SOME RESULTS OF PRIME AND SEMIPRIME RINGS WITH DERIVATIONS

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Abstract

The main purpose of this paper is to study prime and semiprime rings admitting a derivation d satisfying new conditions when d acts as a homomorphism on non-zero ideal. we give some results about that.

Keywords: prime and semiprime rings, derivations, homomorphism, central ideal

Introduction

Long ago Herstein [1] proved that if R is a prime ring of characteristic not 2 which admits a non-zero derivation such that d(x)d(y) = d(y)d(x) for all x, y $\in$ R , then R is commutative. H.E.Bell and W.S.Martindale [2] proved that let R be a prime ring and U a non–zero right ideal. If R admits a non-zero derivation d such that [x, d(x)] is central for all x $\in$ U, then R is commutative. M. N. Daif [3] proved that, let R be a semprime ring and d a derivation of R with $d^3 \neq 0$. If [d(x), d(y)] = 0 for all x,y $\in$ R , then R contains a non-zero central ideal. M. N. Daif and H. E. Bell [4] proved that, let R be a semiprime ring admitting a derivation d for which either xy + d(xy) = yx + d(yx) for all x, y $\in$ R or xy – d(xy) = yx – d(yx)for all x, y $\in$ R, then R is commutative. V. De Filippis [5] proved that, when R be prime ring let d a non-zero derivation of R, U $\neq$ (0) a two sided ideal of R, such that d([x, y]) = [x, y] for all x, y $\in$ U, then R is commutative. A. H. Majeed [6] proved that, let R be a prime ring and U be a non-zero ideal of R. If R admits derivations d and g with d (U) $\neq$ {0}, such that d(xy) = g(yx) for all x, y $\in$ U, then R is commutative. Recently Mehsin Jabel [7] proved that, let R be a 2-torsion free semiprime ring and U a non-zero ideal of R. If R admits a non-

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zero derivation d such that d(xoy) $\mp$ d([x,y]) $\in$ Z(R) for all x, y $\in$ U, then R contains a non-zero central ideal. We write xoy = xy+ yx. Our purpose is to study semiprime rings and prime rings admitting a derivation d satisfying new conditions when d acts as homomorphism on non-zero ideal.

Preliminaries

Throughout this paper, R denoted a semiprime ring if a Ra = (0), with a $\in$ R implies a = 0, and called a prime ring if a Rb = (0), a, b, $\in$ R, implies that a =0 or b=0. A ring R is said to be n-torsion free, where n $\neq$ 0 is an integer, if whenever nx = 0, with x $\in$ R then x = 0. If U is a non empty subset of R, then the centralizer of U in R, denoted by C_R (U), is defined by : C_R (U) = {a $\in$ R\ax=xa for all x $\in$ U}. If a $\in$ C_R (U) we say that a centralizes U. An additive map d from R to R is called a derivation if d(xy)= d(x)y+ xd(y)for all x, y $\in$ R. Let U be a subset of R, a map d: R $\longrightarrow$ R is said to centralizing on U if [x, d(x)] $\in$ Z(R) for all x $\in$ U. and is said to skw-centrlaizing on U if d(x)x+ xd(x) $\in$ Z(R) for all x $\in$ U. We say a derivation d acts as a homomorphism on U if d(xy)= d(x) d(y) for all x, y $\in$ U.

Let U be an ideal of R, then U is a central ideal if xy=yx for all x, $\in$ U , y $\in$ R, and U is commutative ideal, if xy= yx for all x, y $\in$ U . It is clear that any central ideal is commutative. We write [x, y]= xy-yx and note that important identities [x,yz] =y[x,z]+[x,y]z and [xy,z] = x[y,z]+[x,z]y. To achieve our purposes, we mention the following results.

Lemma 1[9]

Let n be a fixed integer, let R be n!-torsion free semiprime ring and U be a non-zero left ideal of R. If R admits a derivation d which is non-zero on U and n centralizing on U, then R contains a non-zero central ideal.

