\sigma(v) + d(u)\,d(v) = d(u)\,d(v)$, for all $u,v \in J$,

and this implies that

$d(u)\,\sigma(v) = 0$, for all $u,v \in J$.

Replacing v by vw, $w \in J$, we get

$d(u)\,\sigma(v)\,\sigma(w) = 0$, for all $u,v,w \in J$,

that is, $\sigma^{-1}(d(u))\,vw = 0$, for all $u,v,w \in J$, and hence

$\sigma^{-1}(d(u))\,Jw = \{0\}$, for all $u,v,w \in J$.

Hence by lemma (2.5), we get either $d(u) = 0$ or $w = 0$, for all $u,w \in J$.

Since J is a nonzero Jordan ideal of R we have $d(u) = 0$,

for all $u \in J$, then by lemma a (2.7), we get d = 0 on R.

§ 3 Left (σ,τ) – (J,R) Derivations:

We will study the behaviour of a left (σ,τ) – (J,R) derivation which acts either as a homomorphism or as an anti-homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring.

Now we introduce the following new definition which a generalize of definition 1.12

Definition 3.1:

Let J be a Jordan ideal of a ring R. An additive mapping $\delta:R \rightarrow R$ is called a left (σ,τ) – (J,R) derivation where $\sigma,\tau:R \rightarrow R$ are two mappings of R, if

$\delta(xy) = \sigma(x)\ \delta(y) + \tau(y)\ \delta(x)$, for all $x\in J$, $y\in R$ and we say that δ is a Jordan left (σ,τ) – (J,R) derivation if $\delta(x^2) = \sigma(x)\ \delta(x) + \tau(x)\ \delta(x)$, for all $x\in R$.

Example 3.2:

$$\text{Let } R=\left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x,\ y\in N\right\}$$

where N is the ring of integers be a ring of 2x2 matrices with respect to the usual addition and multiplication.

$$\text{Let } J=\left\{\begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix} : y\in N\right\}$$

It is clear that J is a Jordan ideal of R.

Let $\delta:R\rightarrow R$, defined by $\delta\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$, for all $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\in R$.

and let $\sigma,\tau:R\rightarrow R$ be two mappings, such that

$$\sigma\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} x & -y \\ 0 & 0 \end{pmatrix},\ \tau\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & -x \\ 0 & 0 \end{pmatrix}.$$

For all $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\in R$.

Then δ is a left (σ,τ) – (J,R) derivation.

The following lemmas help us to prove the main theorems of this section:

Lemma 3.3: [8]

Let R be a 2-torsion-free ring, J a Jordan ideal and a subring of R. Suppose that σ is an endomorphism of R and $\delta:R\rightarrow R$ is an additive mapping satisfying $\delta(u^2) = 2\sigma(u)\ \delta(u)$, for all $u\in J$, then

(i) $\delta(uv + vu) = 2\sigma(u)\ \delta(v) + 2\sigma(v)\ \delta(u)$ for all $u,v\in J$.

(ii) $\delta(uvu) = \sigma(u^2)\ \delta(v) + 3\ \sigma(u)\ \sigma(v)\ \delta(u) - \sigma(v)\ \sigma(u)\ \delta(u)$, for all $u,v\in J$.

(iii) $\delta(uvw + wvu) = (\sigma(u)\ \sigma(w) + \sigma(w)\ \sigma(u))\ \delta(w) + 3\ \sigma(u)\ \sigma(v)\ \delta(w) + 3\ \sigma(w)\ \sigma(v)\ \delta(u) - \sigma(v)\ \sigma(u)\ \delta(w) - \sigma(v)\ \sigma(w)\ \delta(u)$, for all $u,v\in J$.

(iv) $[\sigma(u), \sigma(v)]\ \sigma(u)\ \delta(u) = \sigma(u)\ [\sigma(u), \sigma(v)]\ \delta(u)$, for all $u,v\in J$.

(v) $[\sigma(u), \sigma(v)]\ (\delta(uv) - \sigma(u)\ \delta(u) - \sigma(v)\ \delta(u) = 0$ for all $u,v\in J$.

