Oscillations of First Order Neutral Differential Equations with Positive and Negative Coefficients

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Abstract:
Oscillation criterion is investigated for all solutions of the first-order linear neutral differential equations with positive and negative coefficients. Some sufficient conditionsare established so that every solution ofeq.(1.1) oscillate.Generalizing of some results in [4] and [5] are given.Examples are given toillustrated our main results.

Keywords: Neutral differential equations oscillations.

Introduction:
The study of neutral differential equationswith positive and negative coefficients has been recentlyconsideredthe attention of many authors all over the world for the last several years, see [1]-[6] a few of them have been investigated the case with variablemixed coefficients, that is the coefficients are variable positive and negative,see [1],[4]and [6]. The authors in [1] investigated the first order delay differential equations with positive and negative coefficients . While in [2],[5] and [6] the authors gave some sufficient conditions for the oscillation of neutral differential equation with positive and negative coefficients and constant delays . In this paper we give a generalization tosome results in [4] and [5]where we have used a variable delays. Consider the linear neutral differential equation with positive and negative coefficients:

$$\left[y(t) - P(t)y\big(\tau(t)\big) \right]' + Q(t)y\big(\sigma(t)\big)$$
$$- R(t)y\big(\alpha(t)\big) = 0 \quad (1.1)$$

Where $P, Q, R, \in C[[t_0, \infty); R^+]$,   and $\tau, \sigma, \alpha$are   continuous   strictly increasing   functions   with$\lim_{t\to\infty} \tau(t) = \infty$,$\lim_{t\to\infty} \sigma(t) = \infty$,

$\sigma(t) < \alpha(t)$and$\tau(t), \sigma(t), \alpha(t) < t$.(1.2)

By a solution of eq.(1.1) we mean a function $y \in \big([t_y, \infty), R\big)$ such that $y(t) - P(t)y\big(\tau(t)\big)$is continuously differentiable and $y(t)$satisfies eq. (1.1), $t_y = \max\{\tau(t), \sigma(t), \alpha(t)\}$in   the initial interval.A solution of eq.(1.1) is said to be oscillatory if it has arbitrarily large zeros , otherwise is said to be nonoscillatory.The purpose of this paper is to obtain sufficient conditions for the oscillation of all solutions of eq. (1.1).

Some Basic Lemmas:
The following lemmas will be useful in the proof of the main results:

Lemma 1 (Theorem 2.1.1 [4])
Suppose that $p, q \in C[R^+, R^+]$,$q(t) < t$for $t \geq t_0$,$\lim_{t\to\infty} q(t) = \infty$ and $\liminf_{t\to\infty} \int_{q(t)}^{t} p(s)ds > \frac{1}{e}$(2.1)
Then   the   inequality $y'(t) + P(t)y(q(t)) \leq 0$hasno   eventually positive solutions.

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Lemma 2.

Let $y(t)$ be an eventually positive solution of (1.1) and set

$$W(t) = y(t) - P(t)y\big(\tau(t)\big) - \int_{\alpha^{-1}(\sigma(t))}^{t} R(s)y\big(\alpha(s)\big)ds, \ t \geq \alpha^{-1}\big(\sigma(t)\big) \quad (2.2)$$

and the following assumptions are hold.

H1: $Q(t) - R(\alpha^{-1}(\sigma(t)))[\alpha^{-1}(\sigma(t))]'$

≥ 0 H2: $\limsup_{t\to\infty}[P(t) + \int_{\alpha^{-1}(\sigma(t))}^{t} R(s)ds] \leq 1$

Then $W(t)$ is eventually positive and non increasing function.

Proof.Suppose that $y(t) > 0, y\big(\tau(t)\big) > 0, y\big(\sigma(t)\big) > 0$ and $y\big(\alpha(t)\big) > 0$, $t \geq t_0$

Differentiate (2.2) and use (1.1) we get
$W'(t)$
$= \big[y(t) - P(t)y\big(\tau(t)\big)\big]' - R(t)y\big(\alpha(t)\big)$
$+ R\big((\alpha^{-1}(\sigma(t)))\big) y\big(\sigma(t)\big) \big(\alpha^{-1}(\sigma(t))\big)'$
$= -Q(t)y\big(\sigma(t)\big) + R(t)y\big(\alpha(t)\big)$
$- R(t)y(\alpha(t))$
$+ R(\alpha^{-1}(\sigma(t)))y\big(\sigma(t)\big)\big(\alpha^{-1}(\sigma(t))\big)$
$= -[Q(t) - R(\alpha^{-1}(\sigma(t)))\big(\alpha^{-1}(\sigma(t))\big)']y\big(\sigma(t)\big) \leq 0 \quad (2.3)$