Lemma 2[10:Lemma 3]

If the prime ring R contains a commutative non-zero right ideal, then R is commutative.

Lemma 3[11:Main Theorem]

Let R be a semiprime ring, d a non zero derivation of R, and U a non-zero left ideal of R. If for some positive integers t_0, t, ..., t_n and all

$x \in U$, the identity $[[\ldots[[d(x^{t_0}), x^{t_1}], x^{t_2}], \ldots], x^{t_n}]=0$ holds, then either d(U) =0 or else d(U) and d(R)U are contained in a non-zero central ideal of R. In particular when R is a prime ring, R is commutative

The Main Results

The main results of this paper contain three sections.

3.1-On Semiprime Rings

Theorem 3.1.1

Let R be a 2- torsion free semiprime ring and U a non-zero ideal of R, R admitting a non-zero derivation d to satisfying $[d^2(x), x]=0$ for all $x \in U$. If d acts as a homomorphism on U, then R contains a non zero central ideal.

Proof: We have the following relation $[d^2(x),x] = 0$ for all $x \in U$. Replacing x by xy, we obtain $[d^2(x)y,xy]+2[d(x)d(y), xy] +[xd^2(y), xy]=0$ for all $x,y \in U$. Then we have $d^2(x)[y, xy] + [d^2(x), xy]y +2[d(x)d(y), xy] +x[d^2(y), xy] +[x,xy]d^2(y) = 0$ for all $x, y \in U$. $d^2(x)[y,x]y+x[d^2(x),y]y+[d^2(x),x]y^2 +2[d(x)d(y),xy]+x^2[d^2(y),y] +x[d^2(y),x]y+x[x, y]d^2(y) = 0$ for all $x, y \in U$. Now replacing y by x, we obtain $x[d^2(x),x]x+[d^2(x), x]x^2 +2[d(x)^2, x^2]+x^2 [d^2(x), x]+ x[d^2(x), x]x = 0$ for all $x \in U$. According to the relation $[d^2(x), x]= 0$, we obtain $2[d(x)^2, x^2] = 0$ for all $x \in U$. Since R is 2-torsion free, we have $[d(x)^2, x^2] = 0$ for all $x \in U$. Since d acts as homomorphism on U, we obtain $[d(x^2), x^2] = 0$ for all $x \in U$. Thus by Lemma (3), we have either d(U) = 0 or else d(U) and d(R)U are contianed in non-zero central ideal of R. Since d is a non-zero of U, then R contains a non-zero central ideal. This complete the proof.

Theorem 3. 1. 2

Let R be a 2-torsion free semiprime ring and U a non-zero ideal of R. R admitting a non-zero derivation d to satisfying $[d^2(x),x^2] = 0$ for all $x \in U$. If d acts as a homomorphism on U, then R contains a non-zero central ideal.

Proof: We have $[d^2(x),x^2] = 0$ for all $x \in U$. Replacing x by xy, we obtain: $[d^2(x), x^2y^2]+ 2[d(x)d(y), x^2y^2]+ [xd^2(y), x^2y^2] = 0$ for all $x \in U$. Then $d^2(x)[y,x^2y^2]y+[d^2(x),x^2y^2]y+2[d(x)d(y),x^2y^2]+x[d^2(y),x^2y^2]+[x,x^2y^2]d^2(y) = 0$ for all $x,y \in U$.

$d^2(x)[y,x^2]y^2+x^2[d^2(x),y^2]y+[d^2(x),x^2]y^3+$ $2[d(x)d(x),x^2y^2]+x^3[d^2(y),y^2]+$ $x[d^2(y),x^2]y^2+x^2[x,y^2]d^2(y)=0$ for all $x,y \in U$.

