Approximate Regular Modules

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Abstract:

There are two (non-equivalent) generalizations of Von Neuman regular rings to modules; one in the sense of Zelmanowize which is elementwise generalization, and the other in the sense of Fieldhowse. In this work, we introduced and studied the approximately regular modules, as well as many properties and characterizations are considered, also we study the relation between them by using approximately pointwise-projective modules.

Kay words: approximately regular modules, approximately Z-regular, approximately F-regular, approximately direct-summand, approximately pointwise-projective modules

Introduction:

Let R be a ring with 1, and let M be a unitary (left) R-module. Recall that R regular ring, if for each element x in R, there exists an element y in R such that x = xyx. In the sense of Zelmanowitze the module M is called Z-regular, if for each element m in M, there exits $f \in M^*=Hom_R(M,R)$ such that m= f(m)m [1]. In this paper we introduce the concept of approximately Z-regular modules, we call an element m in an R-module M is approximately regular, if there exists $\alpha \in M^*$ such that $m-\alpha(m)m \in J(R)M$ and $\alpha(m)=(\alpha(m))^2$. An R-module M is said to be approximately Z-regular module if each of its element is approximately regular. A ring R is approximately Z-regular if it is approximately Z-regular R-module. We obtain that approximately Z-regular modules is closed under direct sums and direct summands. Recall that an R-module M is said to be an approximately pointwise-projective module, if given R-epimorphism $\alpha : A \rightarrow B$ (where A and B are R-modules) and R-homomorphism $f : M \rightarrow B$, for each $a \in M$, there exists an R-homomorphism $g_a : M \rightarrow A$ (may depend on a) such that $(\alpha \circ g_a)(a)-f(a) \in J(R)B$ [2].We obtain that every approximately Z-regular module is approximately pointwise-projective projective and we consider versus conditions. In the sense of Fieldhouse the module M is called F-regular if every submodule of M is pure [3]. We introduce the concept of approximately F-regular modules, we call an R-module M is approximately F-regular if each submodule of M is approximately-pure. In (6) we proved that every approximately Z-regular module is approximately F-regular, recall that a submodule N of an R-module M is said to lie over a direct summand of M, if there exists a direct decomposition $M=P \oplus Q$ with $P \subseteq N$ and $N \cap Q$ is small in M [4], this concept leads us to introduce the concept of lie over approximately direct-summand. We call a submodule N of an R-module M lies over approximately direct-summand of M, if there exists a direct decomposition $M=P \oplus Q$ with $P \subseteq N$

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and $N \cap Q \subseteq J(R)M$. It is clear that a submodule which lies over approximately direct-summand lies over direct-summand.

We introduce a generalization of the following:

Proposition(1)[4]: *If M is any R-module, then the following conditions are equivalent for an element x in M:-*

(1) Rx lies over a projective direct summand of M.

(2) There exists $\alpha \in M^*$ s.t. $\alpha(x)=(\alpha(x))^2$ and $x-\alpha(x)x \in J(M)$.

(3) There exists a regular element $y \in Rx$ such that $x-y \in J(M)$ and $Rx=Ry \oplus R(x-y)$.

(4) There exists a regular element $y \in M$ such that $x-y \in J(M)$.

(5) There exists $\rho : M \longrightarrow Rx$ such that $\rho^2 = \rho$, $\rho(M)$ is projective and $x-\rho(x) \in J(M)$.

Results:

We recall that the dual basis lemma of approximately pointwise-projective modules in the flowing lemma which appear in [2]:

Lemma (2): *Let M be an R-module. Then the following statements are equivalent:*

1) M is approximately pointwise-projective.

2) Every R-epimorphism $\alpha : A \longrightarrow M$ is approximately pointwise *spilt for each R-module A.*

3) Every R-epimorphism $\alpha : F \longrightarrow M$ is approximately pointwise-projective *spilt for each free R-module F.*

4) For each $m \in M$, there exist families $\{x_i\}_{i=1}^{n}, x_i \in M$ and $\{\varphi_i\}_{i=1}^{n}$, $\varphi_i \in M^*=Hom_R(M,R)$ such that $\sum_{i=1}^{n} \varphi_i(m)x_i - m \in J(R)M$.

