Approximate Solution of Delay Differential Equations Using the Collocation Method Based on Bernstien Polynomials

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Abstract:

In this paper a modified approach have been used to find the approximate solution of ordinary delay differential equations with constant delay using the collocation method based on Bernstien polynomials.

Key words: Bernstien polynomial, Delay differential equation.

Introduction:

The definition of the n-th order linear delay differential equations with one constant delay may be written as:

$F(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_k), y'(t), y'(t-\tau_1), y'(t-\tau_2), \ldots, y'(t-\tau_k), y''(t-\tau_1), y''(t-\tau_2), \ldots, y''(t-\tau_k), y^{(n)}(t-\tau_1), y^{(n)}(t-\tau_2), \ldots, y^{(n)}(t-\tau_k) = g(t)$ …(1)

where $g(t)$ is a given continuous function and the time lag τ is constant and a_0, a_1, b_0 and b_1 are constant coefficients.

Delay differential equations have a great importance in real life problems which found many applications in mechanics, physics, engineering, economics, biology and especially in the theory of automatic control, [1].

For this importance of delay differential equations many scientists and mathematicians worked on this field of mathematics applied by using several methods of solution like (the method of steps, Laplace transformation method, etc.), see [1], [2].

In this paper, we will solve the first order linear delay differential equations approximated by the collocation method with the Bernstien polynomials as a basis functions.

1- Bernstien Polynomials:

Now, the fundamental definition and some basic properties of Bernstain polynomials are given that will be used later in the definition and construction of the collocation method for solving delay differential equations.

1.1 Definition: [3]

The Bernstien polynomials of degree $n \in \mathbf{N}$ are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad , 0 \le t < \infty$$

For $i = 0, 1, \ldots, n$, where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$

We usually set $B_{i,n} = 0$ if $i < 0$ or $i > n$.

1.2 Converting Bernstien Basis to Power Basis:[4]

Since the power basis $\{1, t, t^2, \ldots, t^n\}$ form a basis for the space of polynomials of degree less than or equal to n, then any Bernstien polynomial of degree n may be rewritten in terms of the power basis, as follows:

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

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$$= \binom{n}{k} t^k \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} t^i$$

$$= \sum_{i=0}^{n-k} (-1)^i \binom{n}{k}\binom{n-k}{i} t^{i+k}$$

$$= \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{k}\binom{n-k}{i-k} t^i$$

or

$$B_{k,n}(t) = \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{i}\binom{i}{k} t^i \quad ,$$

$0 \le t < \infty$, k=0,1,…,n.

1.3 Differentiation of Bernstien Polynomials: [5]

Derivation of the n-th degree of Bernstien polynomials are polynomials of degree $n - 1$. By using the definition of the Bernstien polynomial, we can show that this derivative may be written also as a linear combination of Bernstien polynomials, as:

$$\frac{d}{dt} B_{k,n}(t) = n\,(B_{k-1,n-1}(t) - B_{k,n-1}(t)),\ 0 \le k \le n.$$

This can be shown by direct differentiation as following:

$$\frac{d}{dt} B_{k,n}(t) = \frac{d}{dt} \binom{n}{k} t^k (1-t)^{n-k}$$

$$= \frac{k\,n!}{k!(n-k)!} t^{k-1}(1-t)^{n-k} + \frac{(n-k)\,n!}{k!(n-k)!} t^k (1-t)^{n-k-1}$$

$$= \frac{n(n-1)!}{(k-1)!(n-k)!} t^{k-1}(1-t)^{n-k} + \frac{n(n-1)!}{k!(n-k-1)!} t^k (1-t)^{n-k-1}$$

$$\frac{d}{dt} B_{k,n}(t) = n\left(\frac{(n-1)!}{(k-1)!(n-k)!} t^{k-1}(1-t)^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} t^k (1-t)^{n-k-1}\right)$$

$$= n\,(B_{k-1,n-1}(t) - B_{k,n-1}(t)) \quad \ldots(2)$$

2- The Collocation Method:

The collocation method is one of the most usually common methods used to approximate the solution of ordinary differential equations, integral equations, partial differential equations, etc., [Delves L.1985] and [Doyc D., 2001], see [6] and [7].

