Approximate Solution of Some Classes of Integral Equations Using Bernstein Polynomials of Two-Variables

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Abstract:
The research aims to find approximate solutions for two dimensions Fredholm linear integral equation. Using the two-variables of the Bernstein polynomials we find a solution to the approximate linear integral equation of the type two dimensions. Two examples have been discussed in detail.

Key Word: Bernstein Polynomials, Two-Variables Fredholm linear integral equation

Introduction:

New methods are always needed to solve integral equation because no single method works well for all such equations. There has been considerable interest in solving differential and integral equations using techniques which involve new formal two-variables Bernstein polynomials method.

Now consider the following linear two-dimensional Fredholm integral equations of the second kind [1-3]

$$u(x,y)=f(x,y)+\int_0^1\int_0^1 k(x,y,t,s,u(t,s))dsdt \quad \ldots (1)$$

$$(x,y)\in D=[0,1]\times[0,1]$$

Where $k: D\times D\times R\to R$ is a continuous linear given function, $f: D\to R$ is also continuous given function and the two-variables function u(x,y) is the unknown function.

In this paper we introduce an approximate approach to solve two-dimensional linear Fredholm integral equations of the second kind given in (1) using new formal of two-variables Bernstein polynomials method.

1. Bernstein Polynomials Method with Two-Variables.

The Bernstein polynomials of two-variables degree of (n+m) can be defined[4-6]:

$$B_{ij}^{n+m}(x,t)=\binom{n}{i}\binom{m}{j}x^i t^j(1-x)^{n-i}(1-t)^{m-j} \quad \ldots (2)$$

where

$$\binom{n}{i}\binom{m}{j}=\frac{n!}{i!(n-i)!}\frac{m!}{j!(m-j)!}$$

and (n,m) are the degree of polynomials and i,j are the index of polynomials

Proposition

The Bernstein polynomials of degree n+m in the terms of the power basis is given by the following formula [7]

$$B_{ij}^{n+m}(x,t)=\sum_{k=i}^{n}\sum_{l=j}^{m}(-1)^{k-i}(-1)^{l-j}\binom{n}{k}\binom{k}{i}\binom{m}{l}\binom{l}{j}x^k t^l \quad \ldots (3)$$

2. General Form of the Double integral Using Two- Variables Bernstein Polynomials Method

In this section the Bernstein polynomials of two-variables of degree (n+m) may be used define, the general form, of the double integral as the following lemma

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Lemma

The double integral in the interval [0,1]x[0,1], Bernstein polynomials of degree n,m is given by the following formula

$$\int_0^1\int_0^1 B_i^{n+m}(x,t)dxdt = \frac{1}{(n+1)(m+1)} \quad \ldots (4)$$

Proof

Directs calculating using the definition of the Bernstein polynomials of two variables and the binomial theorem, as follows

$$B_{ij}^{n+m}(x,t) = \binom{n}{i}\binom{m}{j} t^j x^i (1-x)^{n-i}(1-t)^{m-j}$$

$$= \binom{n}{i}\binom{m}{j} t^j x^i \sum_{k=0}^{n-i}\sum_{l=0}^{m-j}(-1)^k(-1)^l\binom{n-i}{k}\binom{m-j}{l}x^k t^l$$

Any Bernstein polynomial of degree n can be written in terms of the power basis $\{1, t, t^2, t^3, \ldots, t^n\}$, $\{1, x, x^2, x^3, \ldots, x^n\}$

$$= \sum_{k=0}^{n-i}\sum_{l=0}^{m-j}(-1)^k(-1)^l\binom{n}{i}\binom{m}{j}\binom{n-i}{k}\binom{m-j}{l}x^{i+k}t^{j+l}$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{k-i}(-1)^{l-j}\binom{n}{i}\binom{m}{j}\binom{n-i}{k-i}\binom{m-j}{l-j}x^k t^l$$

$$B_{ij}^{n+m}(x,t) = \sum_{k=i}^{n}\sum_{l=j}^{m}(-1)^{k-i}(-1)^{l-j}\binom{n}{k}\binom{k}{i}\binom{m}{l}\binom{l}{j}x^k t^l$$

