

Convergence of the Generalized Homotopy Perturbation Method for Solving Fractional Order Integro-Differential Equations

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Abstract:

In this paper, the homotopy perturbation method (HPM) was applied to obtain the approximate solutions of the fractional order integro-differential equations. The fractional order derivatives and fractional order integral are described in the Caputo and Riemann-Liouville sense respectively. We can easily obtain the solution from convergent the infinite series of HPM. A theorem for convergence and error estimates of the HPM for solving fractional order integro-differential equations was given. Moreover, numerical results show that our theoretical analysis are accurate and the HPM can be considered as a powerful method for solving fractional order integro-differential equations.

Key words: Homotopy perturbation method, fractional calculus, integro-differential equations.

Introduction:

In recent years various analytical and numerical methods have been applied for approximating the solutions of fractional order differential equations (FDEs). Since exact solutions of most of fractional differential equations do not exist, approximation and numerical methods are used for the solutions of the FDEs, [1-3]. He [1,2,3] was first propose the homotopy perturbation method (HPM) for finding the solutions of linear and nonlinear problems. The (HPM) is the traditional perturbation method and homotopy in topology. This method has been successfully applied by many authors [4,2,5,6] for finding the approximate solutions as well as numerical solutions of functional equations which arise in scientific and engineering problems. Fractional order integro-differential equations arise in modeling processes in applied sciences like physics, engineering, chemistry, and other sciences [7], which can be described very successfully by models

using mathematics tools from fractional calculus, such as, frequency dependent damping behavior of materials, diffusion process and motion of a large thin plate in a Newtonian fluid creeping, etc., [8]. The fractional integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. And there are few techniques for solving fractional integro-differential equations, such as, the Adomian decomposition method, the collocation method and the fractional differential transform method, [9],[6]. The purpose of this paper is to extend the analysis of HPM to construct the approximate solutions of fractional order integro-differential equations.

$$D_*^\alpha y(x) = g(x) + I^\beta K(y(x)) \quad (1)$$

where D_*^α indicates the fractional order differential operator in the Caputo sense and I^β is the fractional order integral operator in the

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Riemann-Liouville sense and $K(y(x))$ is any nonlinear continuous function, α, β are real constants and g are given and can be approximated by Taylor polynomials.

Basic definitions

In this section, we shall give some but not all, of the basic definitions and properties of fractional calculus theory which are further used in this paper.

Definition(1), [10]:

The Riemann-Liouville definition of the right side fractional integral which is:

$${}_a I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t) dt, \alpha > 0 \tag{2}$$

while the left hand sided integral:

$${}_x I_b^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} u(t) dt, \alpha > 0 \tag{3}$$

Definition (2), [2]:

The Caputo definition of fractional derivative is given by:

$${}_0 D_x^\alpha u(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} u^m(t) dt$$

where $m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0$

Now, some properties of fractional concerning differintegration are given next, [11],[10]:

I- If $m-1 < \alpha \leq m, m \in \mathbb{N}$ and f is any function, then:

$$D_*^\alpha I_*^\alpha f(x) = f(x)$$

$$I^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!},$$

$x > 0$

Where 0^+ refers to the right hand side limit of $f^{(k)}$ at 0.

2- $I^0 f(x) = D_*^0 f(x) = f(x).$

3- $I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x) = I^{\alpha+\beta} f(x), \forall \alpha, \beta \geq 0.$

4- $I^\alpha f(x) = D_*^{-\alpha} f(x), \alpha > 0.$

5- $I^\alpha(0) = D_*^{-\alpha} 0 = 0, \alpha > 0.$

Analysis of HPM

To illustrate the basic concepts of HPM for fractional order integro-differential equations, consider the fractional order integro-differential equation (1).

