

## Some Probability Characteristics Functions of the Solution of Stochastic Fredholm Integral Equation Contains a Known Sine Wave Function

*Mohammad Wahdan Muflih\**

*Amina Kassim Hussain\*\**

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**Abstract:**

In this paper, some probability characteristics functions (moments, variances, covariance, and spectral density functions) are found depending upon the smallest variance of the solution of some stochastic Fredholm integral equation contains as a known function, the sine wave function.

**Key words:** Sine wave function, stochastic Fredholm integral equation.

**Introduction:**

The aim of this paper is finding some probability characteristics like moments, variances, covariance function and spectral density function,

depending upon the smallest variance of the solution of the following stochastic Fredholm integral equation (SFIE),

$$u(w, t) = f(w, t) + \int_0^{\infty} k(w, t, s) u(w, s) ds \quad \dots\dots\dots(1.1)$$

such that,

- $f(w,t)$  is a sine wave function (sinusoidal wave) which is a

- stochastic process defined for  $t > 0$  with the formula

$$f(w,t) = A \sin(wt + B), t > 0, 0 \leq w \leq 2\pi \quad \dots\dots\dots(1.2)$$

where  $A$  is the peak deviation (amplitude) of the function from its center position,  $w$  is the angular frequency specify how many oscillations occur in a unit time interval (scalling by degrees per second) and  $B$  is the phase shift, [1].

In this paper, two cases of the phase shift  $B$  will be considered,

$$B = \frac{\pi}{3} \text{ and } B = \frac{\pi}{6} \text{ associated with}$$

two values of the amplitude  $A$ . Also, the paper consider that, the kernel  $k(w,t,s)$  have the following negative exponential function,

- $k(w,t,s)$  is known stochastic kernel defined by  $t > 0$  and  $s \in S$ , where  $S$  is a compact metric space.
- $u(s,t)$  is a scalar function defined for the time  $t > 0, s \in S$ .

$$k(w, t, s) = e^{-(st+w)}, 0 < s < 2\pi, t > 0, 0 \leq w \leq \pi \quad \dots\dots\dots(1.3)$$

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\*Department of Mathematics-Ibn-Al-Haitham College of Education - University of Baghdad

\*\*Department of Material Engineering, College of Engineering – Al-Mustansiriya University

**Preliminaries**

By substituting (1.2) and (1.3) into (1.1), we get

$$u(w, t) = A \sin(wt + B) + \int_0^\pi e^{-(st+w)} u(w, s) ds \tag{2.1}$$

To find the solution of this equation, Adomain decomposition method (A.D.M) will be used which briefly depend on the following steps, [2],

$$u_0(w, t) = f(w, t) = A \sin(wt + B) \tag{2.1a}$$

$$u_0(w, s) = f(w, s) = A \sin(ws + B) \tag{2.1b}$$

$$u_{m+1}(w, s) = \int_0^{2\pi} e^{-(st+w)} u_m(w, s) ds, m=0,1,2,\dots \tag{2.1c}$$

that is to obtain the following stochastic solution,

$$u(w, t) = u_0(w, t) + \sum_{n=1}^\infty u_n(w, t) \tag{2.2}$$

So that when  $m = 0$ , by (2.1b) and (2.1c),

$$\begin{aligned} u_1(w, t) &= A \int_0^{2\pi} e^{-(st+w)} \sin(ws + B) ds \\ &= \frac{A}{t} e^{-(w-B)} \int_B^{2\pi t+B} e^{-y} \sin y dy, \quad y = ws + B \end{aligned}$$

or

$$u_1(w, t) = \frac{A}{2t} e^{-w} \left\{ (\sin B + \cos B) - e^{-\pi t} (\sin(2\pi t + B) + \cos(2\pi t + B)) \right\} \tag{2.3}$$

and when  $m = 1$ , by (2.1c) and (2.3),

$$\begin{aligned} u_2(w, t) &= \int_0^\pi e^{-(st+w)} u_1(s, t) ds \\ &= \frac{A}{2t} \left\{ (\sin B + \cos B) - e^{-\pi t} (\sin(\pi t + B) + \cos(\pi t + B)) \right\} \int_0^\pi e^{-s} e^{-(st+w)} ds \\ &= \frac{A}{2t} e^{-w} \left\{ (\sin B + \cos B) - e^{-\pi t} (\sin(\pi t + B) + \cos(\pi t + B)) \right\} \int_0^\pi e^{-(t+1)s} ds \end{aligned}$$

