Jordan left (θ,θ) -derivations Of σ-prime rings

Ikram A.saed*

Received 15, March, 2010
Accepted 27 , September, 2010

Abstract:
It was known that every left (θ,θ) -derivation is a Jordan left (θ,θ) – derivation on σ-prime rings but the converse need not be true. In this paper we give conditions to the converse to be true.

Key words: σ- prime rings , σ- square closed lie idea, left (θ,θ) - derivation , Jordan left (θ,θ) -derivations .

Introduction:
In [1] Ashraf proved that every Jordan left(θ,θ) - derivation on prime ring is a left (θ,θ) - derivation on prime ring . In[2] Oukhtite and Salhi proved that every Jordan left derivation on σ-prime ring is a left derivation on σ-prime ring. In this paper we prove that every Jordan left (θ,θ) - derivation on σ-prime ring is a left (θ,θ) - derivation on σ-prime ring.

§ 1 Basic Concepts:

Definition 1.1 : [2]
A ring R is said to be 2-torsion-free if whenever 2x= 0 with
x$\in$R, then x = 0 .

Definition 1.2 : [3]
Let R be a ring . Define a lie product [. , .] on as follows
[x , y] = xy – yx , for all x,y $\in$R .

Properties 1.3: [3]
Let R be a ring . Then for all x,y $\in$R , we have
1. [x,yz] = y[x,z] + [x,y] z .
2. [xy,z] = x[y,z] + [x,z] y .

Definition 1.4 : [4]
A ring R is called a prime if for any a,b $\in$R,
aRb = {o} implies that either a = o or b = o

Definition 1.5 : [2]
A ring R with involution σ is said to be σ- prime if aRb = aR σ (b) = {o} implies that a = o or b = o

Definition 1.6 : [5]
A ring R with involution σ , we define
Sa_{σ} (R) = {r$\in$R / σ (r) = $\pm$r} .

Definition 1.7 : [3]
A Lie ideal of a ring R is an additive subgroup
U of ring R satisfying [U,R] $\subset$U .

Definition 1.8 : [4]
A Lie ideal U of a ring R is said to be
σ - lie ideal , if σ (U) = U

Definition 1.9 : [2]
If U is a σ - Lie ideal of a ring R such that

*University of Technology

u^2 ∈U for all u∈U , then U is called a σ – square closed Lie ideal .

Definition 1.10 : [1]

Let R be a ring . An additive mapping d:R→R is called a left (θ,θ) -derivation where θ : R → R is a mapping of R , if

d(xy) = θ (x) d (y) + θ (y) d (x) , for all x,y ∈R and

we say that d is a Jordan left (θ,θ) -derivation

If d(x^2) = θ (x) d(x) + θ (x) d(x), for all x ∈ R.

= 2θ(x) d(x), for all x ∈ R.

It is clear that every left (θ,θ)-derivation of R is a Jordan left (θ,θ)-derivation , but the converse is not true as the following example, shows:

Example 1.11:-

Let R be a commutative ring and let a∈R

Such that $\theta(x)\, a\theta(x) = 0, for\ all$ x∈R.

but $\theta(x)\ a\theta(y) \neq 0$, for some x and y ∈ R, such that x ≠ y.

Define a map d:R→ R as follows

d(x) = θ(x) a, for all x∈R

Where θ:R→ R is an endomorphism mapping.

Then d is a Jordan left (θ,θ)-derivation but not a left (θ,θ)-derivation.

It is clear that d is an additive mapping.

Now, we have to show that d is satisfies

d(x^2) = $\theta(x)$ d(x) + $\theta(x)$ d(x) = 2 $\theta(x)$ d(x), for all x ∈ R

d(x^2) = θ(x^2)a

= $\theta(x)$a $\theta(x)$ = 0, for all x ∈ R

2 $\theta(x)$ d(x) = 2 $\theta(x)$ $\theta(x)$a

=2 $\theta(x)$a $\theta(x)$ = 0, for all x ∈ R

∴ d(x^2) = $2\ \theta(x)\ d(x)$, for all x ∈ R

∴ d is a Jordan left (θ,θ)- derivation of R.

We must prove that d is not a left (θ,θ)- derivation of R.

d(xy) = $\theta(xy)$a

= $\theta(x)$a $\theta(y)$, for all x, $y \in$ R.

but

$\theta(x)\ d(y)$ + $\theta(y)\ d(x)$ = $\theta(x)\ \theta(y)$a + $\theta(y)\ \theta(x)$a = $\theta(x)$a $\theta(y)$ + $\theta(x)$a $\theta(y)$= 2 $\theta(x)$a $\theta(y)$ for all x, y ∈ R.