In view of HPM [2,3,6], construct the following homotopy for equation (1):

$$(1-p)D_*^\alpha y(x) + p(D_*^\alpha y(x) - g(x) - I^\beta K(y(x))) = 0 \tag{4}$$

or

$$D_*^\alpha y(x) = p(g(x) + I^\beta K(y(x))) \tag{5}$$

where $p \in [0,1]$ is an embedding parameter. If $p = 0$, then equation (5) becomes a linear equation

$$D_*^\alpha y(x) = 0 \tag{6}$$

and when $p = 1$, then equation (5) turns out to be the original equation (1).

In view of basic assumption of homotopy perturbation method, solution of equation(1) can be expressed as a power series in p :

$$y(x) = y_0(x) + py_1(x) + p^2y_2(x) + p^3y_3(x) \dots$$

(7)

setting $p = 1$, in (5) then we get an approximate solution of equation(1)

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \dots$$

(8)

Substitution (7) into (5), then equating the terms with identical power of p , we obtain the following series of linear equations.

$$D_*^\alpha(y_0(x) + py_1(x) + p^2y_2(x) + p^3y_3(x) + \dots) = p(g(x) + I^\beta K(y_0(x) + py_1(x) + p^2y_2(x) + \dots)), \text{ implies to:}$$

$$D_*^\alpha y_0 + pD_*^\alpha y_1 + p^2D_*^\alpha y_2 + p^3D_*^\alpha y_3 + \dots = pg(x) + pI^\beta K_1 y_0 + p^2I^\beta K_2 y_1 + p^3I^\beta K_3 y_2 + p^4I^\beta K_4 y_2 + \dots, \text{ thus}$$

$$p^0: D_*^\alpha y_0 = 0$$

$$p^1: D_*^\alpha y_1 = g(x) + I^\beta K_1(y_0)$$

(9)

$$p^2: D_*^\alpha y_2 = I^\beta K_2(y_1)$$

(10)

$$p^3: D_*^\alpha y_3 = I^\beta K_3(y_2)$$

(11)

(12) : where the functions K_1, K_2, \dots satisfy the following condition:

$$K(y_0(x) + py_1(x) + p^2y_2(x) + \dots) = K_1(y_0(x)) + pK_2(y_1(x)) + p^2K_3(y_2(x)) + \dots, x \in [0, T]$$

Equations (9)-(12) can be solved by applying the operator I^α , which is inverse of the operator D_*^α and then by simple computation, we approximate the series solution of HPM by the following n-term truncated series:

$$\varphi_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \dots + y_n(x)$$

(13)

Now, we show that the series defined by (13) with $y_0(x) = y_0$ converges to the solution

of (1). To do this we state and prove the following theorem.

Theorem(1):

Let $y \in C^1[0, T]$ which defined with maximam norm $\| \cdot \|_\infty$.

$$D_*^\alpha y(x) = g(x) + I^\beta K(y(x)), \quad y(0) = y_0,$$

(14)

and $y_n \in (C^1[0, T], \| \cdot \|_\infty)$ be obtained solution of the sequence defined by

$$\varphi_{n+1}(x) = \varphi_n(x) + I^{\alpha+\beta} K_{n+1}(y_n(x)), \quad n \geq 1.$$

(15)

$$\varphi_1(x) = \varphi_0(x) + I^\alpha g(x) + I^{\alpha+\beta} K_1(y_0(x))$$

$$\varphi_0(x) = y_0(x).$$

If $E_n(x) = y_n(x) - y(x)$ and K in equation (15) satisfies Lipschitz condition with constant $L_n, n \geq 1$ such that $L = \max\{L_n, n \geq 1\}$ and $L < \Gamma(\alpha + \beta)$, then the sequence of approximate solution $\{\varphi_n\}, n = 0, 1, \dots$, converges to the exact solution y .