or

$$u_2(w, t) = \frac{A}{2t} e^{-w} \left\{ (\sin B + \cos B) - e^{-\pi t} (\sin(\pi t + B) + \cos(\pi t + B)) \right\} \left( \frac{1 - e^{-\pi(t+1)}}{t+1} \right) \tag{2.4}$$

and when  $m = 2$ , by (2.1c) and (2.4)

$$u_3(w, t) = \frac{A}{2t} e^{-w} \left\{ (\sin B + \cos B) - e^{-\pi t} (\sin(\pi t + B) + \cos(\pi t + B)) \right\} \left( \frac{1 - e^{-\pi(t+1)}}{t+1} \right) \int_0^\pi e^{-(t+1)s} ds$$

or

$$u_3(w, t) = \frac{A}{2t} e^{-w} \left\{ (\sin B + \cos B) - e^{-\pi t} (\sin(\pi t + B) + \cos(\pi t + B)) \right\} \left( \frac{1 - e^{-\pi(t+1)}}{t+1} \right)^2 \tag{2.5}$$

By iteration for  $m = 3, 4, 5, \dots$  and adding (2.3) – (2.5), we get,

$$\sum_{n=1}^{\infty} u_n(w, t) = \frac{A}{2t} e^{-w} \left\{ (\sin B + \cos B) - e^{-\pi t} (\sin(\pi t + B) + \cos(\pi t + B)) \right\} \sum_{k=0}^{\infty} \left( \frac{1 - e^{-\pi(t+1)}}{t+1} \right)^k$$

$$= \frac{A}{2t} e^{-w} \left\{ (\sin B + \cos B) - e^{-\pi t} (\sin(\pi t + B) + \cos(\pi t + B)) \right\} \left( \frac{t+1}{t + e^{-\pi(t+1)}} \right)$$

or

$$\sum_{n=1}^{\infty} u_n(w, t) = AM_{(B)} e^{-w} \dots\dots\dots(2.6)$$

where

$$M_{(B)} = \left\{ \frac{(\sin B + \cos B) - e^{-\pi t} (\sin(\pi t + B) + \cos(\pi t + B)) A}{2t} \right\} \left( \frac{t+1}{t + e^{-\pi(t+1)}} \right)$$

Finally, by substituting (2.1a), (2.6) into (2.2) which represents the stochastic solution of (2.1) we get,

$$u_{(B)}(w, t) = A \left[ \sin(wt + B) + M_{(B)} e^{-w} \right], 0 < w < \pi, t > 0 \dots\dots\dots(2.7)$$

Moreover, this solution can be considered as a solution for the time  $t \in T$  such that  $T = \{0.1, 0.2, 0.3, \dots, 2.0\}$ .

the solution by one and find the value of the amplitude A, (i.e.), we write,

$$\int_0^{\pi} u_{(B)}(w, t) dw = \int_0^{\pi} A \left[ \sin(wt + B) + M_{(B)} e^{-w} \right] dw = 1$$

**Moments, Variances of the Solution:**

In order to find the moments, variances of the solution (2.7), it must be that this solution is a probability density function of  $w$ . So, we equate

which yields to,

$$A = \frac{1}{0.975M_{(B)} - \frac{1}{t} [\cos(\pi t + B) - \cos B]} \dots\dots\dots(3.1)$$

Hence the solution (2.7) with the following formula

$$u_{(B)}(w, t) = \frac{\sin(wt + B) + M_{(B)} e^{-w}}{0.975M_{(B)} - \frac{1}{t} [\cos(\pi t + B) - \cos B]} \dots\dots\dots(3.2)$$

is a probability density function.

**First moment:**

$$\begin{aligned}
 E_{u_{(B)}} [w, t] &= \int_0^\pi w [u_{(B)}(w, t)] dw \\
 &= \frac{\int_0^\pi w [\sin(wt + B) + M_{(B)} e^{-w}] dw}{0.975M_{(B)} - \frac{1}{t} [\cos(\pi t + B) - \cos B]} \\
 &= \frac{\frac{1}{t^2} [\sin(\pi t + B) - \sin B] - \frac{\pi}{t} \cos(\pi t + B) + (1 - e^{-\pi} - \pi e^{-\pi}) M_{(B)}}{0.975M_{(B)} - \frac{1}{t} [\cos(\pi t + B) - \cos B]}
 \end{aligned}$$

or

$$E_{u_{(B)}} [w, t] = \frac{\frac{1}{t^2} [\sin(\pi t + B) - \sin B] - \frac{\pi}{t} \cos(\pi t + B) + 0.821M_{(B)}}{0.975 - \frac{1}{t} [\cos(\pi t + B) - \cos B]} \dots\dots\dots(3.1.1)$$