Since $\theta(x)$a $\theta(y)$ ≠ 0, for some x and y ∈ R.

∴ d(xy) ≠ $\theta(x)\ d(y)$ + $\theta(y)\ d(x)$, for some x and y ∈ R.

∴ d is not a left (θ,θ)- derivation of R.

Lemma 1.12: [2]

If U⊄ Z (R) is a σ - Lie ideal of a 2- torsion – free σ- Prime ring R and a,b ∈ R such that a U b = σ (a) U b = {0}, then a=0 or b=0.

Lemma 1.13:

Let R be a 2- torsion-free σ - prime ring and U be a σ - square closed Lie ideal of R. suppose that θ is an endomorphism of R. If d:R→ R is an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$, for all u, v $\in$ U then

(i) $d(uv+vu) = 2\theta(u)d(v) + 2\theta(v)d(u)$, for all u, v $\in$ U.

(ii) $d(uvu) = \theta(u^2)d(v) + 3\theta(u)\theta(v)d(u) - \theta(v)\theta(u)d(u)$, for all u, v $\in$ U

(iii) $d(uvw + wvu) = \{\theta(u)\theta(w) + \theta(w)\theta(u)\}d(v) + 3\theta(u)\theta(v)d(w) + 3\theta(w)\theta(v)d(u) - \theta(v)\theta(u)d(w) - \theta(v)\theta(w)d(u)$, for all u, v, w $\in$ U

(iv) $[\theta(u), \theta(v)]\theta(u)d(u) = \theta(u)[\theta(u), \theta(v)]d(u)$, for all u, v $\in$ U.

(v) $[\theta(u), \theta(v)]d([u,v]) = 0$, for all u, v $\in$ U.

(vi) $d(vu^2) = \theta(u^2)d(v) + (3\theta(v)\theta(u) - \theta(u)\theta(v))d(u) - \theta(u)d([u,v])$, for all u, v $\in$ U.

Proof:

(i) Since $uv + vu = (u + v)^2 - u^2 - v^2$, we find that uv + vu $\in$ U . for all u,v $\in$ U

Hence by linearizing

$d(u^2) = 2\theta(u)d(u)$ on u, we get

$d(uv + vu) = 2\theta(u)d(v) + 2\theta(v)d(u)$, for all u, v $\in$ U. ———1

(ii) Replacing v by uv + vu in 1, we get

$d(u(uv + vu) + (uv + vu)u) = 4\theta(u^2)d(v) + 6\theta(u)\theta(v)d(u) + 2\theta(v)\theta(u)d(u)$ ————2

On ht other hand,

$d(u(uv + vu) + (uv + vu)u) = d(u^2v + vu^2) + 2d(uvu)$

$= 2\theta(u^2)d(v) + 4\theta(v)\theta(u)d(u) + 2d(uvu)$.

Combining the above equation with 2, we get

$d(uvu) = \theta(u^2)d(v) + 3\theta(u)\theta(v)d(u) - \theta(v)\theta(u)d(u)$, for all u, v $\in$ U.

(iii)By linearizing (ii) on u, we get

$d((u + w)v(u + w)) = \theta(u^2)d(v) + \theta(w^2)d(v) + \{\theta(u)\theta(w) + \theta(w)\theta(u)\}d(v) + 3\theta(u)\theta(v)d(w) + 3\theta(u)\theta(v)d(u) + 3\theta(w)\theta(v)d(w) + 3\theta(w)\theta(v)d(u) - \theta(v)\theta(u)d(u) - \theta(v)\theta(u)d(w) - \theta(v)\theta(w)d(u) - \theta(v)\theta(w)d(w)$. ————3

on the other hand,

$d[(u + w)v(u + w)] = d(uvu) + d(wvw) + d(uvw + wvu)$

$= \theta(u^2)d(v) + 3\theta(u)\theta(v)d(u) - \theta(v)\theta(u)d(u) + \theta(w^2)d(v) + 3\theta(w)\theta(v)d(w) - \theta(v)\theta(w)d(w) + d(uvw + wvu)$. ————4

Combining 3 and 4, we get

$d(uvw + wvu) = \{\theta(u)\theta(w) + \theta(w)\theta(u)\}d(v) + 3\theta(u)\theta(v)d(w) + 3\theta(w)\theta(v)d(u) - \theta(v)\theta(u)d(w) - \theta(v)\theta(w)d(u)$, for all u, v $\in$ U. ————5

(iv)Since uv + vu and uv – vu both belong to U

we find that 2uv $\in$ U for all u, v $\in$ U.