$$\int_0^1\int_0^1 B_{ij}^{n+m}(x,t)dxdt$$

$$= \int_0^1\int_0^1 \sum_{k=i}^{n}\sum_{l=j}^{m}(-1)^{k-i}(-1)^{l-j}\binom{n}{k}\binom{k}{i}\binom{m}{l}\binom{l}{j}x^k t^l dxdt$$

$$\int_0^1\int_0^1 B_{ij}^{n+m}(x,t)dxdt =$$

$$\int_0^1 \sum_{k=i}^{n}(-1)^{k-i}\binom{n}{k}\binom{k}{i}x^k \left[\int_0^1 \sum_{l=j}^{m}(-1)^{l-j}\binom{m}{l}\binom{l}{j}t^l dt\right]dx$$

$$\int_0^1\int_0^1 B_{ij}^{n+m}(x,t)dxdt = \int_0^1 \sum_{k=i}^{n}(-1)^{k-i}\binom{n}{k}\binom{k}{i}x^k\left[\frac{1}{m+1}\right]dx$$

Therefore,

$$\int_0^1\int_0^1 \sum_{k=i}^{n}\sum_{l=j}^{m}(-1)^{k-i}(-1)^{l-j}\binom{n}{k}\binom{k}{i}\binom{m}{l}\binom{l}{j}x^k t^l dxdt$$

$$= \frac{1}{(n+1)(m+1)}$$

3. Solution of Two-Dimensional Linear Fredholm integral equation with Two-variables Bernstein Polynomials Method

In this section, Bernstein polynomials have been used to find the approximate solution for the two-dimensional linear Fredholm integral equations, as follows.

Recall the Fredholm integral equation of the second kind given in equation.(1).

$$u(x,y) = f(x,y) + \int_0^1\int_0^1 k(x,y,t,s,u(t,s))dsdt$$

Let $u(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} P_{ij}B_{ij}^{n+m}(x,y)$ … (5)

where

$$B_{ij}^{n+m}(x,y) = \sum_{k=i}^{n}\sum_{l=j}^{m}(-1)^{k-i}(-1)^{l-j}\binom{n}{k}\binom{k}{i}\binom{m}{l}\binom{l}{j}x^k y^l$$

And P_{ij} control points unknown.

Substitution of the relation in equation(5) in equation (1)gives rise to the relation

$$\sum_{i=0}^{n}\sum_{j=0}^{m}P_{ij}B_{ij}^{n+m}(x,y) = f(x,y) + \int_0^1\int_0^1 k\left(x,y,t,s,\sum_{i=0}^{n}\sum_{j=0}^{m}P_{ij}B_{ij}^{n+m}(t,s)\right)dsdt$$

$$= f(x,y) + \int_0^1\int_0^1 k\left(x,y,t,s,\sum_{i=0}^{n}\begin{matrix}P_{i0}B_{i0}^{n+m}(t,s)+\\ P_{i1}B_{i1}^{n+m}(t,s)+\cdots+\\ P_{im}B_{im}^{n+m}(t,s)\end{matrix}\right)dsdt$$

$$= f(x,y) + \int_0^1\int_0^1 k(x,y,t,s)\begin{pmatrix} P_{00}B_{00}^{n+m}(t,s) + P_{10}B_{10}^{n+m}(t,s) + \cdots \\ + P_{n0}B_{n0}^{n+m}(t,s) \\ + P_{01}B_{01}^{n+m}(t,s) + P_{11}B_{11}^{n+m}(t,s) + \cdots \\ + P_{m1}B_{m1}^{n+m}(t,s) \\ \cdots + P_{0m}B_{0m}^{n+m}(t,s) + P_{1m}B_{1m}^{n+m}(t,s) + \cdots \\ P_{nm}B_{nm}^{n+m}(t,s) \end{pmatrix} dsdt$$

…(6)]

now to find all integration in equation(6).

Then in order to determine $P_{00}, P_{01}\cdots, P_{0m}, P_{10}, P_{11}, \ldots P_{1m}, \cdots P_{nm}$, we need n equations;

Now Choose $x_i, i = 1,2,3,\ldots n$ and $y_j, j = 1,2,3,\ldots m$ in the interval [0,1x[0,1], which give (n) equations. Solve the (n) equations by Gauss elimination to find the values $P_{00}, P_{01}\cdots, P_{0m}, P_{10}, P_{11}, \ldots P_{1m}, \cdots P_{nm}$.