Proposition (3): *Every approximately Z-regular module is* approximately pointwise-projective.

Proof: Let M be approximatelt Z-regular, then $\forall m \in M, \exists \alpha \in M^*$ such that $m - \alpha(m)m \in J(R)M$ and $\alpha(m) = (\alpha(m))^2$. i.e. there exist families $\{x_i\}_{i=1}^{n} \begin{cases} = m & \text{where } i=1 \end{cases}$ and $\{\alpha_i\}_{i=1}^{n} \begin{cases} \alpha & \text{where } i=1 \\ 0 & \text{where } i>1 \end{cases}$

$\phantom{\{x_i\}} \begin{cases} 0 & \text{where } i>1 \end{cases}$

s.t.

$$\sum_{i=1}^{n} \alpha_i(x_i)m - m = \alpha(m)m - m \in J(R)M$$

.So by Dual-Basis Lemma (2) M is approximately pointwise-projective.

Recall that an R-module M is said to be an approximately-projective module, if for each R-epimorphism $\alpha: A \to B$ (where A and B are R-modules) and every R-homomorphism $f: M \to B$, there exists an R-homomorphism $g: M \to A$ such that $(\alpha \circ g)(a)-f(a) \in J(R)B$ $\forall a \in M$ [5].

Now, we are in a position to give an example of approximately pointwise-projective module, but it is not approximately-projective by using approximately Z-regular module.

Example (4): Let K be a field, and I be an infinite index set. For each $i \in I$, let $K=K_i$. Let $R = \prod_{i=1}^{\infty} K_i$ ith coordinate operations R is ring. R is a regular ring [6]. Let $P = \oplus_{i \in I} K_i$, it is clear that P is an ideal. P is a regular [6]. So P is approximately Z-regular and by (3) we have P is approximately pointwise-projective module. P is a submodule of a free R-module which is not direct summand of R. So P is not projective and J(R)=0, then P is not approximately -projective.

Proposition (5):*Let M approximately Z-regular R-module and N be a submodule of M with $J(R)M \cap N \subseteq J(R)N$, then N is* *approximately Z-regular (and hence* approximately pointwise-projective*)* .

Proof: Let N be a submodule of approximately Z-regular R-module M and let $n \in N$, then $n \in M$, so there exists $f \in M^*$ s.t. $f(n)n-n \in J(R)M$ and $(f(n))^2=f(n)$. Let $\alpha = f|_N$. be the restriction of N to R, since N is a submodule of M, then $\alpha \in N^*$ and $\alpha(n)n-n \in J(R)M \cap N \subseteq J(R)N$ and $(\alpha(n))^2 = \alpha(n)$. So N is approximately Z-regular.

It is known that every Z-regular module is F-regular [7], but the converse is not true, if M is projective R-module, then every F-regular module is Z-regular [8].

Recall that a submodule N of an R-module M is approximately-pure submodule, if $N \cap IM = IN + J(R)M \cap (N \cap IM)$, for each ideal I of R [5].

Proposition (6): *Every approximately Z-regular module is approximately F-regular.*

Poof: Let M be approximately Z-regular module, P be a submodule of M and I be an ideal in R., let $x \in P \cap IM$, then $x \in P$ and $x = \sum_{i=1}^{n} r_i m_i$ where $r_i \in I$, $m_i \in M$. Since M is approximately Z-regular, then there exists $h \in M^*$ s.t. $h(x)x-x \in J(R)M$ and $(h(x))^2=h(x)$, then $h(\sum_{i=1}^{n} r_i m_i)x - x \in J(R)M$. i.e. $x=\sum_{i=1}^{n} r_i h(m_i)x+t$ where $t \in J(R)M$, it is clear that $\sum_{i=1}^{n} r_i h(m_i) \in I$, then $x \in IP+J(R)M$ and $t=x-\sum_{i=1}^{n} r_i h(m_i)x$. i.e. $t \in P \cap IM$, then $x \in IP+J(R)M \cap (P \cap IM)$.Then $P \cap IM \subseteq IP+J(R)M \cap (P \cap IM)$,so$P \cap IM = IP+J(R)M \cap (P \cap IM)$. Then M is approximately F-regular.