Proof

Consider the fractional order integro-differential equation of fractional order $D_*^\alpha y(x) = g(x) + I^\beta K(y(x)), y(0) = y_0$, $x \in [0, T]$

where the approximate solution using HPM is given by:

$$\varphi_{n+1}(x) = \varphi_n(x) + I^{\alpha+\beta} K_{n+1}(y_n(x)), \quad n \geq 1.$$

$$\varphi_1(x) = \varphi_0(x) + I^\alpha g(x) + I^{\alpha+\beta} K(y_0(x))$$

$$\varphi_0(x) = y_0(x).$$

Since y is the exact solution of the integro-differential equation of fractional order, hence:

$$y(x) = y(x) + I^\alpha g(x) + I^{\alpha+\beta} K(y(x))$$

hence subtracting y from φ_{n+1} yields to:

$$\begin{aligned} \varphi_{n+1}(x) - y(x) &= \varphi_n(x) - y(x) + I^\alpha g(x) - I^\alpha g(x) + I^{\alpha+\beta} (K(\varphi_n(x)) - K(y(x))) \\ \varphi_{n+1}(x) - \varphi_n(x) &= I^{\alpha+\beta} (K(\varphi_n(x)) - K(y(x))) \\ \varphi_{n+1}(x) - \varphi_n(x) &= I^\gamma [K(\varphi_n(x)) - K(y(x))], \quad \text{set } \gamma = \alpha + \beta \\ &= \frac{1}{\Gamma(\gamma)} \int_0^x (x-s)^{\gamma-1} [K(\varphi_n(s)) - K(y(s))] ds \end{aligned}$$

Now, taking the maximum-norm of the two sides of $\varphi_{n+1}(x) - \varphi_n(x)$ will give:

$$\begin{aligned} \|\varphi_{n+1}(x) - \varphi_n(x)\| &= \left\| \frac{1}{\Gamma(\gamma)} \int_0^x (x-s)^{\gamma-1} [K(\varphi_n(s)) - K(y(s))] ds \right\|_\infty \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^x \|x-s\|^{\gamma-1} \|K(\varphi_n(s)) - K(y(s))\|_\infty ds \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^x \max_{s \in [0,x]} |x-s|^{\gamma-1} L \|\varphi_n(s) - y(s)\|_\infty ds \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^x x^{\gamma-1} L \|E_n(s)\|_\infty ds, \quad \forall n = 0, 1, \dots \end{aligned}$$

From (9-12), we have that

$$\begin{aligned} \|E_{n+1}(s)\|_\infty &= \|y_{n+1}(x) - y(x)\|_\infty = \\ \|I^{\alpha+\beta} [K(y_n(x)) - K(y(x))]\|_\infty &\leq \|I^{\alpha+\beta} (y_n(x) - y(x))\|_\infty \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^x \max_{s \in [0,x]} |x-s|^{\gamma-1} L \|y_n(s) - y(s)\|_\infty ds \leq \frac{1}{\Gamma(\gamma)} \int_0^x x^{\gamma-1} L \|E_n(s)\|_\infty ds, \end{aligned}$$

Hence

$$\|E_{n+1}(s)\|_\infty \leq \frac{1}{\Gamma(\gamma)} \int_0^x x^{\gamma-1} L \|E_n(s)\|_\infty ds$$

Now if $n = 0$, then:

$$\begin{aligned} \|\varphi_1(x) - \varphi_0(x)\|_\infty &= \frac{1}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \|E_0(s)\|_\infty ds \\ &\leq \frac{L}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \max_{s \in [0,x]} |E_0(s)|^{\gamma-1} ds \leq \frac{L}{\Gamma(\gamma)} x^{\gamma-1} \max_{s \in [0,x]} |E_0(s)| \int_0^x ds \\ &\leq \frac{L}{\Gamma(\gamma)} x^\gamma \max_{s \in [0,x]} |E_0(s)|, \end{aligned}$$

also, for $n = 1$, we have

$$\begin{aligned} \|\varphi_2(x) - \varphi_1(x)\|_\infty &\leq \frac{1}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \|E_1(s)\|_\infty ds \\ &\leq \frac{L}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \left[\frac{L}{\Gamma(\gamma)} x^\gamma \max_{s \in [0,x]} |E_0(s)| \right] ds \\ &\leq \left(\frac{L}{\Gamma(\gamma)} \right)^2 x^{2\gamma-1} \max_{s \in [0,x]} |E_0(s)| \int_0^x ds \leq \left(\frac{L}{\Gamma(\gamma)} \right)^2 \frac{x^{2\gamma}}{\gamma+1} \max_{s \in [0,x]} |E_0(s)| \end{aligned}$$