Here, we will use the collocation method to solve ordinary and linear delay differential equations with Bernstien polynomials as basis functions. For this objective consider the retarted delay differential equation of the form

$$y'(t) + y(t) + y(t-\tau) = g(t) \quad \ldots(3)$$

With the initial condition $y_0(t) = \phi(t)$ where $t_0 - \tau \le t \le t_0$ and the solution of this equation will be formed by using the collocation method

Let $$y(t) = \psi(t) + \sum_{i=0}^{n} a_i B_{i,n}(t) \quad \ldots(4)$$

where ψ is an arbitrary function satisfying the non homogenous initial conditions, then

$$y'(t) = \psi'(t) + \sum_{i=0}^{n} a_i B'_{i,n}(t)$$

from (2.2), we have

$$y'(t) = \psi'(t) + \sum_{i=0}^{n} n\,a_i\,(B_{i-1,n-1}(t) - B_{i,n-1}(t))$$

$$y'(t) = \psi'(t) + \sum_{i=1}^{n} n\,a_i\,B_{i-1,n-1}(t) - \sum_{i=0}^{n-1} n\,a_i\,B_{i,n-1}(t) \quad \ldots(5)$$

and $y(t-\tau) = \psi(t-\tau)$ …(6)

By substituting (4), (5) and (6) in (3), we have;

$$\psi(t) + \psi'(t) + \psi(t-\tau) + \sum_{i=1}^{n} n\,a_i\,B_{i-1,n-1}(t) - \sum_{i=0}^{n-1} n\,a_i\,B_{i,n-1}(t) + \sum_{i=0}^{n} a_i\,B_{i,n}(t) = g(t)$$

From (2.1), we have

$$\psi(t) + \psi'(t) + \psi(t-\tau) + \sum_{i=1}^{n} n\,a_i \sum_{k=i-1}^{n-1} (-1)^{k-i+1} \binom{n-1}{k}\binom{k}{i-1} t^k - \sum_{i=0}^{n-1} n\,a_i \sum_{k=i}^{n-1} (-1)^{k-i} \binom{n-1}{k}\binom{k}{i} t^k + \sum_{i=0}^{n} a_i \sum_{k=i}^{n} (-1)^{k-i} \binom{n}{k}\binom{k}{i} t^k = g(t) \ldots(7)$$

Simplifying equation (7) and substituting on t by $n + 1$ points that lies in the interval $[t_0 - \tau, t_0]$, we have $n + 1$ algebraic equations with $n + 1$ unknowns $a_0, a_1, \ldots, a_n$ and in matrix form as:

A X = B

where A is an $(n + 1) \times (n + 1)$ constant matrix, X is the column of the unknown elements $a_0, a_1, \ldots, a_n$ and B is a given vector, [8].

This system may be solved by using computer programs to get the values of $a_0, a_1, \ldots, a_n$.

Then substituting this values back into the function $y(t)$ to get the approximate solution of the delay differential equation.

The next examples illustrate the above method of solution;

2.1 Example :

Consider the retarted delay differential equation

$y'(t) + y(t) + y(t-1) = t^2$ …(8)

with the initial condition $y_0(t) = t$, $-1 \le t \le 0$.

Then by letting

$$y(t) = t + \sum_{i=0}^{n} a_i \, B_{i,n}(t)$$

Hence, from the initial condition at $t = 0$, we have

$y(0) = a_0 \Rightarrow a_0 = 0$.

$$y'(t) = 1 + \sum_{i=1}^{n} n\, a_i \, B_{i-1,\, n-1}(t) - \sum_{i=1}^{n-1} n\, a_i \, B_{i,\, n-1}(t)$$

and

$$y(t-1) = t - 1 + \sum_{i=0}^{n} a_i \, B_{i,n}(t-1)$$

Substituting in the delay differential equation yields t_0:

$$2\,t + \sum_{i=1}^{n} n\, a_i \, B_{i-1,\, n-1}(t) - \sum_{i=1}^{n-1} n\, a_i \, B_{i,\, n-1}(t) + \sum_{i=1}^{n} a_i \, B_{i,n}(t) = t^2$$