Remark (7): The converse of above proposition is not true for example Z_8 is approximately F-regular, but it is not approximately Z-regular.

An element x in an R-module M is said to be semi-regular, if the conditions in the proposition (1) are satisfies. An R-module M is called semi-regular, if each of its elements is semi-regular [4].

We need the following lemma which appears in [4].

Lemma (8) [4]: *Let M be an R-module and let $x \in M$ be a regular element, if $\alpha \in M^*$ satisfies $x=\alpha(x)\, x$ and if $e=\alpha(x)$. Then:*

(1) $e^2 = e$ and $x = ex$.

(2) $Rx \cong Re$, so Rx is projective.

(3) $M=Rx \oplus W$, where $W = \{w \in M \mid \alpha(w)\, x = 0\}$.

We need the following lemma which appears in [8].

Lemma (8)[8]: *Let M be a projective R-module and N be a submodule of M. Then M/N is flat if and only if given $x \in N$, there exists an R-homomorphism $\alpha : M \longrightarrow N$ such that $x=\alpha(x)$.*

Proposition (10): *If M is* approximately pointwise-projective *semi-regular R-module, then every approximately F-regular is approximately Z-regular.*

Proof: Let $x \in M$, then by Dual-Basis Lemma (2), there exist families $\{x_i\}_{i=1}^{n}$,$x_i \in M$ and $\{\varphi_i\}_{i=1}^{n}$, $\varphi_i \in M^*$ s.t. $\sum_{i=1}^{n} \varphi_i(x)\, x_i - x \in J(R)M$. Hence $x = \sum_{i=1}^{n} \varphi_i(x)\, x_i + \sum_{j=1}^{k} s_j m_j$ s.t. $s_j \in J(R)$ and $m_j \in M$. Let I be an ideal of R generated by $\{\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), s_1, s_2, \ldots, s_k\}$, then $x \in P \cap IM$

where P is a submodule of M generated by x. Since M is approximately F-regular, then $x \in IP+J(R)M \cap (P \cap IM)$. Hence x=cx such that $c \in I$, then $c=\sum_{i=1}^{n} t_i \varphi_i (x)+\sum_{j=1}^{k} r_j s_j$ i.e. $a=\sum_{i=1}^{n} t_i \varphi_i (x) x+\sum_{j=1}^{k} r_j s_j x$ s.t. $r_j, t_i \in R$. Then $\sum_{i=1}^{n} t_i \varphi_i (x) x-x \in J(R)M$. Put $h=\sum_{i=1}^{n} t_i \varphi_i$, it is clear that $h \in M^*$, then $h(x)x-x \in J(R)M$ and since M is semi-regular, then $(h(x))^2=h(x)$, so M is approximately Z-regular.

Let R be regular ring. Then every R-module M is F-regular [9].

Corollary (11): *Let R be a regular ring. Then every* approximately pointwise-projective *semi-regular R-module is approximately Z-regular*.

Proof: Let M be approximately pointwise-projective R-module. Then M is F-regular, and hence is approximately F-regular. The conclusion follows by proposition (10).

In the following theorem we give several characterizations of approximately regular modules.

Theorem (12): *Let M be an R-module. Then the following conditions are equivalent for an element x in M:-*

(1) Rx lies over a projective approximately direct- summand of M.

(2) x is approximately regular element in M.

(3) There exists a regular element $y \in Rx$ such that $x-y \in J(R)M$ and $Rx=Ry \oplus R(x-y)$.

(4) There exists a regular element $y \in M$ such that $x-y \in J(R)M$.

(5) There exists $\rho : M \longrightarrow Rx$ suct t hat $\rho^2=\rho$, $\rho(M)$ is a projective and $x-\rho(x) \in J(R)M$.