Hence, by our hypothesis we find that

$d((2uv)^2) = 2\theta(2uv)d((2uv))$

$4d((uv)^2) = 8\theta(uv)d(uv)$. Since char R$\neq$ 2, we have

$d((uv)^2) = 2\theta(u)\theta(v)d(uv)$. Replace w by 2uv in 5, and use the fact that char R$\neq$ 2, to get

$d(uv(uv) + (uv)vu) = \{\theta(u^2)\ \theta(v) + \theta(u)\ \theta(v)\ \theta(u)\}\ d(v) + 3\,\theta(u)\ \theta(v)\ d(uv) + 3\,\theta(u)\ \theta(v^2)\ d(u) - \theta(v)\ \theta(u)\ d(uv) - \theta(v)\ \theta(u)\ \theta(v)\ d(u).$ ——6

On the other hand,

$d((uv)^2 + uv^2u) = 2\,\theta(u)\ \theta(v)\ d(uv) + 2\,\theta(u^2)\ \theta(v)\ d(v) + 3\,\theta(u)\ \theta(v^2)\ d(u) - \theta(v^2)\ \theta(u)\ d(u).$ ——7

Combining 6 and 7, we get

$[\theta(u), \theta(v)]\ d(uv) = \theta(u)\ [\theta(u), \theta(v)]\ d(v) + \theta(v)\ [\theta(u), \theta(v)]\ d(u)$ ——8

Replacing u + v for v in 8, we have

$2[\theta(u), \theta(v)]\ \theta(u)\ d(u) + [\theta(u), \theta(v)]\ d(uv) =$

$2\,\theta(u)\ [\theta(u), \theta(v)]\ d(u) + \theta(u)\ [\theta(u), \theta(v)]\ d(v) +$

$\theta(v)\ [\theta(u), \theta(v)]\ d(u).$

Now application of 8 yields (iv)

(v) linearize (iv) on u, to get

$[\theta(u), \theta(v)]\ \theta(u)\ d(u) + [\theta(u), \theta(v)]\ \theta(v)\ d(v) +$

$[\theta(u), \theta(v)]\ \theta(u)\ d(v) + [\theta(u), \theta(v)]\ \theta(v)\ d(u) =$

$\theta(u)\ [\theta(u), \theta(v)]\ d(u) + \theta(u)\ [\theta(u), \theta(v)]\ d(v) +$

$\theta(v)\ [\theta(u), \theta(v)]\ d(u) + \theta(v)\ [\theta(u), \theta(v)]\ d(v)$, for all $u, v \in U$.

Now application of 8 and (iv) yields that

$[\theta(u), \theta(v)]\ \theta(u)\ d(v) + [\theta(u), \theta(v)]\ \theta(v)\ d(u) =$

$[\theta(u), \theta(v)]\ d(uv)$ and hence

$[\theta(u), \theta(v)]\ \{d(uv) - \theta(u)\ d(v) - \theta(v)\ d(u)\} = 0$ for all $u, v \in U$. ——9

Combining 1 and 9 we find that,

$[\theta(u), \theta(v)]\ \{d(vu) - \theta(u)\ d(v) - \theta(v)\ d(u)\} = 0$ for all $u, v \in U$. ——10

Further, combining of 9 and 10 yields the required result.

(vi) replace v by 2vu in 1, and use the fact that char $R \neq 2$, to get

$d(uvu + vu^2) = 2\,\theta(\theta(u)\ d(uv) + \theta(v)\ \theta(u)\ d(u))$ for all $u, v \in U$. ——11

Again replacing v by 2uv in 1, we get

$d(u^2v + uvu) = 2(\theta(u)\ d(uv) + \theta(u)\ \theta(v)\ d(u))$ for all $u, v \in U$. ——12

Now, combining 11 and 12, we get

$d(u^2v - vu^2) = 2(\theta(u)\ d([u,v]) + [\theta(u), \theta(v)]\ d(u))$, for all $u, v \in U$. ——13

Replacing u by u^2 in 1, we have

$d(u^2v + vu^2) = 2(\ \theta(u^2)\ d(v) + 2\,\theta(v)\ \theta(u)\ d(u))$, for all $u, v \in U$. ——14

Hence, subtracting 13 from 14 and using the fact that char $R \neq 2$, we find that

$d(vu^2) = \theta(u^2)\ d(v) + \{3\theta(v)\ \theta(u) - \theta(u)\ \theta(v)\}\ d(u) - \theta(u)\ d([u, v])$, for all $u, v \in U$.