Similarly, for $n = 2$, then

$$\begin{aligned} \|\varphi_3(x) - \varphi_2(x)\|_\infty &\leq \frac{1}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \|E_2(s)\|_\infty ds \\ &\leq \frac{L}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \left[\left(\frac{L}{\Gamma(\gamma)}\right)^2 \frac{s^{2\gamma}}{\gamma+1} \max_{s \in [0,x]} |E_0(s)| \right] ds \\ &= \left(\frac{L}{\Gamma(\gamma)}\right)^3 x^{\gamma-1} \max_{s \in [0,x]} |E_0(s)| \int_0^x \left[\frac{s^{2\gamma}}{\gamma+1} \right] ds \\ &= \left(\frac{L}{\Gamma(\gamma)}\right)^3 x^{\gamma-1} \max_{s \in [0,x]} |E_0(s)| \frac{x^{2\gamma}}{(\gamma+1)(2\gamma+1)} \\ &= \left(\frac{L}{\Gamma(\gamma)}\right)^3 \frac{x^{3\gamma}}{(\gamma+1)(2\gamma+1)} \max_{s \in [0,x]} |E_0(s)| \end{aligned}$$

⋮
⋮
⋮

$$\begin{aligned} \|\varphi_n(x) - \varphi_{n-1}(x)\|_\infty &\leq \left(\frac{L}{\Gamma(\gamma)}\right)^n \frac{x^{n\gamma}}{(\gamma+1)(2\gamma+1) \dots ((n-1)\gamma+1)} \max_{s \in [0,x]} |E_0(s)| \\ &\leq \left(\frac{L}{\Gamma(\gamma)}\right)^n \frac{T^{n\gamma}}{(\gamma+1)(2\gamma+1) \dots ((n-1)\gamma+1)} \max_{s \in [0,x]} |E_0(s)| \quad \text{for } n \geq 1, \end{aligned}$$

and since $L < \Gamma(\gamma)$, so as $n \rightarrow \infty$, we have $\|\varphi_{n+1}(x) - \varphi_n(x)\| \rightarrow 0$.

Now, we have that

$$y(x) = \varphi_0(x) + \sum_{n=0}^{\infty} (\varphi_{n+1}(x) - \varphi_n(x))$$

Now we have that $\varphi_j(x) = \varphi_0(x) + \sum_{m=1}^{j-1} (\varphi_{m+1}(x) - \varphi_m(x))$

$$y(x) - \varphi_j(x) = \sum_{n=j}^{\infty} (\varphi_{n+1}(x) - \varphi_n(x))$$

$$\begin{aligned} \|y(x) - \varphi_j(x)\|_\infty &\leq \sum_{n=j}^{\infty} \|\varphi_{n+1}(x) - \varphi_n(x)\|_\infty \\ &\leq \sum_{n=j}^{\infty} \left(\frac{L}{\Gamma(\gamma)}\right)^n \frac{T^{n\gamma}}{(\gamma+1)(2\gamma+1) \dots ((n-1)\gamma+1)} \max_{s \in [0,x]} |E_0(s)| \\ &\leq T^j \left(\frac{L}{\Gamma(\gamma)}\right)^j \max_{s \in [0,x]} |E_0(s)| \end{aligned}$$

$$\sum_{n=0}^{\infty} \left(\frac{L}{\Gamma(\gamma)}\right)^n \frac{T^{n\gamma}}{(\gamma+1)(2\gamma+1) \dots ((n-1)\gamma+1)}$$

Since $L < \Gamma(\gamma)$ therefore $T^{j\gamma} \left(\frac{L}{\Gamma(\gamma)}\right)^j \rightarrow 0$ as $j \rightarrow \infty$. Hence $\varphi_j \rightarrow y$.