4. Numerical Examples:

Example(1)

Consider the following two-dimensional linear Fredholm integral equation of the second kind:

$$u(x,y) = x + y + 2(x+y)(1-e^1) + \int_0^1\int_0^1 (x+y)e^{s+t}u(s,t)dsdt \quad \ldots(7)$$

with the exact solution $u(x,y) = x + y$

we choose uniform partition with m=n=1,2,3. Approximated solution for some values of (x,y) by using two-variables Bernstein polynomials method and exact values $u(x,y) = x + y$.

by using equation(5) let n=m, for n=1 we get

$$u(x,y) = \sum_{i=0}^{1}\sum_{j=0}^{1} p_{ij}B_{ij}^2$$

$$u(x,y) = \sum_{i=0}^{1}\sum_{j=0}^{1} p_{ij}\binom{1}{i}\binom{1}{j}x^i y^j (1-x)^{1-i}(1-y)^{1-j}$$

$$u(x,y) = p_{00}B_0^1(x)B_0^1(y) + p_{01}B_0^1(x)B_1^1(y) + p_{10}B_0^1(y)B_1^1(x) + p_{11}B_1^1(x)B_1^1(y)$$

$$u(x,y) = p_{00}(1-x)(1-y) + p_{01}(1-x)y + p_{10}(1-y)x + p_{11}xy \quad \ldots(8)$$

Substitution of the relation in equation(8) in equation (7)gives rise to the relation

$$p_{00}(1-x)(1-y) + p_{01}(1-x)y + p_{10}(1-y)x + p_{11}xy = x + y + 2(x+y)(1-e^1) + \int_0^1\int_0^1 (x+y)e^{s+t}\begin{bmatrix} p_{00}(1-s)(1-t) + p_{01}(1-s)t \\ + p_{10}(1-t)s + p_{11}st \end{bmatrix} dsdt$$

Where The control points p_{ij} , i=0,1 j=0,1 are found as follows:

Find all integration in equation. Then in order to determine control points p_{ij} , i=0,1 , j=0,1 we need n equations; now choose $x_i = 0,1$ and $y_j = 0,1$ in the interval $[0,1]x[0,1]$, which gives (n)equations . solve the (n) equations by Gauss elimination to find the values p_{ij}, i=0,1 , j=0,1. we obtain the approximate solution as

$$u(x,y) = 0(1-x)(1-y) + 1(1-x)y + 1(1-y)x + 2xy$$

when n=m=2 in equation (5)

$$u(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} p_{ij}B_{ij}^4$$

$$u(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} p_{ij}\binom{2}{i}\binom{2}{j}x^i y^j (1-x)^{2-i}(1-y)^{2-j}$$

$$\begin{aligned} u(x,y) = {} & p_{00}B_0^2(x)B_0^2(y) + 2p_{01}B_0^2(x)B_1^2(y) + 2p_{10}B_1^2(x)B_0^2(y) \\ & + p_{20}B_2^2(x)B_0^2(y) + p_{02}B_0^2(x)B_2^2(y) + 4p_{11}B_1^2(x)B_1^2(y) \\ & + 2p_{21}B_2^2(x)B_1^2(y) + 2p_{12}B_1^2(x)B_2^2(y) + p_{22}B_2^2(x)B_2^2(y) \end{aligned}$$

$$\begin{aligned} u(x,y) = {} & p_{00}(1-x)^2(1-y)^2 + 2p_{01}y(1-y)(1-x)^2 + 2p_{10}x(1-x)(1-y)^2 \\ & + p_{20}x^2(1-y)^2 + p_{02}y^2(1-x)^2 + 4p_{11}xy(1-x)(1-y) \\ & + 2p_{21}x^2y(1-y) + 2p_{12}x(1-x)y^2 + p_{22}x^2y^2 \end{aligned}$$

. . .(9)

