

Solving Parabolic Partial Differential Equations using Modified Bellman's Method with G-Spline Interpolation

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Abstract:

The aim of this paper is to approximate the solution of the parabolic partial differential equations (heat equations) using Bellman's method with the cooperation of the G-spline interpolation formula. The partial differential equation will then be changed into a system of the first order ordinary differential equation. The resulting system may be then solved easily by using the fundamental matrix solution. In this paper, the Bellman's method may be considered as a generalization to the usual Bellman's method with an arbitrary ordinary derivative.

Key words: Bellman's Method, G-spline Interpolation.

Introduction:

The mathematical formulation of most problems in science involving rates of change with respect to two or more independent variables, usually representing time, length or angle, lead either to a partial differential equation or to a set of such equations.

The general two dimensional second order partial differential equation:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu + G = 0$$

where A, B, C, D, E, F and G may be constants or functions of the independent variables x and y, [1].

As a special case, this equation is said to be elliptic when $B^2 - 4AC < 0$, parabolic when $B^2 - 4AC = 0$ and hyperbolic when $B^2 - 4AC > 0$.

Problems involving time t as one independent variable lead usually to parabolic or hyperbolic equations.

The simplest parabolic equation, $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ derives from the theory of heat conduction and its solution gives, for example the temperature u at a distance x units of length from one end of a thermally insulated bar after t seconds of heat conduction.

In such a problem, the temperatures at the ends of a bar of length ℓ (say) are often known for all time. In other words, the boundary conditions are known. It is also usual for temperature distribution along the bar to be known at some particular instant. This instant is usually taken as zero time and the temperature distribution is called the initial condition. The solution gives u for values of x between 0 and ℓ and values of t from zero to infinity [1].

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There are several numerical methods for solving the partial differential equations, such as the finite difference method, finite-element method, variational methods, spline interpolation method, the collocation method, the method of lines and the method which we shall used in this paper which is so called the Bellman's method, [1,2]. Really, Bellman's method depends on polynomial interpolation in order to interpolate the functions $u_i(t)$, which considered to be known, but in fact they are unknowns, therefore, the interpolation method which will be used here is the G-spline interpolation because of its simplicity and efficiency in evaluating the approximate solution.

G-spline interpolation was first introduced by I. J. Schoenberg [3] as a tool used to specify interpolatory conditions:

$$f^{(j)}(x_i) = y_i^{(j)}, \text{ for } (i, j) \in e$$

which is called the Hermite-Birkhoff problem, where e is a certain set of order pairs (defined later in section two) and he proved that this tool (G-spline interpolation) gives the best approximation for linear functionals, [3].

G-Spline Interpolation, [3],[4],[5]:

In 1968 Schoenberg [3] extended the idea of Hermite for splines to specify that the orders of the derivatives specified may vary from node to node.

As usual let $I = [a, b]$ be an interval partitioned by the nodes:

$$a = x_1 < x_2 < \dots < x_n = b$$

and let α be the maximum of the orders of the derivatives to be specified at the nodes, we introduce an incidence

matrix E , where the incidence matrix is defined by:

$$E = [a_{ij}], i = 1, 2, \dots, n; j = 0, 1, \dots, \alpha$$

where:

$$a_{ij} = \begin{cases} 1, & \text{if } (i, j) \in e \\ 0, & \text{if } (i, j) \notin e \end{cases}$$

Here $e = \{(i, j), i = 1, 2, \dots, n; j = 0, 1, \dots, \alpha\}$ has been chosen in such a way that i takes the values $1, 2, \dots, n$; one or more times, while $j \in \{0, 1, \dots, \alpha\}$ and $j = \alpha$ is attained in at least one element

(i, j) of e , assume also that each row of the incidence matrix E and last column of E should contain some element equals 1.

The Hermite-Birkhoff problem is to find $f(x) \in C^\alpha$, which satisfies the interpolatory conditions:

$$f^{(j)}(x_i) = y_i^{(j)}, \text{ for } (i, j) \in e \dots (1)$$

Definition (1), [3],[4],[5]:

Let m be a natural number, then the Hermite-Birkhoff problem (1) is said to be m -poised provided that:

$$p(x) \in \Pi_{m-1} \\ p^{(j)}(x_i) = 0 \text{ if } (i, j) \in e$$

then:

$$p(x) = 0.$$

where Π_{m-1} is the class of polynomial of degree $m - 1$ or less.

The definition of G-spline is facilitated by defining a matrix E^* which is obtained from the incidence matrix E by adding $m - \alpha - 1$ columns of zeros to the matrix E .