Applications

In this section the (HPM) has been applied to linear and nonlinear fractional order integro-differential equations in order to illustrate the validity of the proposed method.

The following examples had been studied and discussed by using iteration variational method in [6].

Example(1)

Consider the following linear fractional order integro-differential equation

$$D_*^{0.75} y(x) = \frac{6}{\Gamma(3.25)} x^{2.25} - \frac{6}{\Gamma(4.75)} x^{3.75} + I^{0.75} y(x)$$

$$(16) y(0) = 0, x \in [0,2]$$

for comparison purpose, the exact solution of equation(16) is given by $y(x) = x^3$.

According to HPM, we construct the following homotopy:

$$D_*^{0.75}y(x) = p \left(\frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y(x) \right)$$

Substitution of (5) into (14) and then equating the terms with same powers of p yield the following series of linear equations:

$$p^0: D_*^{0.75}y_0 = 0 \tag{17}$$

$$p^1: D_*^{0.75}y_1 = \frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y_0(x) \tag{18}$$

$$p^2: D_*^{0.75}y_2 = I^{0.75}y_1(x) \tag{19}$$

$$p^3: D_*^{0.75}y_3 = I^{0.75}y_2(x) \tag{20}$$

⋮

Applying the operator $I^{0.75}$ to the above series of linear equations, we get $y_0(x) = y(0) = 0$

$$I^{0.75}D_*^{0.75}y_1(x) = I^{0.75} \left[\frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y_0(x) \right]$$

$$y_1(x) - y_1^{(0)}(0^+) \frac{x^0}{0!} = I^{0.75} \left[\frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y_0(x) \right]$$

$$y_1(x) = I^{0.75} \left[\frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y_0(x) \right]$$

$$y_2(x) = I^{0.75}I^{0.75}y_1(x)$$

$$y_3(x) = I^{0.75}I^{0.75}y_2(x)$$

⋮

Hence we find that

$$y_1(x) = x^3 - \frac{6x^{4.5}}{\Gamma(5.5)}, y_2(x) = \frac{\Gamma(4)}{\Gamma(5.5)}x^{4.5} - \frac{x^6}{120}, y_3(x) = \frac{\Gamma(4)}{\Gamma(7)}x^6 - \frac{\Gamma(7)x^{7.5}}{120\Gamma(8.5)}$$

$$y_4(x) = \frac{\Gamma(4)}{\Gamma(8.5)}x^{7.5} - \frac{6}{\Gamma(10)}x^9, y_5(x) = \frac{\Gamma(4)}{\Gamma(10)}x^9 - \frac{6}{\Gamma(11.5)}x^{10.5}$$

and

$$y_6(x) = \frac{\Gamma(4)}{\Gamma(11.5)}x^{10.5} - \frac{6}{\Gamma(13)}x^{12},$$

⋮

and according to equation (8) the approximate solution of equation (16) can be written as

$$\varphi_n(x) = y_1(x) + y_2(x) + y_3(x) + \dots + y_n(x)$$

.

Thus therefor the approximate solution up to seven terms given as

$$\varphi_7(x) = x^3 + 1.10^{-23}x^{7.5} - \frac{1}{79833600}x^{12}$$

Follows table (1) represents a comparison between the approximate solution and the exact solution

Table(1):The absolute error between the exact and approximate solutions of example (1).

x	$y(x)$	$\varphi_7(x)$	$ y(x) - \varphi_7(x) $
0	0	0.0000000000000000	0
0.2	$8 \cdot 10^{-3}$	7.999999999999950e-3	0
0.4	0.064	0.0639999999999790	$2.102 \cdot 10^{-13}$
0.6	0.216	0.215999999972734	$2.727 \cdot 10^{-11}$
0.8	0.512	0.511999999139216	$8.608 \cdot 10^{-10}$
1	1	0.999999987473946	$1.253 \cdot 10^{-8}$
1.2	1.728	1.727999888316442	$1.117 \cdot 10^{-7}$
1.4	2.744	2.743999289848980	$7.102 \cdot 10^{-7}$
1.6	4.096	4.095996474229187	$3.526 \cdot 10^{-6}$
1.8	5.832	5.831985509467422	$1.449 \cdot 10^{-5}$
2	8	7.999948693282024	$5.131 \cdot 10^{-5}$

