On The Queuing System $M/E_r/1/N$

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Abstract:

In this paper the queuing system ($M/E_r/1/N$) has been considered in equilibrium. The method of stages introduced by Erlang has been used. The system of equations which governs the equilibrium probabilities of various stages has been given. For general N the probability of j stages of service are left in the system, p_j has been introduced. And the probability for the empty system p_0 has been calculated in the explicit form.

Key words: Queuing system, Erlangian service, steady-state.

Introduction:

The queuing problems in which the service-facility consists of a number of service channels in series have been studied. But only a few have placed the restriction of a finite queue size. One of the leading investigations in this direction was made by Hunt [1]. He obtained the maximum possible utilization of the system and the expected number of customers in the system, assuming exponential services. His results have rather been limited in character. Hollier and Boling [2] presented some new extensions of Hunt's work. More specifically they studied the queuing system which consists of N service channels in series, each channel has either an exponential or Erlangian service-time. They obtained the steady state mean output rate and also mean number of customers in the system. Recently many authors have considered the system $M/E_r/$ 1. Griffiths and his colleagues [3], have obtained the transient phase probabilities in terms of a new generalization of the modified Bessel function, and mean waiting time in the queue. Paoumy [4], has derived the analytic solution of the truncated Erlangian service queue with state-dependent rate, balking and reneging. Shawky [5] has obtained the analytical solution of the queue $M/E_r/$ 1/k/N for machine interference system with balking and reneging. Pourdarvish [6], has obtained $P_{n,s}$ the steady state probabilities with the unit in the service being at stage s of the truncated Erlangian service queuing system with state-dependent with fuzzy arrival rate. It is known that when we use distribution other than exponential, the memoryless property of the exponential will be lost and the analysis becomes more complicated. Special methods in this case have been devised. Some of these special ones are method of imbedded-Markov chain, [7], the method of Lindley's integral equation [8]. These methods do not in general give an explicit solution but give some functions associated with the solution.

In this paper we consider the Erlangian queuing system ($M/E_r/1/N$) in equilibrium, where customers arrive at random at mean rate λ and the

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service times have an Erlangian distribution with parameter r and mean service rate μ. Even though there is a single r-stage Erlangian server, we can consider this service to make up of r exponential services in series, each with a mean service rate rμ. That means Erlang has retained the valuable property of exponential and still allowing for more general distribution.
This consideration will help in formulating the equilibrium. Probabilities equations with the help of state transition rate diagram by using the inspection method. To give a precise description of the state of the system, let then (k) customers in the system out of which (k-1) are in the queue and the one in service is in the stage of service yet to be completed. If (j) denotes the number of stages of service left in the system at this moment, then

j= (k-1)r+(r-i+1)
= rk-i+1, i=1, 2, ... , r , k=1, 2, ... , N+1 … (1)

There is special state j=0 which is not covered by (1) which means that the system is completely empty. the total number of possible states, is therefore (N+1)r +1 which can be designated by the symbols $E_0, E_1, E_{2,\ldots\ldots}, E_{(N+1)r}$.
We shall use the following notation for equilibrium probabilities.

P_j = Probability j stages of service are left in the system, …(2)

and

p_k = probability there are k customers in the system … (3)

with these definitions it is now clear that p_0 , the probability for the empty system, is same as the probability of zero stages of serves are left in the system and

$$p_k = \sum_{j=(k-1)r+1}^{kr} P_j \text{ ,k=1,2,……,N+1} \quad \ldots (4)$$

Equations for Equilibrium Probabilities:

Let us start in this section by drawing the state transition rate diagram for (M/E$_r$/1/N), to help us for writing the equilibrium equation for state probabilities:

