G- Cyclicity And Somewhere Dense Orbit

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Abstract:

let H be an infinite – dimensional separable complex Hilbert space, and S be a multiplication semigroup of $\mathbb{C}$ with 1. An operator T is called G-cyclic over S if there is a non-zero vector $x\in H$ such that $\{\alpha T^n x \mid \alpha\in S, n\geq 0\}$ is norm-dense in H. Bourdon and Feldman have proved that the existence of somewhere dense orbits implies hypercyclicity. We show the corresponding result for G-cyclicity.

Key words:- Hypercyclic operators, hypercyclic vectors, semigroup, somewhere dense set, everywhere dense set.

Introduction:

Let H be an infinite dimensional separable complex Hilbert space, and *B(H)* be the Banach algebra of all linear bounded operator on *H*.

Let S be a multiplication semigroup of $\mathbb{C}$ with 1,an operator $T\in B(H)$ is called G-cyclic over S if there is a vector x in H such that $\{\alpha T^n x \mid \alpha\in S, n\geq 0\}$ is norm-dense in H. In this case x is called a G-cyclic vector for T over S [1]. Clearly, every hypercyclic operator is G-cyclic and every G-cyclic operator is supercyclic. Bourdon and Feldman [2] proved that every somewhere dense orbit is everywhere dense, and they used this result to give another proof of "*Ansari's theorem*" if T is hypercyclic, then for each n≥1, T^n is hypercyclic, moreover T and T^n share the same collection of hypercyclic vectors" [3]. Also they use their theorem and give another proof of "*Multihypercyclicity Theorem*" If T is multihypercyclic then T is hypercylic " [4]. Our purpose in this paper is to obtain the corresponding results for G-cylic over S.

$1 Somewhere dense orbit is everywhere dense:

The aim of this section is to prove that the existence of somewhere dense orbit implies. G-cyclicity. Next we fix notation required for the discussion.

Notation : Let S be a semigroup of $\mathbb{C}$ with 1 then

1. .Sorb$(T,x)\equiv\{\alpha T^n x \mid \alpha\in S, n\geq 0\}$.
2. $\mathbb{C}$ orb$(T,x)=\{\alpha T^n x \mid \alpha\in \mathbb{C}, n\geq 0\}$.
3. F(x)=$\overline{\text{Sorb}(T,x)}$
4. U(x)=int(F(x)).
5. $X^c\equiv$ complement of X in H.

Clearly from the definition of G–cyclic [1], that every G-cyclic operator is supercyclic operator, so we get:

Proposition 1.1: Suppose that x∈H such that Sorb(T,x) is somewhere dense in H, then $\mathbb{C}$orb(T,x) is somewhere dense in H.

Proof: Sinse S is a semigroup of $\mathbb{C}$ with 1, then $\overline{Sorb(T,x)}\subseteq\overline{\mathbb{C}orb(T,x)}$.

Now since U(x)≠∅ and

U(x) $\subseteq$ int ($\overline{\mathbb{C}orb(T,x)}$). Thus $\mathbb{C}$ orb(T,x) is somewhere dense.

From [2] we get immediately the following two lemmas:

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Lemma 1.2: Suppose that $x \in H$ such that Sorb(T,x) is somewhere dense in H. Then T^* may have at most one eigenvalue.

Lemma 1.3: Suppose that Sorb(T,x) is somewhere dense in H, then for each $\alpha \in S$, $j \in$ N, $\alpha T^n{}_j\, x$ is a cyclic vector for T.

Peries in [4] proved the following lemma.

Lemma 1.4 [4]: Let p be a complex polynomial, $p(T)$ has a dense range if and only if $p(\lambda) \neq 0$ for every eigenvalue λ of T^*.

The next lemma provides the crucial element of the argument.

Lemma 1.5: Let $x \in H$, then for every $\lambda \in S$, $U^c(x)$ is invariant under λT.
In addition $U^c(x)$ is invariant under multiplication by any $\alpha \in S$.

Proof: Since U(x) is nonempty, then there is a positive integer j and a non-zero $\beta \in S$ such that $\beta T^j x$ belongs to U(x) and set $x_j = \beta_j T^j x$. For any $k \in N$, Sorb(T,T^k x_j) is dense in U(x), thus x_j is a limit point of Sorb(T, T^k x_j) and $U(x) = U(T^k x_j)$. By lemma (1.3) x_j is cyclic vector for T i.e. $\{p(T)x_j \,|\, p \text{ is polynomail}\}$ is dense in H.

Fix $\alpha \in S$. assume that $U^c(x)$ is not λT-invariant, i.e. there is $y \notin$ U(x) but $Ty \in$ U(x). We may assume $y \notin$ F(x), if not, then $y \in \partial$F(x), also since λT is continues, hence there is point $y' \in$ F(x) close enough to y and λTy to keep it in U(x). Thus rename y' as y.

Because $F^c(x)$ is open and $\{p(T)x_j | p$ is *polynomial*$\}$ is dense in H, Thus there is a polynomial p so that $p(T)x_j$ is closed enough to y ensure $p(T)x_j \in F^c(x)$ and $\lambda Tp(T)x_j \in$ U(x). Since U(x)$\subset$F(x) and F(x) is λT-invariant, then Sorb($T,\lambda Tp(T)x_j$)$\subseteq$F(x). However Sorb($T,\lambda Tp(T)x_j$)=Sorb($T,p(T)Tx_j$).
Because x_j is a limit point of Sorb(T,Tx_j), the continuity of $p(T)$ yield $p(T)x_j \in$F(x). Thus $p(T)x_j \in$F(x) and its complement, a contradiction.