Example(2):

Consider the following linear integro-differential equation

$$D_*^{0.5} y(x) = \frac{2}{\Gamma(2.5)} x^{1.5} - \frac{\Gamma(5)}{\Gamma(5.5)} x^{4.5} + I^{0.5} (y(x))^2$$

(21)

$$y(0) = 0, x \in [0,1]$$

for comparison purpose, the exact solution of equation (21) is given by $y(x) = x^2$.

According to homotopy perturbation method, we construct the following homotopy:

$$D_*^{0.5} y(x) = p \left(\frac{2}{\Gamma(2.5)} x^{1.5} - \frac{\Gamma(5)}{\Gamma(5.5)} x^{4.5} + I^{0.5} (y(x))^2 \right)$$

Substitution (7) into (21) and then equating the terms with same powers of p yield the following series of linear equations:

$$p^0: D_*^{0.5} y_0 = 0$$

(22)

$$p^1: D_*^{0.5} y_1 = \frac{2}{\Gamma(2.5)} x^{1.5} - \frac{\Gamma(5)}{\Gamma(5.5)} x^{4.5} + I^{0.5} (y_0)^2$$

(23)

$$p^2: D_*^{0.5} y_2 = 2I^{0.5} y_0(x)y_1(x)$$

(24)

$$p^3: D_*^{0.5} y_3 = I^{0.5} (2y_0(x)y_2(x) + (y_1(x))^2)$$

(25)

$$p^4: D_*^{0.5} y_4 = 2I^{0.5} (y_1(x)y_2(x) + (y_0(x)y_3(x)))$$

(27)

$$p^5: D_*^{0.5} y_5 = I^{0.5} (2y_0(x)y_4(x) + 2(y_1(x)y_3(x) + (y_2(x))^2))$$

(28)

$$p^6: D_*^{0.5} y_6 = 0$$

(29)

$$p^7: D_*^{0.5} y_7 = I^{0.5} (2y_1(x)y_5(x) + (y_3(x))^2)$$

(30)

$$p^8: D_*^{0.5} y_8 = 0$$

(31)

$$p^9: D_*^{0.5} y_9 = I^{0.5} 2(y_1(x)y_7(x) + y_3(x)y_5(x))$$

(32)

⋮

Applying the operator $I^{0.5}$ to the above series of linear equations:

$$y_0(x) = 0$$

$$y_1(x) = I^{0.5} I^{0.5} y_0^2(x), \text{ in general we obtain that,}$$

cases may be derived from fractional integro-differential equations:

1. If $\alpha= 0$, then eq. (1) will be reduced to:

$$u(x) = g(x) + I^\beta k(u(x)) \quad (35)$$

which is known as the fractional integral equation.

2. If $\beta= 0$, then eq. (1) will be reduced to

$$D^\alpha u(x) = g(x) + k(u(x)) \quad (36)$$

which is known as fractional order differential equation.

3. If $\alpha= 1, \beta= 1$ then eq. (1) will be reduced to

$$u'(x) = g(x) + \int_a^x k[x, t; u(t)] dt \quad (37)$$

which is known as integro-differential equation.

4. If $\alpha= 0, \beta= 1$ then eq. (1) will be reduced to

$$u(x) = g(x) + \int_a^x k[x, t; u(t)] dt \quad (38)$$

which is the usual volterra integral equation.

5. If $\alpha= 1, \beta= 0$ then eq. (1) will be reduced to

$$u'(x) = g(x) + k(u(x)) \quad (39)$$

which is a first order ODE.