Let $E^* = [a_{ij}^*]$, where $(i = 1, 2, \dots, n; j = 0, 1, \dots, m - 1)$, and:

$$a_{ij}^* = \begin{cases} a_{ij}, & \text{if } j \leq \alpha \\ 0, & \text{if } j = \alpha + 1, \alpha + 2, \dots, m - 1 \end{cases}$$

If $m = \alpha + 1$, then $E^* = E$.

Definition (2), [3],[4],[5]:

A function $S(x)$ is called natural G-spline for the knots x_1, x_2, \dots, x_n and the matrix E^* of order m provided that it satisfies the following conditions:

1. $S(x) \in \Pi_{2m-1}$ in (x_i, x_{i+1}) , $i = 1, 2, \dots, n - 1$.
2. $S(x) \in \Pi_{m-1}$ in $(-\infty, x_1)$ and in (x_n, ∞) .
3. $S(x) \in C^{m-1}(-\infty, \infty)$.
4. If $a_{ij}^* = 0$, then $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$.

Let $S(E^*; x_1, x_2, \dots, x_n)$ denotes the class of all G-spline of order m .

At this point, the G-spline interpolation of order m to f may be given in terms of the fundamental G-spline functions L_{ij} , by:

$$S_m(x) = \sum_{(i,j) \in e} L_{ij}(x) y_i^{(j)} \dots (2)$$

Where L_{ij} satisfies:

$$L_{ij}^{(s)}(x_r) = \begin{cases} 0, & \text{if } (r,s) \neq (i,j) \\ 1, & \text{if } (r,s) = (i,j) \end{cases}$$

Bellman's Technique with Cooperation of G-Spline Interpolation to Approximate the Solution of Parabolic PDE's:

Consider the initial-boundary value problem:

$$\text{PDE} \quad u_{xx} - u_t = g(x, t, u, u_x), \\ 0 < x < \ell, 0 < t < \infty$$

$$\text{BC's} \quad \begin{cases} u(0, t) = p(t) \\ u(\ell, t) = p(t) \end{cases}, \quad 0 < t < \infty \dots (3)$$

$$\text{IC} \quad u(x, 0) = w(x), \quad 0 \leq x \leq \ell$$

The idea of this method is that we shall consider the values:

$$\frac{\partial^j u(x_i, t)}{\partial x^j} = u_{ij}(t), \quad (i, j) \in e$$

are known, but as we mention in the introduction of this paper they are unknowns, and we shall interpolate them by the G-spline interpolation formula given by (2), such that:

$$u(x, t) = \sum_{(i,j) \in e} L_{ij}(x) u_{ij}(t) \dots (4)$$

substituting (4) into the initial-boundary value problem given by (3), we have:

$$\sum_{(i,j) \in e} L''_{ij}(x) u_{ij}(t) - \sum_{(i,j) \in e} L_{ij}(x) u'_{ij}(t) =$$

$$g(x, t, \sum_{(i,j) \in e} L_{ij}(x) u_{ij}(t),$$

$$\sum_{(i,j) \in e} L'_{ij}(x) u_{ij}(t)) \dots (5)$$

and in order to evaluate $u_{ij}(t)$ for $(i, j) \in e$, we must make the number of equations equals to the number of unknowns and this will be satisfied by choosing $(r, s) \in e$ and differentiating (5) s -times at the point $x = x_r$, then we shall get a system of the first order ordinary differential equations with the initial conditions:

$$u_{ij}(0) = \frac{\partial^j u(x_i, 0)}{\partial x^j} = \frac{d^j w(x_i)}{dx^j}$$

Solving the resulting system using matrix exponential method, gives the values:

$$u_{ij}(t) = \frac{\partial^j u(x_i, t)}{\partial x^j}, \quad (i, j) \in e \dots (6)$$

which represent the solution of (3) and its j partial derivative with respect to x

at the x_i , where $(i, j) \in e$ and for $0 < t < \infty$.

The reader can notice that if $j = 0$ in (6), which means that we have evaluated the solution u of (3) at the point x_i , and for $0 < t < \infty$ (which is the usual Bellman's method), therefore we have here a modification to the Bellman's method with the aid of G-spline interpolation.