Substitution of the relation in equation(9) in equation (7) gives rise to the relation

$$\begin{aligned} & p_{00}(1-x)^2(1-y)^2 + 2p_{01}t(1-y)(1-x)^2 + 2p_{10}x(1-x)(1-y)^2 \\ & + p_{20}x^2(1-y)^2 + p_{02}y^2(1-x)^2 + 4p_{11}xy(1-x)(1-y) \\ & + 2p_{21}x^2y(1-y) + 2p_{12}x(1-x)y^2 + p_{22}x^2y^2 \\ & = x + y + 2(x+y)(1-e^1) + \\ & \int_0^1\int_0^1 (x+y)e^{s+t}\begin{bmatrix} p_{00}(1-s)^2(1-t)^2 + 2p_{01}t(1-t)(1-s)^2 + 2p_{10}s(1-s)(1-t)^2 \\ + p_{20}s^2(1-t)^2 + p_{02}t^2(1-s)^2 + 4p_{11}st(1-s)(1-t) \\ + 2p_{21}s^2t(1-t) + 2p_{12}s(1-s)t^2 + p_{22}s^2t^2 \end{bmatrix} dsdt \end{aligned}$$

Where The control points p_{ij} , i=0,1,2 j=0,1,2 are found as follows:

Find all integration in equation. Then in order to determine control points

p_{ij} , i=0,1,2 j=0,1,2 we need n equations; now choose $x_i = 0,1,2$ and $y_j = 0,1,2$ in the interval $[0,1]x[0,1]$, which gives (n)equations . solve the (n) equations by Gauss elimination to find the values p_{ij}, i=0,1,2 , j=0,1,2.

we obtain the approximate solution as

$$\begin{aligned} u(x,y) &= 0*(1-x)^2(1-y)^2 \\ &+ 2*0.5*y(1-y)(1-x)^2 + 2*0.5*x(1-x)(1-y)^2 \\ &+ 1*x^2(1-y)^2 + 1*y^2(1-x)^2 + 4*1*xy(1-x)(1-y) \\ &+ 2*1.5*x^2y(1-y) + 2*1.5*x(1-x)y^2 + 2*x^2y^2 \end{aligned}$$

when n=m=3 in equation(5) we get

$$u(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} p_{ij} B_{ij}^6$$

$$u(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} p_{ij} \binom{3}{i}\binom{3}{j} x^i y^j (1-x)^{3-i}(1-y)^{3-j}$$

$$\begin{aligned} u(x,y) &= p_{00}B_0^3(x)B_0^3(y) + 3p_{01}B_0^3(x)B_1^3(y) + 3p_{02}B_0^3(x)B_2^3(y) \\ &+ p_{03}B_0^3(x)B_3^3(y) + 3p_{10}B_1^3(x)B_0^3(y) + 9p_{11}B_1^3(x)B_1^3(y) \\ &+ 9p_{12}B_1^3(x)B_2^3(y) + 3p_{13}B_1^3(x)B_3^3(y) + 3p_{20}B_2^3(x)B_0^3(y) \\ &+ 9p_{21}B_2^3(x)B_1^3(y) + 9p_{22}B_2^3(x)B_2^3(y) + 3p_{23}B_2^3(x)B_3^3(y) \\ &+ p_{30}B_3^3(x)B_0^3(y) + 3p_{31}B_3^3(x)B_1^3(y) + 3p_{32}B_3^3(x)B_2^3(y) \\ &+ p_{33}B_3^3(x)B_3^3(y) \end{aligned}$$

$$\begin{aligned} u(x,y) &= p_{00}(1-x)^3(1-y)^3 + 3p_{01}y(1-y)^2(1-x)^3 + 3p_{02}y^2(1-y)(1-x)^3 \\ &+ p_{03}y^3(1-x)^3 + 3p_{10}x(1-x)^2(1-y)^3 + 9p_{11}xy(1-x)^2(1-y)^2 \\ &+ 9p_{12}xy^2(1-x)^2(1-y) + 3p_{13}xy^3(1-x)^2 + 3p_{20}x^2(1-x)(1-y)^3 \\ &+ 9p_{21}x^2y(1-y)^2(1-x) + 9p_{22}x^2y^2(1-y)(1-x) + 3p_{23}x^2y^3(1-x) \\ &+ p_{30}x^3(1-y)^3 + 3p_{31}x^3y(1-y)^2 + 3p_{32}x^3y^2(1-y) + p_{33}x^3y^3 \end{aligned} \quad ...(10)$$