Hence $W(t)$ is monotone(nonincreasing) function then

$$\lim_{t\to\infty} w(t) = l, \quad -\infty \leq l < \infty$$

we claim that $l \geq 0$ otherwise $l < 0$,we have two cases for $y(t)$ to consider :

Case 1:-
If y (t) is unbounded then there exist a sequence $\{t_n\}$ such that
$$\lim_{n\to\infty} t_n = \infty, \ \lim_{n\to\infty} y(t_n) = \infty, \text{and}$$
$$y(t_n) = \max\{y(t): t_2 \leq t \leq t_n\}$$

from (1.2) we get

$W(t_n)$
$= y(t_n) - P(t_n)y\big(\tau(t_n)\big)$
$- \int_{\alpha^{-1}(\sigma(t_n))}^{t_n} R(s)y(\alpha(s))ds$
$\geq y(t_n) - P(t_n)y\big(\tau(t_n)\big)$
$-y(t_n)\int_{\alpha^{-1}(\sigma(t_n))}^{t_n} R(s)ds$
$\geq y(t_n) - P(t_n)y(t_n)$
$-y(t_n)\int_{\alpha^{-1}(\sigma(t_n))}^{t_n} R(s)ds$
$= y(t\_n)[1 - P(t\_n)$
$- \int \_(\alpha^{(-1)} (\sigma(t\_n)))^{(t\_n)} ⬚ 〚R(s)ds〛〛$
as $n \to \infty$,$\lim_{n\to\infty} W(t_n) = l \geq 0$.

Case 2:-
Let $y(t)$ be bounded, that is $\limsup_{t\to\infty} y(t) = k < \infty$ then there exist a sequence $\{s_n\}$ such that, $\lim_{n\to\infty} s_n = k$, and
$$y(\eta_n) = \max\{y(t): \lambda_1(s_n) \leq t \leq \lambda_2(s_n)\}$$
Where $\lambda_1(t) = \min\{\tau(t), \sigma(t)\}$, $\lambda_2(t) = \max\{\tau(t), \alpha(t)\}$.

$W(s_n)$
$= y(s_n) - P(s_n)y(\tau(s_n))$
$- \int_{\alpha^{-1}(\sigma(s_n))}^{s_n} R(s)y\big(\alpha(s)\big)ds$

$y(s_n) - W(s_n)$
$= P(s_n)y(\tau(s_n))$
$+ \int_{\alpha^{-1}(\sigma(s_n))}^{s_n} R(s)y\big(\alpha(s)\big)ds$
$\leq P(s_n)y(\eta_n)$
$+ y(\eta_n) \int_{\alpha^{-1}(\sigma(s_n))}^{s_n} R(s)ds$

$= y(\eta_n)[P(s_n) + \int_{\alpha^{-1}(\sigma(s_n))}^{s_n} R(s)ds]$

as $n \to \infty$ then $k - l \leq k$ which implies that $l \geq 0$,this is a contradiction

The proof of lemma is complete. ∎

The proof of the following lemma is similar tothe proof of lemma 2, so we state it without proof.

Lemma 3.
Let $y(t)$ be an eventually positive solution of eq.(1.1) and set
$$W(t) = y(t) - P(t)y(\tau(t))$$
$$- \int_{\sigma(t)}^{\alpha(t)} R(\alpha^{-1}(s))y(s)ds \quad (2.4)$$
and the following assumptions are hold.

H1′: $Q(t) - R(\alpha^{-1}(\sigma(t)))\,\sigma'(t) \geq 0$

  H2′: $\limsup_{t\to\infty}[P(t)$

$$+ \int_{\sigma(t)}^{\alpha(t)} R(\alpha^{-1}(s))ds] \leq 1,$$

H3: $\alpha'(t) \geq 1$

Then $W(t)$ is eventually positive and nonincreasing function.

Main results:
The next result providesa sufficient conditions for the oscillation of all solutions of eq.(1.1)

Theorem 1.
Let $W(t)$ defined as in (2.2) and the assumptionsH1 -H2 hold,in addition to the condition

$\liminf_{t\to\infty} \int_{\sigma(t)}^{t}[Q(s) -$
$R(\alpha^{-1}(\sigma(s)))(\alpha^{-1}(\sigma(s)))'][1 + P(\sigma(s)) +$
$\int_{\alpha^{-1}(\sigma(\sigma(s)))}^{s} R(r)dr]ds > \frac{1}{e}$ (3.1)

Then every solution of eq.(1.1) oscillates.