The solution will be found when $n = 3$, from equation (7), we have;

$$2\,t + \sum_{i=1}^{3} 3a_i \sum_{k=i-1}^{2} (-1)^{k-i+1} \binom{2}{k}\binom{k}{i-1} t^k - \sum_{i=1}^{2} 3a_i \sum_{k=i}^{2} (-1)^{k-i} \binom{2}{k}\binom{k}{i} t^k + \sum_{i=1}^{3} a_i \sum_{k=i}^{3} (-1)^{k-i} \binom{3}{k}\binom{k}{i} t^k + \sum_{i=1}^{3} a_i \sum_{k=i}^{3} (-1)^{k-i} \binom{3}{k}\binom{k}{i} (t-1)^k = t^2$$

$2\,t + H_1 - H_2 + H_3 + H_4 = t^2$ …(9)

where

$$H_1 = \sum_{i=1}^{3} 3a_i \sum_{k=i-1}^{2} (-1)^{k-i+1} \binom{2}{k}\binom{k}{i-1} t^k$$

$$H_2 = \sum_{i=1}^{2} 3a_i \sum_{k=i}^{2} (-1)^{k-i} \binom{2}{k}\binom{k}{i} t^k$$

$$H_3 = \sum_{i=1}^{3} a_i \sum_{k=i}^{3} (-1)^{k-i} \binom{3}{k}\binom{k}{i} t^k$$

$$H_4 = \sum_{i=1}^{3} a_i \sum_{k=i}^{3} (-1)^{k-i} \binom{3}{k}\binom{k}{i} (t-1)^k$$

and to simplify H_1, H_2, H_3 and H_4 as follows:

$$H_1 = \sum_{i=1}^{3} 3a_i \sum_{k=i-1}^{2} (-1)^{k-i+1} \binom{2}{k}\binom{k}{i-1} t^k$$

$$H_1 = 3a_1 \sum_{k=0}^{2} (-1)^{k} \binom{2}{k}\binom{k}{0} t^k + 3a_2 \sum_{k=1}^{2} (-1)^{k-1} \binom{2}{k}\binom{k}{1} t^k + 3a_3 \sum_{k=2}^{2} (-1)^{k-2} \binom{2}{k}\binom{k}{2} t^k$$

$$H_1 = 3a_1 \left[\binom{2}{0}\binom{0}{0} t^0 + (-1)\binom{2}{1}\binom{1}{0} t + (-1)^2 \binom{2}{2}\binom{2}{0} t^2\right] + 3a_2 \left[\binom{2}{1}\binom{1}{1} t + (-1)^1 \binom{2}{2}\binom{2}{1} t^2\right] + 3a_3 \binom{2}{2}\binom{2}{2} t^2$$

$= 3a_1\,[\,1 - 2t + t^2] + 3a_2[2t - 2t^2] + 3a_3 t^2$

$= 3(a_1 - 2a_2 + 3a_3)\, t^2 + 6(-a_1 + a_2)t + 3a_1$

$$H_2 = \sum_{i=1}^{2} 3a_i \sum_{k=i}^{2} (-1)^{k-i} \binom{2}{k}\binom{k}{i} t^k$$

$$= 3a_1 \sum_{k=1}^{2} (-1)^{k-1} \binom{2}{k}\binom{k}{1} t^k +$$

$$3a_2 \sum_{k=2}^{2} (-1)^{k-2} \binom{2}{k}\binom{k}{2} t^k$$

$$= 3a_1\left[\binom{2}{1}\binom{1}{1}t - \binom{2}{2}\binom{2}{1}t^2\right] + 3a_2\binom{2}{2}\binom{2}{2}t^2$$

$$= 3\, a_1[2t - 2t^2] + 3a_2 t^2$$

$$= 3(-2a_1 + 3a_2)\, t^2 + 6(-a_0 + a_1)t + 3a_0$$

By the same way, we have

$H_3 = (3a_1 - 3a_2 + a_3)\, t^3 + (-6a_1 + 3a_2)t^2 + (+3a_1)\, t +$

and

$H_4 = (+3a_1 - 3a_2 + a_3)\, t^3 + (-15a_1 + 12a_2 - 3a_3)\, t^2 + (24a_1 - 15a_2 + 3a_3)\, t + (-12a_1 + 6a_2 - a_3)$

By substituting H_1, H_2, H_3 and H_4 in equation (9), yields:

$2t + 3(a_1 - 2a_2 + 3a_3)\, t^2 + 6(-a_1 + a_2)t + 3a_1 - 3(-2a_1 + 3a_2)\, t^2 - 6a_1 t - 3a_0 + (3a_1 - 3a_2 + a_3)\, t^3 + (-6a_1 + 3a_2)t^2 + 3a_1\, t + (3a_1 - 3a_2 + a_3)\, t^3 + (-15a_1 + 12a_2 - 3a_3)t^2 + (24a_1 - 15a_2 + 3a_3)\, t + (-12a_1 + 6a_2 - a_3) = t^2$

Thus;

$(3a_1 - 3a_2 + a_3)\, t^3 + (3a_1 - 6a_2 + 3a_3 - 1)t^2 + (2 - 9a_1 + 6a_2)t + 3a_3 = 0$

To find a_1, a_2, a_3, one may choose three points say: $t_1 = \frac{1}{3}$, $t_2 = \frac{1}{2}$, $t_3 = 1$ to get the following system:

$$0.444a_1 + 1.222a_2 + 0.37a_3 = -0.55$$
$$-0.375a_1 + 1.125a_2 + 0.875a_3 = -0.75$$
$$-3a_2 + 4a_3 = -1$$

and solving of equations produce

$a_1 = -0.084$, $a_2 = -0.245$ and $a_3 = -0.577$

Hence, the solution is

$$y(t) = t + \sum_{i=0}^{3} a_i B_{i,3}(t)$$

$$= t + 0B_{0,3}(t) - 0.084\, B_{1,3}(t) - 0.245\, B_{2,3}(t) - 0.577\, B_{3,3}(t)$$

$y_1(t) = -0.094t^3 - 0.231\, t^2 + 0.748\, t = \psi_1(t)$, where $0 \le t \le 1$

Now, if $1 \le t \le 2$ by the same way above from the initial condition at $t = 1$, we have:

$$y_1(1) = \psi_1(1) + \sum_{i=0}^{3} a_i\, B_{i,3}(t)$$

that yields to $a_3 = 0$ then

$$y_2(t) = \psi_1(t) + \sum_{i=0}^{2} a_i\, B_{i,2}(t)$$

$$y'_2(t) = \psi'_1(t) + \sum_{i=0}^{2} a_i\, B'_{i,2}(t)$$

$y_2(t-1) = y_1(t-1)$

by substituting $y_2(t)$, $y'_2(t)$, $y_2(t-1)$ in the equation (8) we have the following equation

$(-0.137 - 2a_0 + 3a_1) + (1.214 + 3a_0 - 9a_1 + 6a_2)t + (-1.462 + 3a_1 - 6a_2)t^2 + (-0.188 - a_0 + 3a_1 - 3a_2)t^3 = 0$

We can choose three points say $t_1 = 1.25$, $t_2 = 1.5$, $t_3 = 2$ to get the following system:

$-0.203a_0 + 2.296a_1 - 7.743a_2 = 1.271$

$-0.875a_0 + 6.375a_1 - 14.625a_2 = 2.239$

$-4a_0 + 21a_1 - 36a_2 = 5.061$

and solving of equations produce

$a_0 = 0.108$, $a_1 = -0.054$, $a_2 = -0.183$

Hence, the solution is

$y_2(t) = 0.182\, t^3 - 0.132\, t^2 + 0.262\, t + 0.108$

by the same way for the other intervals [2,3], [3,4], …

2.2 Example :

Consider the mixed linear delay differential equation

$$y'(t) + y'(t-1) + y(t) + y(t-1) = 2t \quad \ldots(10)$$

with the initial condition $y_0(t) = t$, where $-1 \le t \le 0$.

Solution: Let $y(t) = t + \sum_{i=0}^{n} a_i\, B_{i,n}(t)$,

then $y'(t) = 1 + \sum_{i=0}^{n} a_i\, B'_{i,n}(t)$

$$= 1 + \sum_{i=1}^{n} n\, a_i\, B_{i-1,\, n-1}(t) - \sum_{i=0}^{n-1} n\, a_i\, B_{i,\, n-1}(t)$$

and $y(t-1) = t-1$, $y'(t-1) = 1$

Substituting in the delay differential equation (8), give:

$$2t + 1 - \sum_{i=0}^{n-1} n\, a_i\, B_{i,\, n-1}(t) + \sum_{i=0}^{n} a_i\, B_{i,\, n}(t) = 2t$$