Proof: $(1) \Rightarrow (2)$. Assume that there exists a direct decomposition $M=P \oplus Q$, where $P \subseteq Rx$ is projective and $Rx \cap Q \subseteq J(R)M$. Since $Rx=M \cap Rx=(P \oplus Q) \cap Rx=P \cap Rx \oplus Q \cap Rx=P \oplus Q \cap Rx$ hence P is finitely generated projective R-module, so by Dual-Basis Lemma there exist $\{x_i\}_{i=1}^{n}$, $x_i \in P$, and $\{\varphi_i\}_{i=1}^{n}$, $\varphi_i \in P^*$.Put $x_i=r_i x$, $r_i \in R$ and define $\alpha : P \longrightarrow R$ by $\alpha(p)=\sum_{i=1}^{n} r_i \varphi_i(p)$ for each $p \in P$. Then α can be extended to M by putting $\alpha(Q)=0$.If $x=p+q$ where $p \in P$ and $q \in Q$ and $\alpha(x)x=\alpha(p+q)x=\alpha(p)x=(\sum_{i=1}^{n} r_i \varphi_i(p))x=\sum_{i=1}^{n} r_i \varphi_i(p)x=\sum_{i=1}^{n} \varphi_i(p)x_i=p$. It is clear that $x-\alpha(x)x=p+q-p=q \in Q \cap Rx \subseteq J(R)M$.i.e.$x-\alpha(x)x \in J(R)M$,and$(\alpha(x))^2=\alpha(\alpha(x)x)=\alpha(p)=\alpha(x)$, so $(\alpha(x))^2=\alpha(x)$.Then x is approximately regular.

$(2) \Rightarrow (3)$. Let $\alpha \in M^*$s.t.$(\alpha(x))^2=\alpha(x)$ and $x-\alpha(x)x \in J(R)M$. Write $y=\alpha(x)x$. Then $\alpha(y)y=\alpha(\alpha(x)x)\alpha(x)x=\alpha(x)\alpha(x)\alpha(x)x=\alpha(x)\alpha(x)x=\alpha(x)x=y$. Hence y is a regular element and $x-y \in J(R)M$, then by lemma (8), we have $M=Ry \oplus W$, where $W=\{w \in M \mid \alpha(w)y=0\}$. We claim that $Rx \cap W=R(x-y)$. Let $w \in R(x-y)$,then $w=r(x-y)=rx-ry$ for some $r \in R$. $w=rx-ry=rx-r\alpha(x)x=[r-r\alpha(x)]x \in Rx$ and $\alpha(w)y=\alpha(rx-y)y=\alpha(rx)y-\alpha(ry)y=\alpha(rx)\alpha(x)x\alpha(r\alpha(x)x)\alpha(x)x=r\alpha(x)x-r\alpha(x)x=0$, so $w \in W$ and hence $w \in Rx \cap W$. This implies that $R(x-y) \subseteq Rx \cap W$. Now let $z \in Rx \cap W$. Then $z=rx$ where $r \in R$ and $0=\alpha(z)y=\alpha(rx)\alpha(x)x=r\alpha(x)\alpha(x)x=r\alpha(x)$