Proof .Suppose $y(t)$be eventually positive solution of eq.(1.1) then by lemma 2 it follows that $W(t)$ is positive nonincreasing function, differentiate (2.2) and use eq.(1.1) we get (2.3)and from (2.2) we obtain $W(t) \leq y(t)$, hence
$$y(t) = W(t) + P(t)y(\tau(t))$$

$$+ \int_{\alpha^{-1}(\sigma(t))}^{t} R(s)y(\alpha(s))ds$$
$$\geq W(t) + P(t)W(\tau(t))$$
$$+ \int_{\alpha^{-1}(\sigma(t))}^{t} R(s)W(\alpha(s))ds$$

$$\geq W(t) + P(t)W(\tau(t))$$
$$+ W(\alpha(t)) \int_{\alpha^{-1}(\sigma(t))}^{t} R(s)ds$$
$$\geq W(t) + P(t)W(t)$$
$$+ W(t) \int_{\alpha^{-1}(\sigma(t))}^{t} R(s)ds$$
$$= W(t)\,[1 + P(t)$$
$$+ \int_{\alpha^{-1}(\sigma(t))}^{t} R(s)ds\ ]$$

$y(\sigma(t)) \geq W(\sigma(t))\,[1 + P(\sigma(t)) + \int_{\alpha^{-1}(\sigma(t))}^{t} R(s)ds\ ]$

$W'(t) + \left[Q(t) - R(\alpha^{-1}(\sigma(t)))\frac{d}{dt}\alpha^{-1}(\sigma(t)) \right][1$
$$+$$
$+P(\sigma(t))] + \int_{\alpha^{-1}(\sigma(\sigma(t)))}^{\sigma(t)} R(s)ds\ ]\,W(\sigma(t)\ ) \leq 0$

bylemma 1, and the condition (3.1) the last inequality cannot has eventually positive solution, which is a contradiction. ∎

Example 1.
Consider the neutral differential equation:

$\left[y(t) - Py\left(t - \frac{\pi}{2}\right) \right]' + \left(e^{-\pi} + 2Pe^{-\frac{3\pi}{4}} \right)y(t - \pi) - \left(P - 2e^{-\frac{3\pi}{4}} \right)y\left(t - \frac{\pi}{4}\right) = 0$ (E1)

$P(t) = P$,where $0 \leq P \leq 1.178082$.
$$Q(t) = e^{-\pi} + 2Pe^{-\frac{3\pi}{4}}, R(t)$$
$$= P - 2e^{-\frac{3\pi}{4}}$$
We can see that:

- $\sigma(t) < \alpha(t)$, let P=0.92　then
- $Q(t) -$
$R(\alpha^{-1}(\sigma(t)))(\alpha^{-1}(\sigma(t)))' =$
　　$2.8552 \geq 0$
- $\lim \sup_{t\to\infty}[P(t) +$
$\int_{\alpha^{-1}(\sigma(t))}^{t} R(s)ds] =$　　$0.94 < 1$
- $\lim \inf_{t\to\infty} \int_{\sigma(t)}^{t}[Q(s) -$
$R(\alpha^{-1}(\sigma(s)))(\alpha^{-1}(\sigma(s)))'][1 +$
$P(\sigma(s)) +$
　　$\int_{\alpha^{-1}(\sigma(\sigma(s)))}^{s} R(r)dr]ds =$
$17.39 > \frac{1}{e}$

all the conditions of theorem 1 are hold,so according to theorem 1every solution of eq.($E1$) is oscillatory, for instance the solution $y(t) = e^{-t}\sin t$ is sucha solution.

Theorem 2.

Let $W(t)$ defined as in (2.4) and the assumptionsH1 -H2 hold and

$\lim \inf_{t\to\infty} \int_{\sigma(t)}^{t}[Q(s) -$

$R(\alpha^{-1}(\sigma(s)))\sigma'(s)]ds > \frac{1}{e}$　(3.2)

Then every solution of equation (1.1) oscillates.

Proof.Assume for the sake of contradiction that $y(t)$ is eventually positive solutionof eq.(1.1).then by lemma 2 it follows that$W(t)$ is eventually positive and non increasing function, differentiate (2.2)and use (1.1) we get

$W'(t)$
$= - [Q(t)$
$- R(\alpha^{-1}(\sigma(t)))\sigma'(t)]y(\sigma(t)) \leq 0$
$\leq -[Q(t)$
$- R(\alpha^{-1}(\sigma(t)))\sigma'(t)]W(\sigma(t))$
≤ 0
then
$W'(t)$
$+ [Q(t)$
$- R(\alpha^{-1}(\sigma(t)))\sigma'(t)]W(\sigma(t))$
≤ 0

it follows from lemma1and condition (3.2) that the last inequality has noeventually positive solution.The proof is complete .