**Second moment:**

$$\begin{aligned}
 E_{u_{(B)}} [w^2, t] &= \int_0^\pi w^2 [u_{(B)}(w, t)] dw \\
 &= \frac{\int_0^\pi w^2 [\sin(wt + B) + M_{(B)} e^{-w}] dw}{0.975M_{(B)} - \frac{1}{t} [\cos(\pi t + B) - \cos B]} \\
 &= \frac{\frac{2}{t^3} [\cos(\pi t + B) - \cos B] + \frac{2\pi}{t^2} \sin(\pi t + B) - \frac{\pi^2}{t} \cos(\pi t + B) + [2 - (\pi^2 + 2\pi + 2)e^{-\pi}] M_{(B)}}{0.975 - \frac{1}{t} [\cos(\pi t + B) - \cos B]}
 \end{aligned}$$

or

$$E_{u_{(B)}} [w^2, t] = \frac{\frac{2}{t^3} [\cos(\pi t + B) - \cos B] + \frac{2\pi}{t^2} \sin(\pi t + B) - \frac{\pi^2}{t} \cos(\pi t + B) + 1.215M_{(B)}}{0.975 - \frac{1}{t} [\cos(\pi t + B) - \cos B]} \dots\dots(3.1.2)$$

Calculations of  $var_{u_{(B)}} [w, t]$  (defined in 2.6), when  $B = \frac{\pi}{6}, \frac{\pi}{3}$  are tabulated for  $t \in T = \{0.1, 0.2, \dots, 2.0\}$  in tables (1), (2) which shown that the smallest variance when the phase shift  $B = \frac{\pi}{6}$  is

(0.05271) at ( $t = 0.9$ ), while the smallest variance when the phase shift  $B = \frac{\pi}{6}$  is (0.02077) at ( $t = 1.10$ ). Therefore, the stochastic solution (3.2) can be rewritten as the following two formulas,

$$u_{\left(\frac{\pi}{3}\right)}(w, 0.9) = \frac{\sin(0.9w + \frac{\pi}{3}) + 1.743e^{-w}}{2.5484}, \quad 0 < w < \pi \dots\dots\dots(3.3)$$

$$u_{\left(\frac{\pi}{6}\right)}(w,1.1) = \frac{\sin(1.1w + \frac{\pi}{6}) + 1.226e^{-w}}{2.6188}, 0 < w < \pi \quad \dots\dots\dots(3.4)$$

(Figure (1) represent the curves of  $u_{\left(\frac{\pi}{3}\right)}(w,0.9), u_{\left(\frac{\pi}{6}\right)}(w,1.1)$ ).

**Covariance Function of  $u_{(B)}(w,t)$ :**

**When  $B = \frac{\pi}{3}$**

For any  $s > t = 0.9$ ,  $\tau = s - 0.9$  and  $-\tau = 0.9 - s$ , the covariance function

$$\begin{aligned} C(-\tau) &= C(0.9 + \tau, 0.9) \\ &= C(0.9, 0.9 + \tau) \\ &= C(\tau) \\ &= E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9] - E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] - E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9]E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] \\ &= E_{u_{\left(\frac{\pi}{3}\right)}}[w^2, 0.9] - E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau]E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] - E_{u_{\left(\frac{\pi}{3}\right)}}[w^2, 0.9] - E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9]E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] \\ &= E_{u_{\left(\frac{\pi}{3}\right)}}[w^2, 0.9] + E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9]E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] - E_{u_{\left(\frac{\pi}{3}\right)}}[w^2, 0.9] - E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9]E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] \\ &= E_{u_{\left(\frac{\pi}{3}\right)}}[w^2, 0.9] + E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9](E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] - E_{u_{\left(\frac{\pi}{3}\right)}}[w^2, 0.9]) - E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9]E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] \\ &= E_{u_{\left(\frac{\pi}{3}\right)}}[w^2, 0.9] + E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9]E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] - (E_{u_{\left(\frac{\pi}{3}\right)}}[w^2, 0.9])^2 - E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9]E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9 + \tau] \\ &= E_{u_{\left(\frac{\pi}{3}\right)}}[w^2, 0.9] + (E_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9])^2 \\ &= \text{Var}_{u_{\left(\frac{\pi}{3}\right)}}[w, 0.9] \end{aligned}$$

