On Primary Multipliction Modules

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Abstract :

Let R be a commutative ring with identity and M be a unitary R- module. We shall say that M is a primary multiplication module if every primary submodule of M is a multiplication submodule of M. Some of the properties of this concept will be investigated. The main results of this paper are, for modules M and N, we have $M \underset{R}{\otimes} N$ and Hom_R (M, N) are primary multiplications R-modules under certain assumptions.

Key words: Multiplication module, Weak multiplication, Primary submodule, Primary multiplication module.

Introduction:

In this paper all rings are commutative rings with identity and all modules are unital. A submodule N of an R-module M is called ***prime (resp. primary)*** if for any $r \in R$ and $m \in M$ such that $r\,m \in N$, either $m \in N$ or $rM \subseteq N$ (resp. either $m \in N$ or $r^n M \subseteq N$ for some positive integer n) [1], [2] & [3]. Note that in this definition we do not require that N is a proper submodule of M as it was define in [4].

An ideal I of a ring R is called ***Primary*** if it is a primary submodule of R when considered as an R-module [2, p.40].

A submodule N of an R-module M is called ***a multiplication submodule*** if for each submodule K of N, there exists an ideal I of R such that K = IN. in this case we can take I = (K: N) = $\{r \in R: rN \subseteq K\}$. [5]

A module M is called ***multiplication module*** if every submodule of M is multiplication submodule of M [1]. As a generalization of multiplication module. Jain in [1] introduced the concept of weak multiplication module as follows:

An R-module M is said to be ***a weak multiplication module*** if every prime submodule of M is a multiplication submodule of M. In this paper we introduce the concept of primary multiplication module as follows:

An R-module M is said to be ***primary multiplication module*** if every ***primary submodule*** of M is ***a multiplication submodule*** of M.

It is clear that every primary multiplication module is weak multiplication module. Also, we give some results concerning this class of module.

1.Primary multiplication modules:

We begin this section with the notion of primary multiplication module, as follows:

Definition (1.1): An R-module M is said to be ***primary multiplication module*** if every primary submodule of M is a multiplication submodule of M.

Lemma (1.2): let f: M $\rightarrow$N be a module epimophism. If L is a primary

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submodule of N then $f^{-1}(L)$ is a primary submodule of M.

Proof: let $r \in R$ and $m \in M$ such that $r\,m \in f^{-1}(L)$ with $m \notin f^{-1}(L)$. We must prove that there exists a positive integer n such that $r^n M \subseteq f^{-1}(L)$.

Now $rm \in f^{-1}(L)$, then $r\,f(m) = f(rm) \in L$, but L is primary submodule of N, and $f(m) \notin L$, then $r^n N \subseteq L$, for some positive integer n, that is $r^n f(m) \in L$ and hence $f(r^n m) = r^n f(m) \in L$, therefore

$r^n m \in f^{-1}(L)$, and hence $r^n M \subseteq f^{-1}(L)$ this implies that $f^{-1}(L)$ is primary submodule of N.

The next proposition shows that a homomorphic image of primary multiplication module is primary multiplication module.

Proposition (1.3): let $f : M \to N$ be an epimorphism. If M is primary multiplication module, then so is N.

Proof: let k be a primary submodule of N and L be a submodule of N such that $L \subseteq k \subseteq N$. it is clear that $f^{-1}(L) \subseteq f^{-1}(k) \subseteq M$. but M is primary multiplication module, and by lemma (1.2) $f^{-1}(k)$ is a primary submodule of M, thus $f^{-1}(k)$ is a multiplication submodule of M, and hence there exits an ideal I of R such that $f^{-1}(L) = I f^{-1}(k)$. Now,

$f(f^{-1}(L)) = f(I f^{-1}(k)) = I(f f^{-1}(k))$. But f is an epimorphism, then

$L = Ik$. Therefore k is a multiplication submodule of N, and hence N is a primary multiplication module.

2.The tensor product of primary multiplication modules:

The basic motivating idea in this section will be to take two primary multiplication modules and show that their tensor product is also a primary multiplication module.

Let us state the following proposition which is needed later.

Proposition (2.1) [6. corollary (1.3)]

Let N be a submodule of an R-module M. if M is a multiplication submodule of M, then the following are equivalent:

1. N a primary submodule of M.
2. [N: M] is a primary ideal of R.
3. N = AM for some primary ideal A of R with $Ann(M) \subseteq A$.

Proposition (2.2): If M is a primary multiplication module and N is a multiplication submodule of N, then $M \otimes N$ is a primary multiplication module.

Proof:

Let K be a primary submodule of $M \otimes N$. since M is a primary multiplication module, then M is a multiplication submodule of M, and hence $M \otimes N$ is a multiplication submodule of $M \otimes N$ by [7.theorem (2.3)]

Thus $K = (K: M\otimes N)(M\otimes N) = [(K: M\otimes N) M] \otimes N$.

But K is a primary submodule of $M \otimes N$, then $(K: M\otimes N)$ is a primary ideal in R by (2.1).

Now, clearly $Ann(M) \subseteq (K: M\otimes N)$, thus again by (2.1) $(K: M\otimes N) M$ is a primary submodule of M, and hence $(K: M\otimes N) M$ is a multiplication submodule of M. therefore by [7, Theorem (2.3)] $K = [(K: M\otimes N) M] \otimes N$ is a multiplication submodule of $M \otimes N$. thus $M \otimes N$ is a primary multiplication module.

Corollary (2.3): If each of M and N is a primary multiplication module, then $M \otimes N$ is a primary multiplication module.