Lemma 3.4: [8]

Let R be a 2-torssion-free prime ring, J a Jordan ideal and a subring of R. Suppose that σ is an endomorphism of R and $\delta:R\rightarrow R$ is an additive mapping satisfying

$\delta(u^2) = 2\sigma(u)\ \delta(u)$, for all $u\in J$, then

(i) $[\sigma(u),\sigma(v)]\ \delta([u,v]) = 0$, for all $u,v\in J$.

(ii) $(\sigma(u^2)\ \sigma(v) - 2\sigma(u)\ \sigma(v)\ \sigma(u)\ \sigma(u^2))\ \delta(v) = 0$, for all $u,v\in J$.

Lemma 3.5: [8]

Let R be a 2-torssion-free prime ring, J a Jordan ideal and a subring. Suppose that σ is an endomorphism of R and $\delta:R\rightarrow R$ is an additive mapping satisfying

$\delta(u^2) = 2\sigma(u)\ \delta(u)$, for all $u\in J$, then

(i) $\delta(u^2v) = \sigma(u^2)\ \delta(v) + (\sigma(u)\ \sigma(v) + \sigma(v)\ \sigma(v))\ \delta(u) + \sigma(u)\ \delta([u,v])$, for all $u,v\in J$.

(ii) $\delta(vu^2) = \sigma(u^2)\ \delta(v) + (3\sigma(v)\ \sigma(u) - \sigma(u)\ \sigma(v))\ \delta(u) - \sigma(u)\ \delta([u,v])$, for all $u,v\in J$.

Lemma 3.6: [8]

Let R be a 2-torssion-free prime ring, J a Jordan ideal and a subring of R. Such that $[u,v]^2 = 0$, for all $u,v \in J$. Then J is commutative and hence central.

In the next theorem, we attempt to generalize the above mentioned result for Jordan left (σ,τ)- (J,R) derivation which acts a Jordan ideal and a subring J of R.

Theorem 3.7: [8]

Let R be a 2-torssion-free prime ring, J a Jordan ideal and a subring. Suppose that σ is an automorphism of R and δ:R→R is an additive mapping satisfying $\delta(u^2) = 2\sigma(u)\,\delta(u)$ for all $u \in J$, then either $J \subseteq Z(R)$ or $\delta(J) = \{0\}$.

Corollary 3.8: [8]

Let R be a 2-torssion-free prime ring, if δ:R→R is a nonzero additive mapping satisfying $\delta(x^2) = 2x\delta(x)$ for all $x \in R$, then R is commutative.

Now, let us take the following theorem:

Theorem 3.9: [8]

Let R be a 2-torssion-free prime ring, J a Jordan ideal and a subring of R. Suppose that σ is an automorphism of R and δ:R→R is a left (σ,σ) – (J,R) derivation

(i) if δ acts as a homomorphism on J, then δ = 0 on R.
(ii) if δ acts as anti-homomorphism on J, then δ = 0 on R.

In the following theorem we will extend the above theorem to a left (σ,τ) – (J,R) derivation of a 2-torsion-free prime ring R which acts as a homomorphism or as an anti-homomorphism on a nonzero Jordan ideal and a subring J of R.

Theorem 3.10:

Let R be a 2-torssion-free prime ring, J a nonzero Jordan ideal and a subring of R. Suppose that σ,τ is are automorphism of R and δ:R→R is a left (σ,τ) – (J,R) derivation

(i) if δ acts as a homomorphism on J, then either δ = 0 on R or $J \subseteq Z(R)$.
(ii) if δ acts as anti-homomorphism on J, then either δ = 0 on R or $J \subseteq Z(R)$.

Proof:

Suppose that $J \not\subseteq Z(R)$.

(i) if δ acts as a homomorphism on J, then we have

$\delta(uv) = \delta(u)\,\delta(v) = \sigma(u)\,\delta(v) + \tau(v)\,\delta(u)$,

for all $u,v \in J$ ………………(1)

replacing u by uv in (1), we get

$(\sigma(u)\,\delta(v) + \tau(v)\,\delta(u))\,\delta(v) = \sigma(u)\,\sigma(v)\,\delta(v) + \tau(v)\,\delta(u)\,\delta(v)$,

for all $u,v \in U$.