Illustrative Examples:

In the present section, some illustrative examples are given in order to check the accuracy of the results:

Example (1):

Consider the initial-boundary value problem:

$$\text{PDE } u_t = u_{xx}, \quad 0 < x < 1, 0 < t < \infty$$

$$\text{BC's } \begin{cases} u(0, t) = 0 \\ u(1, t) = 0 \end{cases} \quad 0 < t < \infty \dots (7)$$

$$\text{IC } u(x, 0) = x, \quad 0 \leq x \leq 1$$

and to construct the approximate solution of (7), the m-poised Hermite-Birkhoff problem must be chosen. The choice is as follows:

The interval $[0, 1]$ is partitioned with $h = 1/4$, where h is the distance between the nodes, as:

$$0 < 0.25 < 0.5 < 0.75 < 1$$

where 0, 0.25, 0.5, 0.75, 1 are taken to be the node points and let:

$$e = \{(0, 0), (0.25, 0), (0.5, 0), (0.75, 0), (1, 0)\}$$

we shall seek $S_4(x) \in \mathcal{S}(E^*; 0, 0.25, 0.5, 0.75, 1)$, where $E^* = [a_{ij}^*]$ is defined by:

$$a_{ij}^* = \begin{cases} 1, & \text{for } j=0, 0 \leq i \leq 4 \\ 0, & \text{for } 0 < j \leq 4, 0 \leq i \leq 4 \end{cases}$$

Therefore, equation (4) becomes:

$$u(x, t) = L_{00}(x)u_{00}(t) + L_{10}(x)u_{10}(t) + L_{20}(x)u_{20}(t) + L_{30}(x)u_{30}(t) + L_{40}(x)u_{40}(t) \quad (8)$$

substituting (8) into (7), gives:

$$\begin{aligned} L''_{00}(x)u_{00}(t) &+ L''_{10}(x)u_{10}(t) &+ \\ L''_{20}(x)u_{20}(t) &+ L''_{30}(x)u_{30}(t) &+ \\ L''_{40}(x)u_{40}(t) &= L_{00}(x)u'_{00}(t) &+ \\ L_{10}(x)u'_{10}(t) &+ L_{20}(x)u'_{20}(t) &+ \\ L_{30}(x)u'_{30}(t) &+ L_{40}(x)u'_{40}(t) \dots (9) \end{aligned}$$

It is clear that from the BC's that $u_{00}(t) = 0$ and $u_{40}(t) = 0$, and in order to evaluate $u_{10}(t)$, $u_{20}(t)$, $u_{30}(t)$, we shall chose (r, s) to be the order pair $(1, 0)$, $(2, 0)$, $(3, 0)$ respectively, hence after substituting the ordered pairs (r, s) into (9) we shall get the following linear system:

$$u'_{10}(t) = L''_{10}(x_1)u_{10}(t) + L''_{20}(x_1)u_{20}(t) + L''_{30}(x_1)u_{30}(t)$$

$$u'_{10}(t) = L''_{10}(x_2)u_{10}(t) + L''_{20}(x_2)u_{20}(t) + L''_{30}(x_2)u_{30}(t)$$

$$u'_{10}(t) = L''_{10}(x_3)u_{10}(t) + L''_{20}(x_3)u_{20}(t) + L''_{30}(x_3)u_{30}(t)$$

writing the above equations in matrix form $u' = Au$, where:

$$u' = \begin{bmatrix} u'_{10}(t) \\ u'_{20}(t) \\ u'_{30}(t) \end{bmatrix}, \quad A = \begin{bmatrix} L''_{10}(x_1) & L''_{20}(x_1) & L''_{30}(x_1) \\ L''_{10}(x_2) & L''_{20}(x_2) & L''_{30}(x_2) \\ L''_{10}(x_3) & L''_{20}(x_3) & L''_{30}(x_3) \end{bmatrix} = \begin{bmatrix} -29.881 & 12.821 & 2.115 \\ 22.834 & -42.252 & 22.834 \\ 2.119 & 12.821 & -29.881 \end{bmatrix}$$

$$u(t) = \begin{bmatrix} u_{10}(t) \\ u_{20}(t) \\ u_{30}(t) \end{bmatrix}, \text{ with } c = u(0) = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.75 \end{bmatrix}$$

and since A have the distinct eigenvalues $\lambda_1 = -9.748$, $\lambda_2 = -32$, $\lambda_3 = -60.266$, which will give an independent eigenvectors and hence we get a unique solution for the above linear system:

$$u(t) = c\Phi(t) = \begin{bmatrix} 0.12550e^{-9.748t} - 0.3535e^{-32t} - 0.258e^{-60.266t} \\ 0.17625e^{-9.748t} + 3.605 \times 10^{-6}e^{-32t} + 0.65475e^{-60.266t} \\ 0.12550e^{-9.748t} + 0.3535e^{-32t} - 0.258e^{-60.266t} \end{bmatrix}$$

where:

$$\Phi(t) = \begin{bmatrix} 0.502e^{-9.748t} & -0.707e^{-32t} & -0.344e^{-60.266t} \\ 0.705e^{-9.748t} & 6.251 \times 10^{-6}e^{-32t} & 0.873e^{-60.266t} \\ 0.502e^{-9.748t} & 0.707e^{-32t} & -0.344e^{-60.266t} \end{bmatrix}$$