$x=ry$. Thus $z=rx-ry=r(x-y)\in R(x-y)$ which implies that $Rx\cap W\subseteq R(x-y)$, hence $Rx\cap W=R(x-y)$.
Then we have
$Rx=Rx\cap M=Rx\cap(Ry\oplus W)=(Rx\cap Ry)\oplus(Rx\cap W)=Ry\oplus R(x-y)$.
(3) $\Rightarrow$(4). This is clear.
(4)$\Rightarrow$(5).
Suppose that there is a regular element $y\in M$ such that $x-y\in J(R)M$ and suppose that $y=\alpha(y)y$, for some $\alpha\in M^*$ see the proof of (2)$\Rightarrow$(3). Write $e=\alpha(y)$, then $x-ex=(1-e)(x-y)\in J(R)M$ and we claim that ex is regular element. $e-\alpha(x)=\alpha(y-x)\in J(R)R$, so if $b(1-e+\alpha(x))=1$ for some $b\in R$, then $b\alpha\in M^*$ and $(b\alpha(ex))(ex)=(be\alpha(x))(ex)=(eb\alpha(x))e(x)= e(1-b+be)(ex)=e(ex)=ex$ which implies that ex is a regular element. Since $y=1.y$ we may assume that $y\in Ry$, then by lemma (8), $M=Ry\oplus W$ where $W=\{w\in M \mid \alpha(y)w=0\}$. If $\rho:M\to Ry$ is the projection map of M onto Ry. To prove that $x-\rho(x)\in J(R)M$. Write $x=ry+w$, where $r\in R$, $w\in W$. Then $\alpha(x-ry)y=\alpha(ry+w-ry)y=\alpha(ry)y+\alpha(w)y-\alpha(ry)y=0$ and $\alpha(x-ry)y=\alpha(x)y-\alpha(ry)y$. Hence $\rho(x)=ry=\alpha(x)y$, thus $x-\rho(x)=(x-y)-(\alpha(x-y))y\in J(R)M+J(R)Ry\subseteq J(R)M\Rightarrow x-\rho(x)\in J(R)M$.
(5)$\Rightarrow$(1). This is clear.

Remark (13): It is known that if M is Z-regular R-module, then J(M)=0 [1] and hence J(R)M=0. Thus an R-module M is Z-regular if and only if M is approximately Z-regular and J(M)=0.

Corollary (14): *Let M be an R-module and let $x,y\in M$. If $x-y\in J(R)M$ and y is approximately regular, then x is approximately regular.*
Proof: Let y be approximately regular of M ,then by theorem (12) (4) there exists a regular element $z\in M$, s.t. $y-z\in J(R)M$, but we have $x-y\in J(R)M$, hence $x-z\in J(R)M$, so again by theorem (12), x is approximately regular element in M.

Corollary (14): *A projective module M is Z-regular if and only if every homomorphik image is flat and J(M)=0.*
Proof: Let M be a projective R-module, $x\in M$ and M/Rx is flat. Then by lemma (8), there exists an R-homomorphism $\alpha:M\to Rx$ such that $x=\alpha(x)$. It is clear that $\alpha=\alpha^2$, since $M=\alpha(M)\oplus(1-\alpha)(M)$, then $Rx=\alpha(M)$ is a projective direct summand, it is clear that M is approximately Z-regular and since J(M)=0, then by remark (13), M is Z-regular. The converse is an immediate from remark (13) and theorem (12) (5).

Remark (16):
By looking at the proof of the a bove corollary, we observe that a module has zero Jacobson radical if each cyclic submodule is a direct summand.

Theorem (17): *The following conditions are equivalent for an R-module M*
1) *M is approximately Z-regular.*
2) *If N is finitely generated submodule of M, then there exists an R-homomorphism $\alpha:M\to N$ such that $\alpha=\alpha^2$, $\alpha(M)$ is projective and $(1-\alpha)(N)\subseteq J(R)M$.*
3) *Every finitely generated submodule of M lies over a projective approximately direct-summand of M.*

Proof: (1)$\Rightarrow$(2). Observe that theorem (12)(5) starts an induction on the number of generators of N. Suppose $N=Rx_0+\ldots+Rx_n$, then theorem (12)(5) implies that there exists $\beta:M\to Rx_n$ s.t. $\beta^2=\beta$, $\beta(M)$ is projective and $(1-\beta)(N)\subseteq J(R)M$. Write $K=(1-\beta)(Rx_0)+(1-\beta)(Rx_1)+\ldots+(1-\beta)(Rx_{n-1})$ and by induction, there exists $\delta:M\to K$ *such that*. $\delta=\delta^2$, $\delta(M)$ is projective and $(1-\delta)(K)\subseteq J(R)M$. Define $\alpha=\beta+\delta-\beta\delta$. Then $\alpha=\alpha^2$

and α(M)=($\beta+\delta-\beta\delta$)(M)=β(M)$\oplus$$\delta$(M). Hence α(M) is projective and since N=K+Rx_n ,it is follows that α(M)$\subseteq$N and (1-α)(N)=(1-β)(1-δ)(N)$\subseteq$J(R)M.