§ 2 Jordan left (θ, θ)- derivations on σ - square closed Lie ideals:

Theorem 2.1:

Let R be a 2-torsion-free σ - prime ring and let U be a σ- square closed Lie ideal of R. Suppose that θ is an automorphism of R. If d: R→ R is an additive mapping satisfying $d(u^2) = 2\,\theta(u)\ d(u)$, for all u, ∈ U, then $d(uv) = \theta(u)\ d(v) + \theta(v)\ d(u)$, for all $u, v \in U$.

Proof

Suppose [U,U] =0 and let $u, v \in U$.

From $d((u+v)^2) = 2\,\theta(u + v)\ d(u + v)$, it follows that

$2d(uv) = 2\,\theta(u)\ d(u) + 2\,\theta(v)\ d(v) - d(u^2) - d(v^2) + 2\,\theta(u)\ d(v) + 2\,\theta(v)\ d(u)$,

In such a way that
$2d(uv) = 2(\theta(u)\ d(v) + 2\theta(v)\ d(u))$, for all u, v $\in$ U.
As char R $\neq$ 2, then
$d(uv) = \theta(u)\ d(v) + \theta(v)\ d(u)$.
Hence we shall assume that $[U,U] \neq 0$
According to Lemma 1.13 (iv) we have
$\{\theta(u^2)\ \theta(v) - 2\ \theta(u)\ \theta(v)\ \theta(u) + \theta(v)\ \theta(u^2)\}\ d(u) = 0$ ———— 1
For all u,v $\in$ U
Replacing [u,w] for u in 1, where w $\in$ U, we get
$[\theta(u), \theta(w)]^2\ \theta(v)\ d([u,w]) - 2[\theta(u), \theta(w)]\ \theta(v)\ [\theta(u), \theta(w)]\ d([u,w]) + \theta(v)\ [\theta(u), \theta(w)]^2\ d([u,w]) = 0$, for all u, v, w $\in$ U.
Now, application of Lemma 1.13 (v) yields that
$\theta^{-1}([\theta(u), \theta(w)]^2)\ U\ \theta^{-1}(d([u,w])) = \{0\}$
which implies that $[u,w]^2\ U\ \theta^{-1}(d([u,w])) = \{0\}$, for all u, w $\in$ U.
let x,y $\in Sa_{\sigma}$ (R) $\cap$ U, we have
$[x,y]^2\ U\ \theta^{-1}(d([x,y])) = \{0\} = \sigma([x,y]^2)\ U\ \theta^{-1}(d([x,y]))$ and by Lemma 1.12 either $[x,y]^2 = 0$ or
$\theta^{-1}(d([x,y])) = 0$
If $\theta^{-1}(d([x,y])) = 0$, then $d([x,y] = 0$, applying Lemma 1.13 (i) together with char R$\neq$ 2, we find that $d(xy) = \theta(x)\ d(y) + \theta(y)\ d(x)$.
Now suppose that $[x,y]^2 = 0$
From Lemma 1.13 (iv) it follows that
$\{\theta(u^2)\ \theta(v) - 2\ \theta(u)\ \theta(v)\ \theta(u) + \theta(v)\ \theta(u^2)\}\ d(v) = 0$, for all u, v $\in$ U.
Linearizing this relation in u, we obtain
$\theta(u)\ \theta(w)\ \theta(v) + \theta(w)\ \theta(u)\ \theta(v) - 2\ \theta(u)\ \theta(v)\ \theta(w) - 2\ \theta(w)\ \theta(v)\ \theta(u) + \theta(v)\ \theta(u)\ \theta(w) + \theta(v)\ \theta(w)\ \theta(u))\ d(v) = 0$, for all u, v, w $\in$ U.