It is easy, by notation, to prove that $U^c(x)$ is invariant under multiplication $\alpha \in$S

Remark: The preceding show that if y$\in$Sorb(T,x), then U(x)=U(y).

Now we will prove the main result.

Somwhere Dense Theorem 1.6: Suppose $T \in B(H)$, and Sorb(T,x) is somewhere dense in H, then T is G-cyclic operator.

Proof: Assume that $\overline{\text{Sorb}(T,x)} \neq H$. Since x is cyclic vector for T (1.3), then $\{p(T)x_j |\ p$ is polynomial$\}$ is dense in H. Then there is a subcollection Q of polynomial such that $\{q(T)x | q \in Q\}$ is dense subset of $U^c(x)$. By (1.5) $U^c(x)$ is λT-invariant for all $\lambda \in$S, so $q(T)orb(T,x) \subset U^c(x)$ for all $q \in Q$, hence, by continuity of T, $q(T)F(x) \subset q(T)$orb$(T,x) \subset U^c(x)$.

Let W denote the collection of non-zero polynomials not having the (possible) eigenvalue of T^* as a zero and let $p \in W$.

Now put D:=U(x)$\cup\{q(T)x | q \in Q\}$, since $\{q(T)x | q \in Q\}$ is dense in $U^c(x)$, hence D is dense set in H. Because $p(T)$ has dense range in H (1.4), therefore $p(T)$ D is dense in H.

Suppose in order to obtain a contradiction, that $p(T)x \in \partial$U(x), hence $p(T)x \notin$U(x), then $p(T)x \in U^c(x)$. Thus $p(T)$U(x) $\subset U^c(x)$. On the other hand, since $U^c(x)$ is λT-invariant for all $\lambda \in$S, thus $p(T)\{q(T)x | q \in Q\} \subset U^c(x)$. Therefore $p(T)D \subset U^c(x)$ which contradicting the density of $p(T)D$. Thus $p(T)x \notin \partial$U(x). Because $\{p(T)x | p \in W\}$ is connected, contains points in U(x) and contains no boundary point of U(x), thus $\{p(T)x | p \in W\} \subset$U($x$). Given a coefficient n-tuple c for any polynomial, there is a sequence for coefficient n-tuple of a polynomials in Q converging componentwise to c, and since x is cyclic vector for T, then $\{p(T)x|p\in W\}$ is dense in H. Since $\{p(T)x|p\in W\}\subset U(x) \subset F(x)$, therefore $F(x)=H$. Thus T is G-cyclic operator.

$2 Applications to the somewhere dense Theorem:

In this section we give two applications to the somewhere dense theorem. First we need the following fact, let X be a topological space and $F_1, F_2, \ldots, F_n$ a finite family of closed subset of X; $X=\bigcup_{i=1}^{n} F_i$,if $int(F_i)=\varnothing$, then $X=\bigcup_{i=2}^{n} F_i$ [4]

Proposition2.1: If T is a G-cyclic operator over S then for every positive integer n, T^n is G-cyclic operator over S. Moreover, T and T^n share the same collection of G-cyclic vectors.

Proof: Let x be a G-cyclic vector for T over S, and fixed n>1, then $Sorb(T,x)=\bigcup_{j=0}^{n-1} Sorb(T^n, T^j x)$ will be dense in H, thus $H=\bigcup_{j=0}^{n-1}\overline{Sorb(T^n, T^j x)}$

Thus at least one of the sets $Sorb(T^n,T^j x)$ must be somewhere dense. Therefore by (1.6) T^n is a G-cyclic operator over S. Now because T must have dense range [1], the set $T^{n-j}[Sorb(T^n,T^j x)]=Sorb(T^n,T^n x)$ will be dense in H, from which it followed that x is also G-cyclic vector for T^n.

An operator $T\in B(H)$ is a multi-G-cyclic operator over S provided there is a finite subset $\{x_i\}_1^n$ in H such that $\bigcup_{1}^{n} Sorb(T, x_i)$ is dense in H. Clearly every G-cyclic operator is multi-G-cyclic operator. A question arises: **Is the converse true?**

Proposition 2.2: Any multi-G-cyclic operator over S is G-cyclic operator over S.

Proof: Let $\{x_j\}$ be a multi-G-cyclic vector for T over S, then $\bigcup_{1}^{n} Sorb(T, x_i)$ is dense in H. By [4], there is at least j; 1<j<n, such that $Sorb(T, x_j)$ has somewhere dense in H. Thus by (1.6) T is G-cyclic operator over S.

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مؤثر الدواري من نمط G والمدار الكثيف في مكان ما

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الخلاصة:

ليكن H فضاء هلبرت على حقل الاعداد العقدية قابل للفصل غير منته البعد وS شبه زمرة جدائية من $\mathbb{C}$تحتوي على 1. يقال للمؤثر الخطي T انه دوري من النمط G على S اذا وجد متجه غير صفري $x\in H$ بحيث ان $\{\alpha T^n x \mid \alpha\in S, n\geq 0\}$ كثيفة في H.

بوردن وفيلدمان برهنا وجود مدار كثيف في مكان ما يؤدي الى فوق الدوارية. في هذا البحث اعطينا نتائج مماثلة في حالة دواري من النمط G.