Corollary (2.4): If the R-module M is a multiplication submodule of M and I is a primary multiplication ideal of R then IM is a primary multiplication module.

Proof:

Define h: $I \otimes M \rightarrow IM$ by $h(\sum_{i=1}^{n}(a_i \otimes m_i)) = \sum_{i=1}^{n} a_i m_i$ for all $a_i \in I$ and for all $m_i \in M$.

One can show that h is an epimorphisim. But $I \otimes M$ is a primary multiplication module by (2.2), hence IM is a primary multiplication module by (1.3)

3.The module Hom (M, N):

This section is devoted to stady when Hom (M, N) is a primary multiplication module, where M and N are modules. We start with the following:

Definition (3.1) [8]: An R-module M is called a ***weak cancellation module*** whenever AM = BM for ideals A and B of R, then

A + Ann (M) = B + Ann (M). In particular if Ann (M) =0, then we call M a cancellation module.

The following lemma is needed later.

Lemma (3.2):

Let M and N be R-modules. If K is a submodule of Hom (M, N), then Ann M $\subseteq$ (K: Hom (M, N))

Proof: straightforward.

Proposition (3.3):

If M is a finitely generated primary module and N is a multiplication submodule of N such that Ann M $\subseteq$ Ann N, then Hom (M, N) is a primary multiplication module.

Proof:

Let K be a primary submodule of Hom (M, N) and L be a subnodule of Hom (M, N) such that $L \subseteq K$ then (L: Hom (M, N)) $\subseteq$ (K: Hom (M, N)) and thus (L: Hom (M, N)) M $\subseteq$ (K: Hom (M, N)) M.

Since K is a primary submodule of Hom (M, N) then (K: Hom (M, N)) is a primary ideal of R by (2.1). But M is a primary multiplication module and M is a primary submodule of M, so M is a multiplication submodule of M. By lemma (3.2) and proposition (2.1) we have (K: Hom (M, N)) M is a primary submodule of M. thus there exist an ideal I in R such that

(L: Hom (M, N)) M = I (K: Hom (M, N)) M.

Now, since M is finitely generated and multiplication submodule of M, then M has the weak cancellation property by [8. Theorem (6.6)] and hence (L: Hom (M, N)) + Ann M = I (K: Hom (M, N)) +Ann M. thus

[(L: Hom (M, N)) + Ann M] Hom (M, N) = [I (K: Hom (M, N)) +Ann M] Hom (M, N)

Since Ann (M) $\subseteq$ Ann (Hom (M, N)), then

(L: Hom (M, N)) Hom (M, N) = I (K: Hom (M, N)) Hom (M, N)

But, Hom (M, N) is a multiplication submodule of Hom (M, N) by [7.Theorem (3.4)], therefore L = IK, and hence K is a multiplication submodule of Hom (M, N) and thus Hom (M, N) is a primary multiplication module.

Corollary (3.4):

(A): If M is finitely generated primary multiplication module, then Hom (M, M) is a primary multiplication module.

(B): If M is faithful, finitely generated and primary multiplication R-module, then Hom (M, R) a primary multiplication module.

The following proposition is a partial converse of (3.3)

Proposition (3.5):

Let each of M and N be R-modules. If M is a multiplication submodule of M such that Ann M = Ann Hom (M, N) and Hom (M, N) is finitely generated primary multiplication module, then M is a primary multiplication module.

Proof:

Let K be a primary submdule of M and L be a submodule of M such that L $\subseteq$ K. then (L: M) $\subseteq$ (K: M) and hence (L: M) Hom (M, N) $\subseteq$ (K: M) Hom (M, N). Since K is a primary

submodule of M, then (K: M) is a primary ideal in R by (2.1) But
Hom (M, N) is a multiplication submodule of Hom (M, N), and hence by the previous similar argument we have, (K:M) Hom (M,N) is a primary submodule of Hom(M,N), and therefor (K:M) Hom(M,N) is a multiplication submodule of Hom(M,N), This implies the existence of an ideal I in R such that (L:M) Hom(M,N) = I (K:M) Hom (M,N).
Now, Hom (M, N) is finitely generated and multiplication submodule of M, so Hom (M, N) has the weak cancellation property by [8, theorem (6.6)]. That is (L: M) + Ann Hom (M, N) = I (K: M) + Ann Hom (M, N)
Thus [(L: M) + Ann Hom (M, N)] M= [I (k: M) + Ann Hom (M, N)] M.
But Ann M = Ann Hom(M, N).Therefore (L: M) M= I (k: M) M.
Also, M is amultiplication submodule of M, so L= I k, then k is a multiplication submodule of M. Therefore M is a primary multiplication module.
We end this paper by the following corollary

Corollary (3.6):

(A): If M is a multiplication submodule of M such that
Ann (M) = Ann (Hom (M, M)) and Hom (M, M) is a finitely generated, primary multiplication module then M is a primary multiplication module.

(B): Let M be a multiplication submodule of M such that
Ann M = Ann (Hom (M, R)) If Hom (M, R) is a finitely generated primary multiplication module, then M is a primary multiplication module.

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المقاسات الجدائية الابتدائية

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الخلاصة:

يقال لمقاس أحادي M معرفا على حلقة ابدالية ذات عنصر محايد R بأنه جدائي ابتدائي أذا كان كل مقاس جزئي ابتدائي منه هو مقاس جزئي جدائي. درسنا بعض خواص هذا المفهوم وبرهنا على انه اذا كان كل من M وN مقاسا فأن $M \underset{R}{\otimes} N$ و $Hom_R (M,N)$ مقاسان جدائيان ابتدائيان تحت شروط معينة.