The exact solution of (7) is given by [2]:

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x), \quad A_n = \frac{1}{2} \int_0^1 x \sin(n\pi x) dx$$

Table (1) presents a comparison between the analytical and the approximate results obtained using computer program written in Mathcad (2001i):

Table (1)The approximate and the analytical solution of example (1)

	<i>Approximate solution</i> <i>x = 0.25</i>	<i>Analytical solution</i> <i>x = 0.25</i>	<i>Approximate solution</i> <i>x = 0.5</i>	<i>Analytical solution</i> <i>x = 0.5</i>	<i>Approximate solution</i> <i>x = 0.75</i>	<i>Analytical solution</i> <i>x = 0.75</i>
<i>t = 0.25</i>	0.011	0.038	0.015	0.045	0.011	0.038
<i>t = 0.5</i>	9.591×10^{-4}	3.237×10^{-3}	1.347×10^{-3}	4.578×10^{-3}	9.592×10^{-3}	3.237×10^{-3}
<i>t = 0.75</i>	9.385×10^{-5}	2.746×10^{-4}	1.178×10^{-4}	3.883×10^{-4}	8.385×10^{-4}	2.746×10^{-4}

Where $L_{ij}(x)$ for $(i, j) \in e$ are given in [6] which are evaluated analytically, as follows:

$$L_{00}(x) = 1 - 7.9581x + 19.9404x^2 - 16.4592x^3 + \frac{34178.543}{7!}x_+^7 - \frac{136714.1722}{7!}(x-0.25)_+^7 + \frac{205071.2583}{7!}(x-0.5)_+^7 - \frac{136714.1722}{7!}(x-0.75)_+^7 + \frac{34178.543}{7!}(x-1)_+^7$$

$$L_{10}(x) = 14.498x - 55.7616x^2 + 55.16998x^3 - \frac{136714.1722}{7!}x_+^7 +$$

$$\frac{546856.6887}{7!}(x-0.25)_+^7 - \frac{820285.0331}{7!}(x-0.5)_+^7 + \frac{546856.6887}{7!}(x-0.75)_+^7 - \frac{136714.1722}{7!}(x-1)_+^7$$

$$L_{20}(x) = -9.7483x + 55.6423x^2 - 66.75497x^3 + \frac{205071.2583}{7!}x_+^7 - \frac{820285.0331}{7!}(x-0.25)_+^7 + \frac{1230427.5497}{7!}(x-0.5)_+^7 -$$

$$\begin{aligned}
 & \frac{820285.0331}{7!} (x-0.75)_+^7 + \\
 & \frac{205071.2583}{7!} (x-1)_+^7 \\
 L_{30}(x) = & 3.8322x - 23.7616x^2 + \\
 & 33.8366x^3 - \frac{136714.1722}{7!} x_+^7 + \\
 & \frac{546856.6887}{7!} (x-0.25)_+^7 - \\
 & \frac{820285.0331}{7!} (x-0.5)_+^7 + \\
 & \frac{546856.6887}{7!} (x-0.75)_+^7 - \\
 & \frac{136714.1722}{7!} (x-1)_+^7 \\
 L_{40}(x) = & -0.6247x + 3.9404x^2 - \\
 & 5.7925x^3 + \frac{34178.543}{7!} x_+^7 - \\
 & \frac{136714.1722}{7!} (x-0.25)_+^7 + \\
 & \frac{205071.2583}{7!} (x-0.5)_+^7 - \\
 & \frac{136714.1722}{7!} (x-0.75)_+^7 + \\
 & \frac{34178.543}{7!} (x-1)_+^7
 \end{aligned}$$

The computation of L_{ij} 's becomes so difficult and expensive when reducing the step size h , therefore the simplicity of the method is to take the step size large enough, which will guarantee the best approximation for the solution, the details for evaluating the L_{ij} 's is found in [4].