Theorem 3.

Let $W(t)$be defined as in (2.2) and the assumptionsH1′,H2′ andH3 hold and suppose that

$\lim \inf_{t\to\infty} \int_{\sigma(t)}^{t}[Q(s) -$

$R(\alpha^{-1}(\sigma(s)))\sigma'(s)][1 + P(\sigma(s)) +$
　　$\int_{\sigma(t)}^{\alpha(t)} R(\alpha^{-1}(u))du]ds > \frac{1}{e}$ (3.3)

Then every solution of equation (1.1) oscillates.

Proof.
　　The proof is similar to the proof of theorem 1 and we omitted it.

Example 2.

　　Consider the neutral differential equation;

$\left[y(t) - (\frac{1}{4} - \frac{1}{4}\sin 2t) y(t - 2\pi)\right]' +$
$\frac{4}{5}\left(\frac{3}{4} - \frac{1}{4}\sin 2t\right) y\left(t - \frac{5\pi}{2}\right) -$
$\frac{1}{5}\left(\frac{3}{4} - \frac{1}{4}\sin 2t\right) y\left(t - \frac{3\pi}{2}\right) = 0,$　$t \geq t_0(E2)$

We can see that
- $Q(t) - R(\alpha^{-1}(\sigma(t))(\alpha^{-1}(\sigma(t))' =$
$\frac{3}{10} \geq 0$
- $P(t) + \int_{\sigma((\alpha^{-1}(\sigma(t))t)}^{t} R(s)ds =$
$0.971238 < 1$
- $\lim \inf_{t\to\infty} \int_{\sigma(t)}^{t}[Q(s) -$
$R(\alpha^{-1}(\sigma(s))\sigma'(s)]ds = \frac{15\pi}{20} > \frac{1}{e}$
all the conditions of theorem 2 or Theorem 3 are hold and so according to Theorem 2 or Theorem 3 every solution of eq.($E2$) are oscillatory, for instance the solution $y(t) = \frac{\cos t}{\frac{3}{4} + \frac{1}{4}\sin 2t}$
is such solution.∎

Theorem 4.

Let $W(t)$ defined as in (2.4) and the assumptions H1', H2', H3 hold, suppose that

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} [Q(s) - R(\alpha^{-1}(\sigma(s)))\sigma'(s)]\, ds > \frac{1}{e} \quad (3.4)$$

Then every solution of equation (1.1) oscillates.

Proof. The proof is similar to the proof of theorem 2. and will be omitted.

Example 3.

Consider the neutral differential equation:

$$\left[y(t) - \frac{1}{2\sqrt{2}}\, y\left(t - \frac{5\pi}{2}\right)\right]' + \frac{3}{2}y\left(t - \frac{\pi}{2}\right) - \frac{1}{2\sqrt{2}}\, y\left(t - \frac{\pi}{4}\right) = 0 \quad (E3)$$

Solution: We can see that

- $Q(t) - R(\alpha^{-1}(\sigma(t))(\alpha^{-1}(\sigma(t))' \geq 0$

- $P(t) + \int_{\sigma((\alpha^{-1}(\sigma(t))t)}^{t} R(s)ds = 0.63123$

$$< 1$$

- $\liminf_{t\to\infty} \int_{\sigma(t)}^{t}[Q(s) - R(\alpha^{-1}(\sigma(s))\sigma'(s)]ds = 1.785 > \frac{1}{e}.$

all the conditions of theorem 4 hold and so according to theorem 4 every solution of eq.(*E3*) are oscillatory for instance the solution $y(t) = \sin t$ is such a solution. ■

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تذبذب المعادلات التفاضلية المحايدة من الرتبة الأولى ذات المعاملات الموجبة والسالبة

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الخلاصة:

في هذا البحث تم استخراج بعض الشروط الضرورية والكافية لتذبذب جميع حلول المعادلات التفاضلية المحايدة من الرتبة الأولى ذات المعاملات الموجبة والسالبة بالشكل

$$[y(t) - P(t)y(\tau(t))]' + Q(t)y(\sigma(t)) - R(t)y(\alpha(t)) = 0 \tag{1.1}$$

وقد أعطينا بعضالأمثلة لتوضيح النتائج المستخلصة.