Substitution of the relation in equation(10) in equation (7)gives rise to the relation

$$\begin{aligned} u(x,y) &= p_{00}(1-x)^3(1-y)^3 + 3p_{01}y(1-y)^2(1-x)^3 + 3p_{02}y^2(1-y)(1-x)^3 \\ &+ p_{03}y^3(1-x)^3 + 3p_{10}x(1-x)^2(1-y)^3 + 9p_{11}xy(1-x)^2(1-y)^2 \\ &+ 9p_{12}xy^2(1-x)^2(1-y) + 3p_{13}xy^3(1-x)^2 + 3p_{20}x^2(1-x)(1-y)^3 \\ &+ 9p_{21}x^2y(1-y)^2(1-x) + 9p_{22}x^2y^2(1-y)(1-x) + 3p_{23}x^2y^3(1-x) \\ &+ p_{30}x^3(1-y)^3 + 3p_{31}x^3y(1-y)^2 + 3p_{32}x^3y^2(1-y) + p_{33}x^3y^3 \\ &= x + y + 2(x+y)(1-e^1) + \\ &\int_0^1\int_0^1 (x+y)e^{s+t} \left[\begin{aligned} & p_{00}(1-s)^3(1-t)^3 + 3p_{01}t(1-t)^2(1-s)^3 + 3p_{02}t^2(1-t)(1-s)^3 \\ &+ p_{03}t^3(1-s)^3 + 3p_{10}s(1-s)^2(1-t)^3 + 9p_{11}st(1-s)^2(1-t)^2 \\ &+ 9p_{12}st^2(1-s)^2(1-t) + 3p_{13}st^3(1-s)^2 + 3p_{20}s^2(1-s)(1-t)^3 \\ &+ 9p_{21}s^2t(1-t)^2(1-s) + 9p_{22}s^2t^2(1-t)(1-s) + 3p_{23}s^2t^3(1-s) \\ &+ p_{30}s^3(1-t)^3 + 3p_{31}s^3t(1-t)^2 + 3p_{32}s^3t^2(1-t) + p_{33}s^3t^3 \end{aligned} \right] dsdt \end{aligned}$$

Where The control points p_{ij} , i=0,1,2,3 j=0,1,2,3 are found as follows:

Find all integration in equation above. Then in order to determine control points p_{ij}, i=0,1,2,3, j=0,1,2,3 we need n equations; now choose $x_i = 0,1,2,3$ and $y_j = 0,1,2,3$ in the interval $[0,1]x[0,1]$, which gives (n)equations . solve the (n) equations by Gauss elimination to find the values p_{ij}, i=0,1,2,3, j=0,1,2,3.

The results depending on the least square error (L.S.E) is presented in Table(1) .

Table (1) The results of Example(1)

i	x	y	*Exact u(x,y)*	*Approximation u(x,y)n=m=1*	*Approximation u(x,y)n=m=2*	*Approximation u(x,y)n=m=3*	*error*
1	0	0	0	0	0	0	0
2	0	0.25	0.2500	0.2500	0.2500	0.2500	0
3	0	0.5	0.5000	0.5000	0.5000	0.5000	0
4	0	0.75	0.75	0.75	0.75	0.75	0
5	0	1	1	1	1	1	0
6	0.25	0	0.25	0.25	0.25	0.25	0
7	0.25	0.25	0.5	0.5	0.5	0.5	0
8	0.25	0.5	0.75	0.75	0.75	0.75	0
9	0.25	0.75	1	1	1	1	0
10	0.25	1	1.25	1.25	1.25	1.25	0
11	0.5	0	0.5	0.5	0.5	0.5	0
12	0.5	0.25	0.75	0.75	0.75	0.75	0
13	0.5	0.5	1	1	1	1	0
14	0.5	0.75	1.25	1.25	1.25	1.25	0
15	0.5	1	1.50	1.50	1.50	1.50	0
16	0.75	0	0.75	0.75	0.75	0.75	0
17	0.75	0.25	0.1875	0.1875	0.1875	0.1875	0
18	0.75	0.5	1	1	1	1	0
19	0.75	0.75	1.5	1.5	1.5	1.5	0
20	0.75	1	1.75	1.75	1.75	1.75	0
21	1	0	1	1	1	1	0
22	1	0.25	1.25	1.25	1.25	1.25	0
23	1	0.5	1.5	1.5	1.5	1.5	0
24	1	0.75	1.75	1.75	1.75	1.75	0
25	1	1	2	2	2	2	0