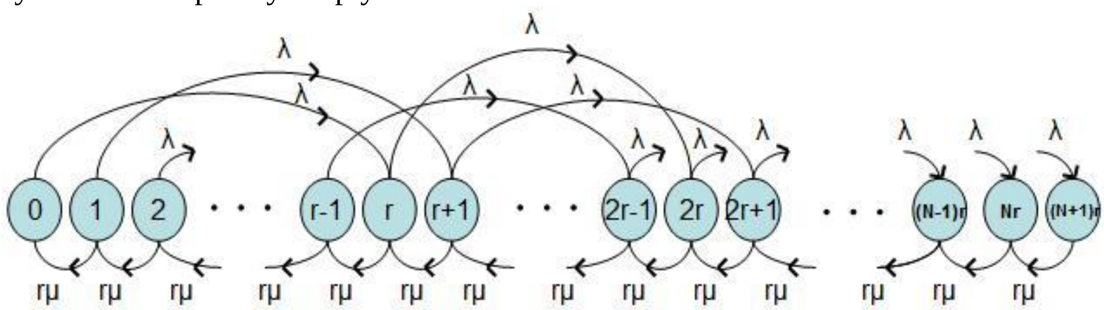

Fig. (1) State transition rate diagram

Since we are considering only the equilibrium of the system, the rates at which probabilities flow into a state must balance with the probabilities which flow out from that state. Thus, it is clear that from fig.1 we have the following equations

$$\lambda P_{\circ} = r\mu P_1, \quad \ldots(5)$$

$$(\lambda + r\mu)P_i = r\mu P_{i+1} \text{ ,i=1,2,…,r-1} \quad \ldots (6)$$

$$(\lambda + r\mu)P_i = \lambda P_{i-r} + r\mu P_{i+1} \text{ ,i=r, r+1,…,Nr} \quad \ldots(7)$$

$r\mu P_i = \lambda P_{i-r} + r\mu P_{i+1}$, i=Nr+1, Nr+2,…,(N+1)r-1 … (8)

and

$r\mu P_{(N+1)r} = \lambda P_{Nr}$, …(9)

The equations (5) to (9) are in fact (N+1)r+1 equations in as many unknown which are state probabilities. These equations are all linear and homogenous, and they are consistent. These equations are such combination of the others. So in essence they are only (N+1)r unknowns in terms of one of them, say P_0 , this last unknown P_0 should then be determined with the help of the conservation equation, viz.,

$$\sum_{i=0}^{(N+1)r+1} P_i = 1 \quad \ldots(10)$$

Obtaining P_0 will immediately give all the other probabilities. Once this has been done we can then find any other desired statistical property of the queuing system. Let us define

$\rho = \dfrac{\lambda}{\mu}$ and $q = 1 + \dfrac{\rho}{r}$, … (11)

and rewrite the equations (5) to (9) as below:

$P_1 = (q-1)P_\circ$, …(12)

$P_{i+1} = qP_i$, i=1,2,….,r-1 …(13)

$P_{i+1} = qP_i - (q-1)P_{i-r}$,i=r,r+1,...,Nr… (14)

$P_{i+1} = P_i - (q-1)P_{i-r}$, i=Nr+1 , Nr+2 ,…,(N+1)r-1 … (15)

and

$P_{(N+1)r} = (q-1)P_{Nr}$ …(16)

Now, we shall study the special case N=0, i.e., no queue-length is allowed. In this case the state –transition-rate diagram reduces to

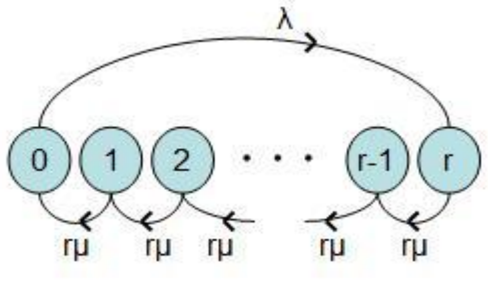

Fig. (2) State transition rate diagram for N=0

and the equilibrium equations for state probabilities are

$P_{i+1} = P_i$, i=1,2,…..,r-1 … (17)

And the last state E_r gives the equation

$P_r = (q-1)P_\circ$ …(18)

Solving these equations one can have

$$P_1 = P_2 = ... = P_r = \frac{\rho}{r(1+\rho)} \quad \ldots(19)$$

Using the conservation equation (10) one can easily show that the probability of empty system is

$$P_\circ = \frac{1}{1+\rho} \quad \ldots (20)$$

Now we are in the position to give our main statements and their prove. In the first theorem, we give the probability of j stages of service are left in the system.