Example (2):

Consider the initial-boundary value problem:

PDE $u_t = u_{xx}, \quad 0 < x < 1, 0 < t < \infty$

BC's $\begin{cases} u(0,t) = 0 \\ u(1,t) = \sin t \end{cases}, \quad 0 < t < \infty \dots (10)$

IC $u(x, 0) = 0, \quad 0 \leq x \leq 1$

A Hermite-Birkhoff problem also must be chosen suppose the choice is also similar to example (1) the node points is then 0, 0.25, 0.5, 0.75, 1 and let $e = \{(0, 0), (0.25, 0), (0.5, 0), (0.75, 0), (1, 0)\}$, then $u(x, t)$ is as given in the form (8), and from the BC's $u_{00}(t) = 0, u_{40}(t) = \sin t$; and to find $u_{10}(t), u_{20}(t), u_{30}(t)$, we pick (r, s) to be (1, 0), (2, 0), (3, 0), respectively. Therefore, we have the nonhomogeneous linear system of the first order ordinary differential equations after substituting (8) into (10) as follows:

$$\begin{aligned}
 u' &= \begin{bmatrix} u'_{10}(t) \\ u'_{20}(t) \\ u'_{30}(t) \end{bmatrix}, \quad A = \\
 & \begin{bmatrix} L''_{10}(x_1) & L''_{20}(x_1) & L''_{30}(x_1) \\ L''_{10}(x_2) & L''_{20}(x_2) & L''_{30}(x_2) \\ L''_{10}(x_3) & L''_{20}(x_3) & L''_{30}(x_3) \end{bmatrix} = \\
 & \begin{bmatrix} -29.881 & 12.821 & 2.115 \\ 22.834 & -42.252 & 22.834 \\ 2.119 & 12.821 & -29.881 \end{bmatrix}
 \end{aligned}$$

$$u(t) = \begin{bmatrix} u_{10}(t) \\ u_{20}(t) \\ u_{30}(t) \end{bmatrix}, \quad \text{with } c = u(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} -0.529804305013021 \sin t \\ -1.7086205419922 \sin t \\ 15.47017904378255 \sin t \end{bmatrix}$$

Then:

$$u(t) = \Phi(t)\Phi^{-1}(0)c + \Phi(t) \int_0^t \Phi^{-1}(s)B(s)$$

ds where:

$$\Phi(t) = \begin{bmatrix} 0.502e^{-9.748t} & -0.707e^{-32t} & -0.344e^{-60.266t} \\ 0.705e^{-9.748t} & 6.251 \times 10^{-6}e^{-32t} & 0.873e^{-60.266t} \\ 0.502e^{-9.748t} & 0.707e^{-32t} & -0.344e^{-60.266t} \end{bmatrix}$$

a comparison between the analytical and the approximate solution is given in table (2):

Table (2) The approximate and the analytical solution of example (2)

	<i>Approximate solution</i> <i>x = 0.25</i>	<i>Analytical solution</i> <i>x = 0.25</i>	<i>Approximate solution</i> <i>x = 0.5</i>	<i>Analytical solution</i> <i>x = 0.5</i>	<i>Approximate solution</i> <i>x = 0.75</i>	<i>Analytical solution</i> <i>x = 0.75</i>
<i>t = 0.25</i>	0.027	0.027	0.084	0.068	0.139	0.136
<i>t = 0.5</i>	0.068	0.084	0.183	0.183	0.291	0.31
<i>t = 0.75</i>	0.136	0.139	0.310	0.291	0.468	0.468

Where the fundamental functions $L_{ij}(x)$ for $(i, j) \in e$ is given in example (1) and the analytical solution is obtained by the Duhamel's principle method, [2], as:

$$u(x, t) = \int_0^t w(x, t - \tau) d\tau$$

where:

$$w(x, t) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-(n\pi)^2 t} \sin(n\pi x)$$

Conclusions:

1. The advantage of the approximation using G-spline functions is the evaluation of the fundamental functions L_{ij} 's once for all with the same nodes for any type of linear functions.
2. Other numerical methods may give more accurate results but with huge number of calculations which increases running time and computer storage, while G-spline methods require minimum computer memory storage.

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حل المعادلات التفاضلية الجزئية المكافئة باستخدام طريقة بيلمان المطورة مع دوال السبلين-G

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الخلاصة:

إنّ الهدف الرئيس من هذا البحث هو تقريب حل المعادلات التفاضلية الجزئية المكافئة (Parabolic partial differential equations) وذلك باستخدام طريقة بيلمان وبالاعتماد على صيغة دوال السبلين-G للاندراج. ستتحوّل المعادلة التفاضلية الجزئية عندئذ إلى منظومة معادلات تفاضلية إعتيادية من الرتبة الاولى وسيتم حل النظام الناتج بسهولة بوساطة استخدام مصفوفة الحل الاساسية. وفي البحث يمكن عد طريقة بيلمان تعميماً لطريقة بيلمان الاعتيادية ولأي رتبة مشتقة إعتيادية.