This implies that

$\sigma(u)\,\delta(v)\,\delta(v) = \sigma(u)\,\sigma(v)\,\delta(v)$, for all $u,v \in J$,

This implies that $\sigma(u)\,(\delta(v) - \sigma(v))\,\delta(v) = 0$,

for all $u,v \in J$

and hence $\sigma(J)\,(\delta(v) - \sigma(v))\,\delta(v) = \{0\}$, for all $v \in J$.

Since σ is an automorphism of R and J is a nonzero Jordan ideal of R, σ(J) is also a nonzero Jordan ideal of R.

Application of Lemma (2.4) yields that

$(\delta(v) - \sigma(v))\,\delta(v) = 0$, for all $v \in J$

and hence $\delta(v^2) = \sigma(v)\,\delta(v)$, for all $v \in J$.

Since δ is a left (σ,τ) – (J,R) derivation, we have

$\sigma(v)\,\delta(v) + \tau(v)\,\delta(v) = \sigma(v)\,\delta(v)$, for all $v \in J$,

this implies that $\tau(v)\,\delta(v) = 0$, for all $v \in J$,

on linearzing the latter relation, we find that

$0 = \tau(v + u)\,\delta(v + u)$

$= (\tau(v) + \tau(u))\,(\delta(v) + \delta(u))$

$= \tau(v)\,\delta(v) + \tau(v)\,\delta(u) + \tau(u)\,\delta(v) + \tau(u)\,\delta(u)$

$= \tau(v)\,\delta(u) + \tau(u)\,\delta(v)$, for all $u,v \in J$. ……(2)

Replacing u by vu in (2), we get

$0 = \tau(v)\,\delta(v)\,\delta(u) + \tau(v)\,\tau(u)\,\delta(v)$

$= \tau(v)\,\tau(u)\,\delta(v)$, for all $u,v \in J$,

That is, $vu\tau^{-1}(\delta(v)) = 0$, for all $u,v \in J$,

and hence $vJ\tau^{-1}(\delta(v)) = \{0\}$, for all $u,v \in J$.

By Lemma (2.5), we get either $v = 0$ or

$\delta(v) = 0$, for all $v \in J$.

Since J is a nonzero Jordan ideal of R and τ is an automorphism of R, we get

$\delta(v) = 0$, for all $v \in J$.

Replacing v by vor, $r \in R$ in the above relation, we have

$0 = \delta(v \circ r) = \delta(vr + rv)$

$= \delta(vr) + \delta(rv)$

$= \sigma(v)\,\delta(r) + \tau(r)\,\delta(v) + \sigma(r)\,\delta(v) + \tau(v)\,\delta(r)$

$= \sigma(v)\,\delta(r) + \tau(v)\,\delta(r)$

$= (\sigma(v) + \tau(v))\,\delta(r)$, for all $v \in J$ and $r \in R$.

Hence we get $(\sigma(J) - \tau(J))\,\delta(r) = \{0\}$, for all $r \in J$.

Since σ,τ are automorphisms of R and J is a nonzero Jordan ideal of R, we get $\sigma(J)$ and $\tau(J)$ are a nonzero Jordan ideals of R, and hence we get $\sigma(J) + \tau(J)$ is a nonzero Jordan ideal of R , thus by lemma (2.4) we get $\delta(r) = 0$, for all $r \in R$,

this implies that is , $\delta = 0$ on R.

(ii) If δ acts as an anti-homomorphism on J, then we have

$\delta(uv) = \delta(v)\,\delta(u) = \sigma(u)\,\delta(v) + \tau(v)\,\delta(u)$,

for all $u,v \in J$ ………………(3)

Replacing v by uv in (3) , we get

$\delta(uv)\,\delta(u) = \sigma(u)\,\delta(v)\,\delta(u) + \tau(v)\,\delta(u)\,\delta(u)$

$= \sigma(u)\,\delta(v)\,\delta(u) + \tau(u)\,\tau(v)\,\delta(u)$,

for all $u,v \in J$,

or equivalently .