Replacing y by x and according to the relation $[d^2(x),x^2]=0$, we obtain $2[d(x)^2,x^4]=0$ for all $x \in U$. Since R is 2–torsion free semiprime we get $[d(x)^2,x^4]=0$ for all $x,y \in U$. Then Since d acts as a homomorphism on U, we obtain $[d(x^2),x^4]=0$ for all $x,y \in U$. We complete the proof by same method in Theorem 3.1.1

Theorem 3.1.3

Let R be a 2–torsion free semiprime ring and U a non-zero ideal of R. R admitting a non-zero derivation d to satisfying $[d^2(x^2),x]=0$ for all $x \in U$. If d acts as a homomorphism on U, then R contains a non-zero central ideal.

Proof: We have the relation $[d^2(x^2),x]=0$ for all $x \in U$. Then $[d^2(x)x,x]+2[d(x)^2,x]+[xd^2(x),x]=0$ for all $x \in U$.

$[d^2(x),x]x+2[d(x)^2,x]+x[d^2(x),x]=0$ for all $x \in U$. Thus we get

$[d^2(x),x^2]+2[d(x)^2,x]=0$ for all $x \in U$.

Replacing x by x^2, we obtain

$[d^2(x^2),x^4]+2[d(x^2)^2,x^2]=0$ for all $x \in U$. Then

$x[d^2(x^2),x^3]+[d^2(x^2),x]x^3+2[d(x^2)^2,x^2]=0$ for all $x \in U$. According to the relation $[d^2(x^2),x]=0$, we obtain

$x[d^2(x^2),x^3]+2[d(x^2)^2,x^2]=0$ for all $x \in U$. Then

$x^2[d^2(x^2),x^2]+x[d^2(x^2),x]x+2[d(x^2)^2,x^2]=0$ for all $x \in U$.

Now according to the relation $[d^2(x^2),x]=0$, we have

$x^2[d^2(x^2),x^2]+x[d^2(x^2),x]x=0$ for all $x \in U$. Thus we obtain

$2[d(x^2)^2,x^2]=0$ for all $x \in U$. Since R is 2-torsion free semiprime, we get $[d(x^2)^2,x^2]=0$ for all $x \in U$.

Since d acts as a homomorphism on U, we obtain

$[d(x^4),x^2]=0$ for all $x \in U$.

Now we complete the proof by same method in theorem 3.1.1

Theorem 3.1.4

Let R be a 2-torsion free semiprime ring and U a non-zero ideal of R. R admitting a non-zero derivation d to satisfying $[d^2(x^2),x^2]=0$ for all $\in U$. If d acts as a homomorphism on U, then R contains a non-zero central ideal.

Proof: We have $[d^2(x^2), x^2]=0$ for all $x \in U$. Then

$[d^2(x)x, x^2]+2[d(x)^2,x^2]+[xd^2(x),x^2]=0$ for all $x \in U$.

$[d^2(x),x^2]x+x[d^2(x),x^2]+2[d(x)^2,x^2] = 0$ for all $x \in U$. Replacing x by x^2, we obtain

$[d^2(x^2),x^4]x^2+x^2[d^2(x^2),x^4]+2[d(x^2)^2,x^4]$ =0 for all $x \in U$. Then $x^2[d^2(x^2),x^2]x^2+[d^2(x^2),x^2]x^4+x^4[d^2(x^2), x^2]+x^2[d^2(x^2),x^2]x^2+2[d(x^2)^2,x^4]=0$ for all $x \in U$. According to the relation $[d^2(x^2),x^2]=0$, we get $2[d(x^2)^2,x^4]=0$ for all $x \in U$.

Since R is 2–torsion free semiprime ring, we get $[d(x^2)^2,x^4]=0$ for all $x \in U$.

Since d acts as a homomorphism, we obtain $[d(x^4), x^4]=0$ for all $x \in U$.

Now we complete the proof by same method in Theorem 3.1.1.

3.2-On Prime Rings

Theorem 3.2.1

Let R be a prime ring and U a non-zero ideal of R .R admitting a derivation d to satisfying $[d(x), d(y)] =[x, y]$ for all $x, y \in U$. If d acts as a homomorphism on U, then R is commutative.