Example(2)

Consider the following two-dimensional linear Fredholm integral equation of the second kind:

$$u(x,y) = xy - \frac{1}{4}(x+y) - \frac{1}{3} + \int_0^1 \int_0^1 (x+y+s+t)u(s,t)dsdt \quad ...(11)$$

with the exact solution $u(x,y) = xy$ we choose uniform partition with m=n=1,2,3. Approximated solution for some values of (x,y) by using two-variables Bernstein polynomials method and exact values $u(x,y) = xy$ when n=m=3 in equation(5) we get

$$u(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} p_{ij} B_{ij}^{6}$$

$$u(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} p_{ij} \binom{3}{i}\binom{3}{j} x^i y^j (1-x)^{3-i}(1-y)^{3-j}$$

$$\begin{aligned} u(x,y) = {} & p_{00}(1-x)^3(1-y)^3 + 3p_{01}y(1-y)^2(1-x)^3 + 3p_{02}y^2(1-y)(1-x)^3 \\ & + p_{03}y^3(1-x)^3 + 3p_{10}x(1-x)^2(1-y)^3 + 9p_{11}xy(1-x)^2(1-y)^2 \\ & + 9p_{12}xy^2(1-x)^2(1-y) + 3p_{13}xy^3(1-x)^2 + 3p_{20}x^2(1-x)(1-y)^3 \\ & + 9p_{21}x^2y(1-y)^2(1-x) + 9p_{22}x^2y^2(1-y)(1-x) + 3p_{23}x^2y^3(1-x) \\ & + p_{30}x^3(1-y)^3 + 3p_{31}x^3y(1-y)^2 + 3p_{32}x^3y^2(1-y) + p_{33}x^3y^3 \end{aligned} \quad ...(12)$$

Substitution of the relation in equation(12) in equation (11)gives rise to the relation

$$\begin{aligned} u(x,y) = {} & p_{00}(1-x)^3(1-y)^3 + 3p_{01}y(1-y)^2(1-x)^3 + 3p_{02}y^2(1-y)(1-x)^3 \\ & + p_{03}y^3(1-x)^3 + 3p_{10}x(1-x)^2(1-y)^3 + 9p_{11}xy(1-x)^2(1-y)^2 \\ & + 9p_{12}xy^2(1-x)^2(1-y) + 3p_{13}xy^3(1-x)^2 + 3p_{20}x^2(1-x)(1-y)^3 \\ & + 9p_{21}x^2y(1-y)^2(1-x) + 9p_{22}x^2y^2(1-y)(1-x) + 3p_{23}x^2y^3(1-x) \\ & + p_{30}x^3(1-y)^3 + 3p_{31}x^3y(1-y)^2 + 3p_{32}x^3y^2(1-y) + p_{33}x^3y^3 \end{aligned}$$

$$= xy - \frac{1}{4}(x+y) - \frac{1}{3} +$$

$$\int_0^1\int_0^1 (x+y+s+t)\begin{bmatrix} p_{00}(1-s)^3(1-t)^3 + 3p_{01}t(1-t)^2(1-s)^3 + 3p_{02}t^2(1-t)(1-s)^3 \\ + p_{03}t^3(1-s)^3 + 3p_{10}s(1-s)^2(1-t)^3 + 9p_{11}st(1-s)^2(1-t)^2 \\ + 9p_{12}st^2(1-s)^2(1-t) + 3p_{13}st^3(1-s)^2 + 3p_{20}s^2(1-s)(1-t)^3 \\ + 9p_{21}s^2t(1-t)^2(1-s) + 9p_{22}s^2t^2(1-t)(1-s) + 3p_{23}s^2t^3(1-s) \\ + p_{30}s^3(1-t)^3 + 3p_{31}s^3t(1-t)^2 + 3p_{32}s^3t^2(1-t) + p_{33}s^3t^3 \end{bmatrix} dsdt$$

Where The control points p_{ij}, i=0,1,2,3 j=0,1,2,3 are found as follows:

Find all integration in equation. Then in order to determine control points p_{ij}, i=0,1,2,3, j=0,1,2,3 we need n equations; now choose $x_i = 0,1,2,3$ and $y_j = 0,1,2,3$ in the interval $[0,1]x[0,1]$, which gives (n)equations . solve the (n) equations by Gauss elimination to find the values p_{ij}, i=0,1,2,3, j=0,1,2,3. we obtain the approximate solution as

$$\begin{aligned} u(x,y) = {} & 0*(1-x)^3(1-y)^3 + 3*0*y(1-y)^2(1-x)^3 + 3*0*y^2(1-y)(1-x)^3 \\ & + 0*y^3(1-x)^3 + 3*0*x(1-x)^2(1-y)^3 + 9*\left(\frac{1}{9}\right)*xy(1-x)^2(1-y)^2 \\ & + 9*\left(\frac{2}{9}\right)*xy^2(1-x)^2(1-y) + 3*\left(\frac{1}{3}\right)*xy^3(1-x)^2 + 3*0*x^2(1-x)(1-y)^3 \\ & + 9*\left(\frac{2}{9}\right)*x^2y(1-y)^2(1-x) + 9*\left(\frac{4}{9}\right)*x^2y^2(1-y)(1-x) + 3*\left(\frac{2}{3}\right)*x^2y^3(1-x) \\ & + 0*x^3(1-y)^3 + 3*\left(\frac{1}{3}\right)*x^3y(1-y)^2 + 3*\left(\frac{2}{3}\right)*x^3y^2(1-y) + 1*x^3y^3 \end{aligned}$$

The results depending on the least square error (L.S.E) is presented in Table(2)

Table (2) The results of Example(2)

i	*x*	*y*	*Exact u(x,y)*	*Approximation u(x,y)n=m=1*	*Approximationu(x,y)n=m=2*	*Approximatio u(x,y)n=m=3*	*Error*
1	0	0	0	0	0	0	0
2	0	0.25	0	0	0	0	0
3	0	0.5	0	0	0	0	0
4	0	0.75	0	0	0	0	0
5	0	1	0	0	0	0	0
6	0.25	0	0	0	0	0	0
7	0.25	0.25	0.0625	0.0625	0.0625	0.0625	0
8	0.25	0.5	0.1250	0.1250	0.1250	0.1250	0
9	0.25	0.75	0.1875	0.1875	0.1875	0.1875	0
10	0.25	1	0.25	0.25	0.25	0.25	0
11	0.5	0	0	0	0	0	0
12	0.5	0.25	0.1250	0.1250	0.1250	0.1250	0
13	0.5	0.5	0.2500	0.2500	0.2500	0.2500	0
14	0.5	0.75	0.3750	0.3750	0.3750	0.3750	0
15	0.5	1	0.50	0.50	0.50	0.50	0
16	0.75	0	0	0	0	0	0
17	0.75	0.25	0.1875	0.1875	0.1875	0.1875	0
18	0.75	0.5	0.3750	0.3750	0.3750	0.3750	0
19	0.75	0.75	0.5625	0.5625	0.5625	0.5625	0
20	0.75	1	0.75	0.75	0.75	0.75	0
21	1	0	0	0	0	0	0
22	1	0.25	0.25	0.25	0.25	0.25	0
23	1	0.5	0.5	0.5	0.5	0.5	0
24	1	0.75	0.75	0.75	0.75	0.75	0
25	1	1	1	1	1	1	0

Conclusion:

This paper presents the use of the two-variables Bernstein polynomials method, for solving linear Fredholm two-dimensional integral equation of the second kind. From solving some numerical examples the following points have been identified:

1. This method can be used to solve of linear Fredholm integral equation.
2. It is clear that using the two-variables Bernstein polynomial basis function to approximate when the n^{th} degree of Bernstein polynomial increases the error is decreases.

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حل تقريبي لبعض أنواع المعادلات التكاملية باستخدام متعدد حدود برنستن ذات متغيرين

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الخلاصة:

يهدف البحث الى إيجاد الحلول التقريبية لمعادلة فريدهوم الخطية من النوع الثاني ذات البعدين. باستخدام متعدد حدود برنستن ذات المتغيرين تم ايجاد الحل التقريبي للمعادلة التكاملية الخطية من النوع الثاني ذات البعدين. تمت مناقشة مثالين بالتفصيل.