So, by table (2) when  $t = 0.9$ ,

$$C(-\tau) = C(\tau) = 0.05271, \tau > 0 \quad \dots\dots\dots(4.1.1)$$

(

**When  $B = \frac{\pi}{6}$**

For any  $s > t = 1.1$ , as previous, the covariance function of  $u_{\left(\frac{\pi}{6}\right)}$  is

$$C(-\tau) = C(\tau) = \text{var}_{u_{\left(\frac{\pi}{6}\right)}}[w, 1.1]$$

and by table (3) when  $t = 1.10$

$$C(-\tau) = C(\tau) = 0.02077, \tau > 0 \quad \dots\dots\dots(4.1.2)$$

**Spectral Density Function of  $u_{(B)}(w,t)$ :**

**When  $B = \frac{\pi}{3}$**

The spectral density function of  $u_{\left(\frac{\pi}{3}\right)}(w,0.9)$  for known  $C(\tau) = 0.05271$

can be found by Khinchine's formula as following, [4],

$$f(u_{(\frac{\pi}{3})}, \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\tau) e^{-i\lambda\tau} d\tau$$

$$= \frac{0.5271}{2\pi} \int_{-\infty}^{\infty} (\cos \lambda\tau - i \sin \lambda\tau) d\tau$$

$$f(u_{(\frac{\pi}{3})}, \lambda) = 0.01678 \frac{\sin[\lambda(s-0.9)]}{\lambda}, \lambda \leq n\pi, n \in \mathbb{R}^+ \cup \{0^+\} \dots\dots\dots(5.1.1)$$

(Figure (2) represents the curves of  $f(u_{(\frac{\pi}{3})}, \lambda), 0^+ \leq \lambda \leq 2\pi$  and  $s = 5.9, 10.9$ ).

**When  $B = \frac{\pi}{6}$**

The spectral density function of  $u_{(\frac{\pi}{6})}(w, 1.1)$  can be found as previously

or

$$f(u_{(\frac{\pi}{6})}; \lambda) = 0.00661 \frac{\sin[\lambda(s-1.1)]}{\lambda}, \lambda \leq n\pi, n \in \mathbb{R}^+ \cup \{0^+\} \dots\dots\dots(5.2.1)$$

Figure (3) represents the curves of  $f(u_{(\frac{\pi}{6})}; \lambda), 0^+ \leq \lambda \leq 2\pi$  and  $s = 6.9, 11.1$ .

**Conclusions :**

- 1- Any increase in the shift phase B followed by an increase in the value of the variance of the of the solution.
- 2- The speed V of the sine wave equation  $V = \frac{w}{t}$  when th shift

and for  $\tau = s - 0.9, -\tau = 0.9 - s$

$$f(u_{(\frac{\pi}{3})}, \lambda) = \frac{0.05271}{\pi} \int_0^{s-0.9} \cos \lambda\tau d\tau$$

or

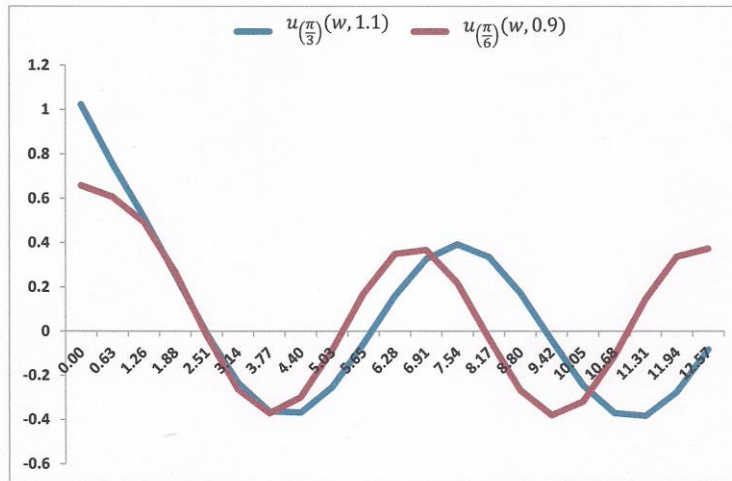
for known  $C(\tau) = 0.02077$  as following,

$$f(u_{(\frac{\pi}{6})}; \lambda) = \frac{0.02077}{2\pi} \int_{-\infty}^{\infty} (\cos \lambda\tau - i \sin \lambda\tau) d\tau$$

and for  $\tau = s - 1.1, -\tau = 1.1 - s$

$$f(u_{(\frac{\pi}{6})}; \lambda) = \frac{0.02077}{\pi} \int_0^{s-1.1} \cos \lambda\tau d\tau$$

- phase  $B = \frac{\pi}{6}$  ( $t = 1.10$ , variance = 0.02077) is less than the speed when  $B = \frac{\pi}{3}$  ( $t = 0.90$ , variance = 0.05271), that is meaning, any increase in the shift phase followed by an increase in the speed V.
- 3- For future work, we recommend to find variances, spectral density functions for different values of the phase shift B ( $B = 0, \frac{\pi}{12}, \dots, \frac{\pi}{2}$ ).