When $n = 3$, then from equation (7), we have;

$$1 + \sum_{i=1}^{3} 3a_i \sum_{k=i-1}^{2} (-1)^{k-i+1} \binom{2}{k} \binom{k}{i-1} t^k - \sum_{i=0}^{2} 3a_i \sum_{k=i}^{2} (-1)^{k-i} \binom{2}{k} \binom{k}{i} t^k + \sum_{i=0}^{3} a_i \sum_{k=i}^{3} (-1)^{k-i} \binom{3}{k} \binom{k}{i} t^k = 0$$

From this equation and the initial condition that produce $a_0 = 0$, at $t = 0$, we get:

$$(1 + 3a_1) + (-9a_1 + 6a_2)t + (3a_1 - 6a_2 + 3a_3)t^2 + (3a_1 - 3a_2 + a_3)t^3 = 0 \qquad \ldots(11)$$

To find a_1, a_2, a_3 on the interval [0,1], we choose three points say: $t_1 = \frac{1}{3}$, $t_2 = \frac{1}{2}$, $t_3 = 1$ substituting in (11) we have:

$$0.444a_1 + 1.222a_2 + 0.73a_3 = -1$$
$$-0.375a_1 + 1.125a_2 + 0.875a_3 = -1$$
$$-3a_2 + 4a_3 = -1$$

by solving this system we have:

$a_1 = -0.33$, $a_2 = -0.508$ and $a_3 = -0.63$

Then $y_1(t) = -0.096\, t^3 + 0.456\, t^2 + 0.01\, t$

and for the interval [1,2] from the new initial condition $y_1(t)$ for $t = 1$ we will get $a_3 = 0$ we choose the points $t_1 = 1.25$, $t_2 = 1.5$, $t_3 = 2$, by the same way above we get:

$$-0.203a_0 + 2.296a_1 - 7.734a_2 = 0.113$$
$$-0.875a_0 + 6.375a_1 - 14.625a_2 = -0.244$$
$$-4a_0 + 21a_1 - 36a_2 = -1.066$$

and by solving this system we have:

$a_0 = -3.626$, $a_1 = -0.743$ and $a_2 = -0.14$

Then $y_2(t) = 1.721\, t^3 - 6.384\, t^2 + 8.659\, t - 3.626$

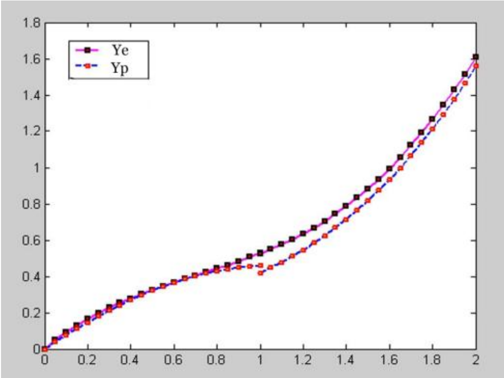

Fig. (1) Illustrate the approximate solution y_p and exact solution y_e of the retarted delay differential equation (8)

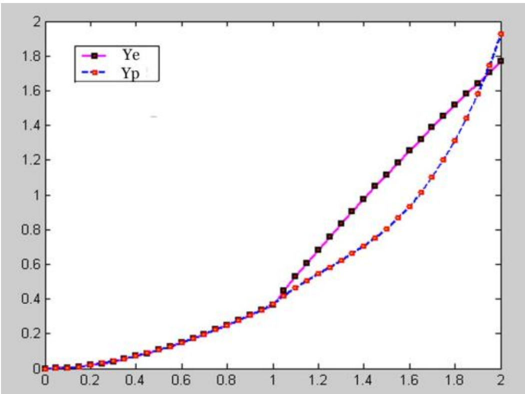

Fig. (2) illustrate the approximate solution y_p and exact solution y_e of the mixed delay differential equation (10)

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الحل التقريبي للمعادلات التفاضلية التباطؤية باستخدام طريقةالحشد المعتمدة على متعددات حدود بيرنشتاين

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الخلاصة:

في هذا البحث تم استخدام تقارب مُعَدل لايجاد حل تقريبي للمعادلات التفاضلية الاعتيادية التباطؤية ذات تباطؤ ثابت باستخدام طريقة الحشد معتمدة على متعددات حدود بيرنشتاين كأساس للحل.