The following examples are designed to illustrate the above concluding remark

Example(3):

Consider the following linear fractional order integral equation of fractional order

$$y(x) = x - \frac{2}{\Gamma(4.5)} x^{3.5} + I^{1.5} (y(x))^2, \quad y(0) = 0 \quad (40)$$

for comparison purpose, the exact solution of equation(40) is given by $y(x) = x$.

According to HPM, we construct the following homotopy :

$$y(x) = p \left(x - \frac{2}{\Gamma(4.5)} x^{3.5} + I^{1.5} (y(x))^2 \right)$$

Substitution of (5) into (40) and then equating the terms with same powers of p yield the following series of linear equations:

$$p^0: y_0(x) = 0$$

(41)

$$p^1: y_1 = x - \frac{2}{\Gamma(4.5)} x^{3.5} + I^{1.5} (y_0(x))^2$$

(42)

$$p^2: y_2 = 2I^{1.5} y_0(x) y_1(x) \quad (43)$$

$$p^3: y_3 = I^{1.5} \left(2y_0(x) y_2(x) + (y_1(x))^2 \right)$$

(44)

$$p^4: y_4 = 2I^{1.5} (y_1(x) y_2(x) + (y_0(x) y_3(x)))$$

(45)

$$p^5: y_5 = I^{1.5} \left(y_0(x) y_4(x) + 2(y_1(x) y_3(x) + (y_2(x))^2) \right)$$

(46)

$$p^6: y_6 = 0$$

(47)

$$p^7: y_7 = I^{1.5} \left(y_1(x) y_5(x) + (y_3(x))^2 \right)$$

(48)

⋮

Hence,

$$y_0(x) = y(0) = 0,$$

$$y_1(x) = x - \frac{2}{\Gamma(4.5)} x^{3.5}$$

$$y_2(x) = 0$$

,

$$y_3(x) = -2.5 \cdot 10^{-2} x^6 + 0.1719 \cdot x^{\frac{7}{2}} + 1.249 \cdot 10^{-3} x^{\frac{17}{2}}$$

$$y_4(x) = 0$$

⋮

and hence the approximate solution of example (2) up to 4-terms may be given as

$$\varphi_4(x) = y_1(x) + y_2(x) + y_3(x) + y_4(x)$$

The comparison between the exact and approximate solution up to 4-terms of example (3) is given in table (3):

Table(3):The absolute error between the exact and approximate solutions of example (3).

x	$y(x)$	$\varphi(x)$	$ y(x) - \varphi(x) $
0.000	0.000	0.0000000000000000	0.000
0.100	0.100	0.099999987540024	1.246e-8
0.200	0.200	0.199999210062249	7.899e-7
0.300	0.300	0.299991141535752	8.858e-6
0.400	0.400	0.399951302058240	4.870e-5
0.500	0.500	0.499819391869387	1.806e-4
0.600	0.600	0.599479087530889	5.209e-4
0.700	0.700	0.698739660702434	1.260e-3
0.800	0.800	0.797323641026904	2.676e-3
0.900	0.900	0.894863519789006	5.136e-3
1.000	1.000	0.990907308909784	9.093e-3

Example(4):

Consider the following linear integro-differential equation

$$D_*^{0.75}y(x) = x^5 - \frac{2}{\Gamma(2.25)}x^{1.25} - (y(x))^2, \quad y(0) = 0 \tag{49}$$

for **comparison** purpose, the exact solution of equation (49) is given by $y(x) = x^2$.