Theorem (1)

For fixed N, where N is the number of customers are permitted in the queue of the system, the probabilities of j stages of services are left in the system is

$$P_{mr+i} = c\left\{\sum_{k=0}^{m}(-1)^k \sum_{h=0}^{k}(-1)^h \binom{(m-k)r+i-1}{h}\binom{(m-k)r+i-h}{k-h} q^{(m-k)r+i-h-1}\right\}$$

where $c = (q-1)P_\circ$, m=0,1,2,…,N-1, and i=1,2,...,r

Proof

Using equation (12) in equation (13) recursively, one can find that:

$$P_i = cq^{i-1} \quad , \text{i=1, 2, …, r} \qquad \text{….(21)}$$

Making use of (21) and recursively using the first (r+1) equations from (14) we can get

$$P_{r+i} = c\left[q^{r+i-1} - iq^{i-1} + (i-1)q^{i-2} \right]$$

, i=1,2,…,r+1 … (22)

Now using the results in (21) and (22) and recursively using the next r+1 equations from (14) one can have

$$= P_0 \left\{ 1 + (q-1) + (q-1)^2 + ... + (q-1)^{N+1} \right\}$$

$$= P_0 \frac{(q-1)^{N+2} - 1}{q-2}$$

Thus

$$P_0 = \frac{q-2}{(q-1)^{N+2} - 1}$$

Note that at this point we apply the following check for validity of our result. Putting r =1 and therefore q = ρ +1. Hence

$$P_{2r+i} = c\left[q^{2r+i-1} - (r+i)q^{r+i-1} + (r+i-1)q^{r+i-2} + \frac{i(i-1)}{2}q^{i-1} - (i-1)^2 q^{i-2} + \frac{(i-1)(i-2)}{2}q^{i-3} \right]$$

i=2,3,….,r+2 …(23)

If we continue in this way and using induction on (i) for fixed N, result follows.

Theorem (2)

The probabilities for stages Nr+2,...and (N+1)r-1 are left in the system are

$$P_0 = \frac{\rho - 1}{\rho^{N+2} - 1} = \frac{1}{1 + \rho + \rho^2 + ... + \rho^{N+1}}$$

And this is the right value of P_0 in the case of (M/M/1/N) queue to which the present queue ($M/E_r/1/N$) degenerates.

$$P_{Nr+i} = c\left[\left\{ \sum_{k=0}^{N-1} (-1)^k \sum_{h=0}^{k} (-1)^h \left[\binom{(N-k)r}{k} \binom{k}{h} . q^{(N-k)r-h} + \binom{(N-k)r+i-1}{k-1} \binom{k-1}{h} . q^{(N-k)r+i-h-1} \right] \right\} \right.$$
$$\left. + (-1)^N \binom{i-1}{N-1} q^{i-N} (q-1)^{N-1} \right]$$

Where $c = (q-1)P_\circ$ and i=2,…,r-1.

Proof: By using the same procedure as in theorem (1), the result follows.

Calculation of P_0

We proceed to calculate P_0, using the conservation equation, that means

$$P_0 + P_1 + P_2 + ... + P_N + P_{N+1} = 1$$

Since

$$P_N = (q-1)P_{N-1} = (q-1)^2 P_{N-2} = ... = (q-1)^N P_0 .$$

Therefore

$$1 = P_0 + \sum_{k=1}^{N+1} P_k$$

$$= P_0 + \sum_{k=1}^{N+1} (q-1)^k P_0$$

Conclusion

In this paper the queue ($M/E_r/1/N$) has been considered and we have found the probabilities of j stages of service are left in the system and we calculate the probabilities for the empty system P_0 , and with this value of P_0 we can calculate all the desired equilibrium – state – probabilities, and any statistical properties of the system.

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حول نظام الطوابير $M/E_r/1/N$

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الخلاصة:

في هذا البحث تمت دراسة نظام الطوابير (**$M/E_r/1/N$**) في حالة الاستقرار لعدد محدد (N) من الزبائن في الطابور. وتم احتساب احتمالية j من مراحل الخدمة المتبقية في النظام ومن ثم احتساب احتمالية ان يكون النظام خالياً من الزبائن وقد اعطيت النتائج بصيغة صريحة.