Replacing v by [x,y] and using Lemma 1.13 (v), we conclude that
$(-2\ \theta(u)\ \theta([x,y])\ \theta(w) - 2\ \theta(w)\ \theta([x,y])\ \theta(w) - 2\ \theta(w)\ \theta([x,y])\ \theta(u) + \theta([x,y])\ \theta(w)\ \theta(u))\ d([x,y]) = 0$ ———— 2
Write u[x,y] instead of u in 2, since $[x,y]^2 = 0$,
Lemma 1.13 (v) leads us to
$\theta([x,y])\ \theta(u)\ \theta([x,y])\ \theta(w)\ d([x,y]) = 0$, for all u, w $\in$ U.
$\theta([x,y])\ \theta(u)\ \theta([x,y])\ U\ d([x,y]) = \{0\}$, for all u $\in$ U.
$\theta^{-1}[\theta([x,y])\ \theta(u)\ \theta([x,y])]\ U$
$\theta^{-1}(d([x,y]) = \{0\}$, for all u $\in$ U.
$[x,y]\ u\ [x,y]\ U\ \theta^{-1}(d([x,y]) = \{0\}$, for all u $\in$ U.
As $[x,y] \in U \cap Sa_{\sigma}$ (R), the fact that $\sigma(U) = U$ yields
$[x,y]\ u\ [x,y]\ U\ \theta^{-1}(d([x,y]) = \{0\} = \sigma([x,y]\ u\ [x,y])\ U\ \theta^{-1}(d([x,y])$
and using Lemma 1.12, either
$\theta^{-1} d([x,y]) = 0$ or
$[x,y]\ u\ [x,y] = 0$, for all u $\in$ U.
If $\theta^{-1}(d([x,y]) = 0$ then $d([x,y] = 0$ by Lemma 1.13 (i) together with char R$\neq$2,
we find that
$d(xy) = \theta(x)\ d(y) + \theta(y)\ d(x)$
If $[x,y]\ u\ [x,y] = 0$, for all u $\in$ U, then
$[x,y]\ U\ [x,y] = \{0\} = \sigma([x,y])U[x,y]$.
Once again using Lemma 1.12, we get $[x,y] = 0$ and
Lemma 1.13 (i) forces $d(xy) = \theta(x)\ d(y) + \theta(y)\ d(x)$.
Consequently, in both the cases we find that
$d(xy) = \theta(x)\ d(y) + \theta(y)\ d(x)$, for all x, y $\in U \cap Sa_{\sigma}$ (R) ———— 3
Now, let u,v $\in$ U, if we set

$u_1 = u + \sigma(u)$, $u_2 = u - \sigma(u)$

$v_1 = v + \sigma(v)$, $v_2 = v - \sigma(v)$

then we have $2u = u_1 + u_2$ and $2v = v_1 + v_2$.

Since $u_1, u_2, v_1, v_2 \in U \cap Sa_\sigma(R)$, application of 3 yields

$$d(2u2v) = d((u_1 + u_2)(v_1 + v_2))$$
$$= d(u_1 v_1 + u_1 v_2 + u_2 v_1 + u_2 v_2)$$
$$= \theta(u_1)\,d(v_1) + \theta(v_1)\,d(u_1) + \theta(u_1)\,d(v_2) + \theta(v_2)\,d(u_1) + \theta(u_2)\,d(v_1) + \theta(v_1)\,d(u_2) + \theta(u_2)\,d(v_2) + \theta(v_2)\,d(u_2)$$
$$= 2\,\theta(u)\,d(2v) + 2\,\theta(v)\,d(2u)$$

As char $R \neq 2$, it then follows

$d(uv) = \theta(u)\,d(v) + \theta(v)\,d(u)$, for all $u, v \in U$.

References:

1. Ashraf, M. 2005. On left $(\theta,\emptyset)$ – derivations of prime rings, Archivum mathematicum (Brno). Toms 41:157-166.
2. Oukhtite, L. and Salhi. S. 2008. Centralizing automorphisms and Jordan left derivations of σ - Prime rings, ISSN. 1(1): 19-26.
3. Ashraf, M.and Rehman.N. 2000. On Lie ideals and Jordan left derivation of prime rings, Archivum mathematicum (Brno).36:201-206.
4. Oukhtite, L., Salhi. S. and Tooufiq. L. 2007. Jordan Generalized Derivations on σ - prime Rings,International Journal of Algebra. 1(5):231-234.
5. Herestein, I.N. 1976. Rings with involution. The University of Chicago press,ed. No. 2, Chicago, pp. 34.

جوردان (θ,θ) – مشتقات يسرى على الحلقات σ- اولية

اكرام احمد سعيد*

*الجامعة التكنولوجية.

الخلاصة:

من المعروف ان كل (θ,θ)– مشتقة يسرى هي جوردان (θ,θ)— مشتقة يسرى على الحلقات σ- اولية لكن العكس غير صحيح . في هذا البحث قدمنا الشروط الكافية ليكون الاتجاه المعاكس صحيح .