Proof: When we have, $d \neq 0$, then

(1) $[d(x),d(y)]=[x, y]$ for all $x,y \in U$. Replacing x by x t , we obtain

$[d(x)t,d(y)]+[xd(t),d(y)]=[xt, y]$ for all $x ,y, t \in U$.

$d(x)[t,d(y)]+[d(x),d(y)]t+x[d(t),d(y)]+[x, d(y)]d(t)=x[t, y]+[x, y]t$ for all $x, y, t \in U$. According to (1) we obtain

(2) $d(x)[t, d(y)]+[x, d(y)]d(t)=0$ for all $x, y, t \in U$.

Replacing t and y by x , we obtain

(3) $d(x)[x,d(x)]+[x,(x)]d(x)=0$ for all $x \in U$. Then

(4) $[x, d(x)^2]=0$ for all $x \in U$. Since d act as a homomorphism, then,we get

(5) $[x, d(x^2)]=0$ for all $x \in U$. By Lemmas (1) and (2) we obtain ,R is commutative .

Now, we suppose that d=0, we obtain $[x, y]=0$ for all $x, y \in U$. Then by Lemma (2) , R is commutative .

Theorem 3.2.2

Let R be a prime ring with char.$\neq 2$ and U anon- zero ideal of R.

If R admitting a derivation d satisfying $[d^2(x), d^2(y)]=[x, y]$ for all $x, y \in U$. Then R is commutative.

Proof: We suppose first that, $d \neq 0$, then

(6) $[d^2(x),d^2(y)]=[x, y]$ for all $x, y \in U$. The linearzation (i.e. putting x+y for x) in (6) , we obtain

(7) $[d^2(x),d^2(y)]=[x^2,y]+[xy, y]+[yx, y]$ for all $x, y \in U$.

According to (6), we have

(8) $[x, y]=[x^2,y]+[x,y^2]$ for all $x, y \in U$. Then

(9) $[x,y-y^2]=[x^2, y]$ for all $x, y \in U$. Replacing x by –x, we obtain

(10) $-[x,y-y^2]=[x^2,y]$ for all $x, y \in U$. Then from (9) and (10) , we get

(11) $2[x^2,y]=0$ for all $x, y \in U$. Since char. $R \neq 2$ then, we obtain

(12) $[x^2,y]=0$ for all $x, y \in U$. From (8) , we get

(13) $[x-x^2,y]=[x,y^2]$ for all $x, y \in U$. Replacing y by –y , we obtain

(14) $-[x-x^2,y]=[x,y^2]$ for all $x, y \in U$. Then from (14) and (13), we get

(15) $2[x,y^2]=0$ for all $x, y \in U$. Since char. $R \neq 2$, then

(16) $[x,y^2]=0$ for all $x, y \in U$. Now, substituting (16) and (12) in (8) gives $[x,y]=0$ for all $x , y \in U$. Then by Lemma (3) , R is commutative .

When d=0 , then by Lemma (2),R is commutative.

Theorem 3.2.3

Let R be a prime ring and U a non-zero ideal of R .R admitting a non-zero derivation d satisfying $[d(x), d(y)]=[x^2,y^2]$ for all $x, y \in U$. If d act as a homomorphism on U , then R is commutative.

Proof: At first, we have $d \neq 0$, then

(17) $[d(x),d(y)]=[x^2,y^2]$ for all $x, y \in U$. Replacing x by x r, we obtain

(18) $[d(x)r,d(y)]+[xd(r),d(y)]=[x^2r^2,y^2]$ for all $x, y \in U$. Then

(19) $d(x)[r,d(y)]+[d(x),d(y)]r+x[d(r),d(y)]+[x,d(y)]d(r)=[x^2r^2,y^2]$ for all $x, y \in U$, $r \in R$. Replacing r and y by x , we obtain

(20) $d(x)[x,d(x)]+[x,d(x)]d(x)=0$ for all $x \in U$. Then

(21) $[x, d(x)^2]=0$ for all $x \in U$. Since d acts as a homomorphism, then $[x, d(x^2)]=0$ for all $x \in U$.