According to homotopy perturbation method, we construct the following homotopy:

$$D_*^{0.75}y(x) = p \left(x^5 + \frac{2}{\Gamma(2.25)}x^{1.25} - x(y(x))^2 \right)$$

Substitution (49) into (7) and then equating the terms with same powers of p yield the following series of linear equations:

$$\begin{aligned} p^0: D_*^{0.75}y_0 &= 0 \\ p^1: D_*^{0.75}y_1 &= x^5 + \frac{2}{\Gamma(2.25)}x^{1.25} - (y_0(x))^2 \\ p^2: D_*^{0.75}y_2 &= -2xy_0(x)y_1(x) \end{aligned} \tag{50}$$

$$p^3: D_*^{0.75}y_3 = -x \left(2y_0(x)y_2(x) + (y_1(x))^2 \right) \tag{51}$$

$$p^4: D_*^{0.75}y_4 = -2x(y_1(x)y_2(x) + (y_0(x))y_3(x)) \tag{52}$$

$$p^5: D_*^{0.75}y_5 = -x \left(y_0(x)y_4(x) + 2(y_1(x)y_3(x) + (y_2(x))^2) \right) \tag{53}$$

$$p^6: D_*^{0.75}y_6 = 0 \tag{54}$$

Applying the operator $I^{0.75}$ to the above series of equations:

$$\begin{aligned} y_0(x) &= 0 \\ y_1(x) &= I^{0.75} \left(x^5 + \frac{2}{\Gamma(2.25)}x^{1.25} - (y_0(x))^2 \right) \\ y_2(x) &= I^{0.75} \left(-2xy_0(x)y_1(x) \right) \\ y_3(x) &= I^{0.75} \left(-x \left(2y_0(x)y_2(x) + (y_1(x))^2 \right) \right) \\ \text{hence} \\ y_1(x) &= x^2 + 0.26489x^{5.75}, \\ y_2(x) &= 0, \\ y_3(x) &= -0.26489x^{5.75} - 9.693 \cdot 10^{-2}x^{9.5} - 1.00321 \cdot 10^{-2}x^{13.25} \end{aligned}$$

and hence the approximate solution of example (2) up to 3-terms may be given as

$$\varphi_3(x) = y_1(x) + y_2(x) + y_3(x)$$

The comparison between the exact and approximate solutions up to 3-term of example (4) is given in table (4):

Table (4):The absolute error between the exact and approximate solutions of example (4).

x	$y(x)$	$\varphi(x)$	$ y(x) - \varphi(x) $
0	0	0.0000000000000000	0.0000000000000000
0.1	0.01	9.999999984673746e-3	1.532625588940295e-11
0.2	0.04	0.0399999889096390	1.109036124741225e-8
0.3	0.09	0.089999479112133	5.208878669227257e-7
0.4	0.16	0.159992038914931	7.961085068536322e-6
0.5	0.25	0.249934517057493	6.548294250710574e-5
0.6	0.36	0.359638304475523	3.616955244774567e-4
0.7	0.49	0.488496457657771	1.503542342229325e-3
0.8	0.64	0.635000214445637	4.999785554362868e-3
0.9	0.81	0.796355168210930	0.013644831789070
1	1	0.969904622919535	0.030095377080465

Conclusion:

In this paper, homotopy perturbation method (HPM) has been successfully applied to integro-differential equations. Two examples are presented to illustrate the accuracy of the present schemes of HPM and the efficiency of the methods.

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تقارب طريقة المقلقة الهوموتوبية المعممة لحل المعادلات التكاملية- التفاضلية ذات الرتب الكسرية

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الخلاصة:

في هذا البحث ، تم تطبيق الطريقة المقلقة الهوموتوبية (HPM) للحصول على الحلول التقريبية للمعادلات التكاملية-التفاضلية. وتم وصف المشتقات الكسرية والتكاملات الكسرية بصيغة كابوتو وريمان ليوفيل على التوالي. فاستطعنا بسهولة الحصول على الحل من خلال متسلسلة منتهية تمثل الحل التقريبي مستخدماً فيها تطبيق طريقة المقلقة الهوموتوبية (HPM). كذلك تم اعطاء نظرية التقارب و تقديرات الخطأ لطريقة (HPM) لحل المعادلات التكاملية-التفاضلية. علاوة على ذلك، لقد بينت النتائج العددية دقة الجانب النظري التحليلي وقوة طريقة (HPM) في حل المعادلات التكاملية-التفاضلية.