(2)$\Rightarrow$(3). Let N be any finitely generated submodule of M. Then by (2) there exists α:M$\longrightarrow$N such tha $\alpha=\alpha^2$, α(M) is projective and (1α)(N)$\subseteq$J(R)M. If y$\in$(1-α)(M)$\cap$N, then y$\in$N and y=(1-α)(x), for some x$\in$M. But x=y+α(x)$\in$N which implies that (1-α)(M)$\cap$N $\subseteq$(1-α)(N)$\subseteq$J(R)M.Now M=α(M)$\oplus$(1-α)(M), α(M)$\subseteq$N and (1-α)(M)$\cap$N$\subseteq$J(R)M. So N lies over a projective approximately direct-summand of M.

(3)$\Rightarrow$(1).This is clear.

Lemma (18): *Let N be a direct summand of an R-module M and x$\in$N. Then x is approximately regular element in N if and only if x is approximately regular element in M.*

Proof: Suppose that x is approximately regular element in N, then there exists $\alpha\in N^*$ s.t. $(\alpha(x))^2=\alpha(x)$ and x-α(x)x$\in$J(R)N. Since N is direct summand of M, M=N$\oplus$K for some submodule K of M. Extend α to all M by putting α(K)=0. Then $\alpha\in M^*$, $(\alpha(x))^2=\alpha(x)$ and x-α(x) x$\in$J(R)N$\subseteq$ J(R)M. Hence x is approximately regular element in M. For the converse, let x be approximately regular element in M and x$\in$N, then there exists $\alpha\in M^*$ such that $(\alpha(x))^2=\alpha(x)$ and x-α(x)x$\in$J(R)M.Let $\alpha_1=\alpha|_N : N\to R$,then $\alpha_1\in N^*$,($\alpha_1(x))^2=\alpha_1(x)$ and x-α_1(x)x=x-α(x)x$\in$J(R)M =J(R)(N$\oplus$K)$\subseteq$J(R)N$\oplus$J(R)K,this implies x-α_1(x)x$\in$J(R)M.Hence x is approximately regular element in N.

Theorem (19): *Let M= $\bigoplus_{i\in I}$ M_i be a direct sum of R-modules M_i. Then M is approximately Z-regular if and only if M_i is approximately Z-regular for each $i\in I$.*

Proof: Let N be a direct summand of M and x$\in$N. Then by lemma (18), x is approximately regular in N if and only if x is approximately regular in M, consequently it suffices to prove the theorem for two summands. Hence let M=N$\oplus$K, where N and K are approximately Z-regular R-modules.

Consider m=x+y, where x$\in$N and y$\in$K, since x is approximately regular in N, then there exists $\alpha\in N^*$ s.t. $(\alpha(x))^2=\alpha(x)$ and x-α(x)x$\in$J(R)N. Extend α to all M by putting α(K)=0, then α(m) is an idempotent and m-α(m)m=(x+y)-α(x+y)(x+y)=(x+y)-α(x)(x+y)=x+y-α(x)x-α(x)y=x-α(x)x+y-α(x)y. But x-α(x)x$\in$J(R)N$\subseteq$J(R)M by corollary (14) and since K is approximately Z-regular, then for each element of K is approximately regular, hence y-α(x)y is approximately regular in K and by lemma (18), it is approximately regular in M.

Conversely direct from lemma (18).

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المقاسات المنتظمة تقريبا

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الخلاصة:

هناك تعميمان (غير متكافئان) للحلقات المنتظمة (بحسب فون نيومان) الى المقاسات المنتظمة. احدهم حسب مفهوم زيلمان وست والذي هو تعميم نقطي والاخر حسب مفهوم فيلدهاوس. في هذا البحث قدمنا ودرسنا المقاسات المنتظمة تقريبا وكذلك نعطي عدة خواص وتشخيصات لهذا المفهوم ودرسنا العلاقة بينهم باستخدام المقاسات النقطية الاسقاط تقريبا.