By Lemmas (1) and (2), R is commutative

3.3- On Prime and Semiprime Rings

Theorem 3.3.1

Let R be a 2-torsion free semiprime ring and U a non-zero ideal of R .R admitting a non-zero derivation d satisfying $d([d(x),d(y)])=[x, y]$ for all $x, y \in U$. If d acts as a homomorphism, then R contains anon zero central ideal .

Proof: We have

(22) $d([d(x),d(y)])=[x, y]$ for all $x, y \in U$. Replacing x by x^2, we obtain $d([d(x^2),d(y)])-[x^2,y]=0$ for all $x, y \in U$.Then

$d([d(x)x, d(y)])+d([x d(x),d(y)])-[x^2,y]=0$ for all $x, y \in U$

$d(d(x)[x,d(y)])+d([d(x),d(y)]x)+d(x[d(x), d(y)])+d([x, d(y)]d(x))-[x^2,y]=0$ for all $x, y \in U$. Then

$d^2(x)[x,d(y)]+d(x)d([x, d(y)])+d([d(x),d(y)])x+[d(x),d(y)]d(x)+d(x)[d(x),d(y)]+x d([d(x),d(y)])+d([x, d(y)])d(x)+[x, d(y)]d^2(x)-[x^2,y]=0$ for all $x, y \in U$.

According to (23), we get $d^2(x)[x,d(y)]+d(x)d([x,d(y)])+[x,y]x+[d(x),d(y)]d(x)+d(x)[d(x),d(y)]+x[x,y]+d([x,d(y)])d(x)+[x,d(y)]d^2(x)-[x^2,y]=0$ for all $x, y \in U$.

Replacing y by x, we obtain

(23) $d^2(x)[x, d(x)]+d(x)d([x, d(x)])+d([x, d(x)])d(x)+[x, d(x)]d^2(x)=0$ for all $x, y \in U$. Then

(24) $d^2(x)[x,d(x)]+[x,d(x)]d^2(x)+d(x)(d(xd(x))-d(d(x)x))+(d(xd(x))-d(d(x)x))d(x)=0$ for all $x \in U$. Thus

(25) $d^2(x)xd(x)-d^2(x)d(x)x+xd(x)d^2(x)-d(x)xd^2(x)+d(x)^3+d(x)xd^2(x)-d(x)d^2(x)x-d(x)^3+d(x)^3+xd^2(x)d(x)-d^2(x)xd(x)-d(x)^3=0$ for all $x \in U$. Then we obtain

(26) $[x,d(x)d^2(x)]+[x,d^2(x)d(x)]=0$ for all $x \in U$. Then

(27) $[x,d(d(x)^2)]=0$ for all $x \in U$. Since d acts as a homomorphism, we get $[x,d^2(x^2)]=0$ for all $x \in U$. By Theorem 3.1.3, R contains a non zero central ideal.

Theorem 3.3.2

Let R be a 2- torsion free semiprime ring. R admitting a derivation d satisfying d([d(x),d(y)])=[x, y] for all x, y ∈ R. If d acts as a homomorphism on R, then R is commutative.

proof :

At first , when d ≠ 0 , then from Theorem 3.3.1, we get R is commutative .

When d=0, then it is clear that, R is commutative.

We can easy give the proof of the following corollary:

Corollary 3.3.3

Let R be a prime ring U a non-zero ideal of R.R admitting a derivation d satisfying d([d(x),d(y)])=[x,y] for all x, y ∈ U. If d acts as a homomorphism on U , then R is commutative .

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بعض النتائج حول الحلقات الاولية وشبه الاولية مع الاشتقاقات

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الخلاصة

الغرض الرئيسي من هذا البحث دراسة الحلقات الاوليه و شبه الاوليه التى تسمح للاشتقاق d بتحقيق شروط جديده عندما تكون فعالية d تشاكل على المثالي غير الصفري، نحن نعطي بعض النتائج حول ذلك .