$\tau(v)\,\delta(u)\,\delta(u) = \tau(u)\,\tau(v)\,\delta(u)$, for all $u,v \in J$ ………(4)

Replacing v by tv , $t \in J$ in (4), we get

$\tau(t)\,\tau(v)\,\delta(u)\,\delta(u) = \tau(u)\,\tau(t)\,\tau(v)\,\delta(u)$,

for all $u,v,t \in J$ …………(5)

in view of (4) , the relation (5) yields that

$[\tau(u) , \tau(t)]\,\tau(v)\,\delta(u) = 0$, for all $u,v,t \in J$.

This implies that $[u,t]\,v\,\tau^{-1}(\delta(u)) = 0$, for all $u,v,t \in J$

and hence $[u,t]\,J\,\tau^{-1}(\delta(u)) = \{0\}$, for all $u,t \in J$

By Lemma (2.5), we get either $[u,t] = 0$ or

$\delta(u) = 0$, for all $u,t \in J$.

Now let

$J_1 = \{u \in J \,/\, [u,t] = 0$, for all $t \in J\}$

and

$J_2 = \{u \in J \,/\, \delta(u) = 0\}$

Clearly, J_1 and J_2 are additive proper subgroups of J whose union is J.

Since a group can not be the set theoretic union of two proper subgroups, hence $J = J_1$ or $J = J_2$.

If $J = J_1$, that is , $[u,t] = 0$, for all $u,t \in J$,

This yields that J is commutative, and hence by lemma (2.6) $J \subseteq Z(R)$, which is a contradiction.

Hence, we have remaining possibility that
$\delta(u) = 0$, for all $u \in J$.
Replace u by uor , $r \in R$, in the above relation, we get

$$0 = \delta(uor) = \delta(ur + ru) = \delta(ur) + \delta(ru)$$
$$= \sigma(u)\,\delta(r) + \tau(r)\,\delta(u) + \sigma(r)\,\delta(u) + \tau(u)\,\delta(r)$$
$$= \sigma(u)\,\delta(r) + \tau(u)\,\delta(r)$$
$$= (\sigma(u) + \tau(u))\,\delta(r) ,$$

for all $u \in J$ and $r \in R$
Hence, we have $(\sigma(J) - \tau(J))\,\delta(r) = \{0\}$, for all $r \in R$.
By a similar way in part (i) , we can get our result.

REFERENCES:

1. Ashraf, M. 2005. On left (θ,ϕ) – derivations of prime rings. Archivum Mathematicum (Vron). Tomus 41:157 – 166.
2. Bell, H. E. and Kappe. L. C. 1989. Rings in which derivations satisfy certain algebraic conditions Acta Math. Hungar 53: 339 – 346.
3. Herestein, I. N. 1965. Topics in ring theory. The University of Chicago Press, ed. No.1, Chicago, PP. 74.
4. Herestein, I. N. 1976. Rings with involution. The University of Chicago Press, ed. No.2, Chicago, PP. 34.
5. HAMDI,A. D. 2007. (σ,τ) – Derivations On prime rings. M. Sc. Thesis, Baghdad University.
6. Jung, Y. S. and Park. K. H. 2006. On generalized (α,β) – derivations and commutivity in prime rings. Bull Korean Math. Soc. 43: 10 – 106.
7. Yengul, S. A. and Argac. N. A. 1994. On prime and semiprime rings with α – derivations. Turkish J. Math. 18: 280 – 284.
8. Zaid, S. M. A., Ashraf. M. and Ali. S. 2004. On Jordan ideals and left (θ,θ) – derivations in prime rings. IJMMS 37: 1957 – 1964.

مشتقات – (J,R) – (σ,τ) على مثاليات جوردان

إكرام احمد سعيد*

*الجامعة التكنولوجية

الخلاصة:

لتكن R حلقة تجميعية مركزها Z(R). نتائج Bell و Kappe المعروفة والمتعلقة بالمشتقات على الحلقات الاولية درست بتوسع من قبل العديد من الباحثين، بعض هؤلاء الباحثين عمموا هذه النتائج لمشتقة - α مثلYenigual و Arguc والبعض الاخر عمموا هذه النتائج لمشتقة - (σ,τ) مثل M.Asharf.

ان الهدف الرئيسي لهذا البحث هو دراسة تأثير المشتقة – (J,R) – (σ,τ) والمشتقة من اليسار – (J,R) – (σ,τ) على مثاليات جوردان.