F Fig. (1)

The curves of  $u_{(\frac{\pi}{3})}(w, 1.1)$ ,  $u_{(\frac{\pi}{6})}(w, 0.9)$ ,  $0 \leq w \leq 4\pi$

4-

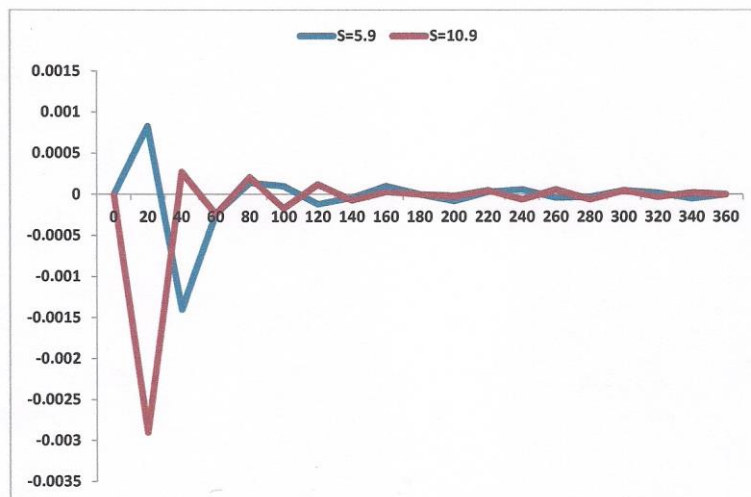


Fig. (2)

The curves of Spectral density function of  $u_{(\frac{\pi}{6})}(w, 0.9)$ ,  $0 \leq \lambda \leq 2\pi$

5-

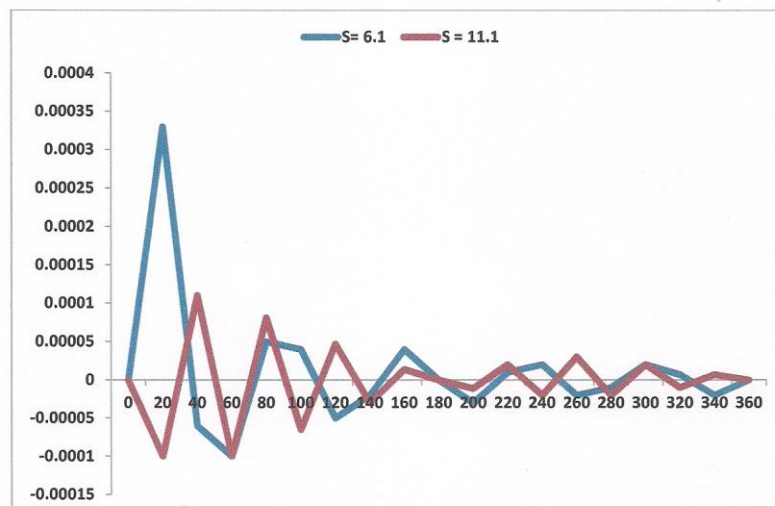


Fig Fig. (3)

The curves of Spectral density function of  $u_{(\frac{\pi}{6})}(w, 1.1)$ ,  $0 \leq \lambda \leq 2\pi$

6-

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## بعض دوال المزاي الاحتمالية لحل معادلة فريدهولم التكاملية والعشوائية الحاوية على دالة موجة الجيب

محمد وهدان مفلح\*      أمينة قاسم حسين\*\*

\*قسم الرياضيات - كلية التربية- ابن الهيثم - جامعة بغداد  
\*\*قسم المواد - كلية الهندسة - الجامعة المستنصرية

### الخلاصة :

في هذا البحث، تم إيجاد بعض دوال المزاي الاحتمالية (العزوم، التباينات، الارتباط والكثافة الطيفية) اعتماداً على أصغر تباين لحل معادلة افتراضية عشوائية لفريدهولم التكاملية التي أحتوت على دالة موجة الجيب.