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Direct method for Solving Nonlinear Variational Problems by Using Hermite Wavelets

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Abstract:

In this work, we first construct Hermite wavelets on the interval [0,1) with it's product, Operational matrix of integration $2^k M \times 2^k M$ is derived, and used it for solving nonlinear Variational problems with reduced it to a system of algebric equations and aid of direct method. Finally, some examples are given to illustrate the efficiency and performance of presented method.

Key words: Hermite wavelets, Operational Matrix of integration, Matrix of product, Nonlinear Variational problems.

Introduction:

Wavelets theory is relatively new in mathematical researches [1], wavelets constitute a family of functions constructed from dilation and traslation of a single function called the mother wavelet [2]. When the dilation parameter a and the translation parameter b vary continuously we have the following family of constituous wavelets:

$$h_{a,b}(x) = |a|^{\frac{1}{2}} h\left[\frac{x-b}{a}\right] \qquad a, b \in R, \quad a \neq 0 \quad \ldots \ (1)$$

An efficien algorthithm based on Chebyshev wavelets as simplest wavelets approach for numerical solution of many problems such that integrel equations, Variational problems, and diffrentional equations is given in [3-5].

Other wavelets families like Legendre wavelets [6], Sine and Cosine wavelets [7], Harr wavelets [8], are used to solve integral equation first and second kind and are used in solving many kinds problems.

Properties of the Hermite Wavelets:-

Hermite wavelets, $h_{n,m}(t) = h(k, n, m, t)$ involve four arguments, $n = 1, 2, \ldots, 2^k$, k is assumed any positive integer, m is the degree of the Hermite polynomials and t independent variable in [0,1].

$$h_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} H_m^*(2^{k+1}t - 2n + 1) & t \in \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right] \\ 0 & o.w \end{cases} \quad \ldots \ (2)$$

where

$$H_m^* = \frac{1}{2^m m! \sqrt{\pi}} H_m \qquad \ldots \ (3)$$

m=0,1,2,...,M-1 $\quad n = 0, 1, 2, \ldots, 2^k$

in eq(3) the coefficients are used for Orthonormality. Her $H_m(t)$ are the Hermite Polynomials of degree m which respect to the weight function or density function e^{-t^2} and if m=n we can normalize the Hermite polynomial so as to obtain an orthonormals set.

Satisfy the following recurrence formulas

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$H_{m+1}(t) = 2tH_m(t) - 2mH_{m-1}(t)$ … (4)

$H'_m(t) = 2mH_{m-1}(t)$

Starting with $H_0 = 1, H_1 = 2t$

- Orthgonality of Hermite polynomials

$$\int_{-\infty}^{\infty} e^{-t^2} H_m(t)H_n(t)dt = \begin{cases} 0 & m \neq n \\ 2^m m!\sqrt{\pi} & m = n \end{cases} \quad \ldots(5)$$

So that the Hermite polynomials are mutually orthogonal with respect to the weight function or density function e^{-x^2} and if m=n we can normalize the Hermite polynomial so as to obtain an orthonormals set.

- Function approxemation:

Afunction f(t) defined over [0,1) may be expanded as

A function f(t) defined over [0,1) as:

$. f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}h_{n,m}(t)$ …(6)

where $C_{n,m} = \left(f(t), h_{n,m}(t)\right) = \int_0^1 w_n(t)h_{n,m}(t)f(t)dt.$

in which (.,.) denoted the inner product in $L^2_{w_n}[0,1]$.

If the infinite series in equation(6) is truncated, then it can be written as:

$f(t) \cong f_{2^k,M-1} = \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} C_{n,m}h_{n,m}(t) = C^T h(t)$ …(7)

where F and $h(t)$ are $2^k M \times 1$ matrices given by:

$F = [f_{1,0}, f_{1,1}, \ldots, f_{1,M-1}, f_{2,0}, \ldots, f_{2,M-1}, \ldots, f_{2^k,0}, \ldots, f_{2^k,M-1}]$

$h(t) = [h_{1,0}(t), h_{1,1}(t), \ldots, h_{1,M-1}(t), h_{2,0}(t), \ldots, h_{2,M-1}(t), \ldots, h_{2^k,0}(t), \ldots, h_{2^k,M-1}(t)]^T$

$C = [C_{10}, C_{11}, \ldots, C_{1(M-1)}, C_{20}, \ldots, C_{2(M-1)}, \ldots, C_{2^{k-1}}, \ldots, C_{2^{k-1}M-1}]^T$

… (8)

Operation Matrix of Integration:

In this section we will know the operational matrix P of integration. the six basis functions when M=3, k=1 are given by:

$$\left.\begin{aligned} h_{1,0}(t) &= \frac{\sqrt{2}}{\sqrt{\pi}} \\ h_{1,1}(t) &= \frac{\sqrt{2}}{2\sqrt{\pi}}(8t-2) \\ h_{1,2}(t) &= \frac{\sqrt{2}}{8\sqrt{\pi}}(64t^2 - 32\mathrm{t} + 2) \end{aligned}\right\} 0 \leq t < \frac{1}{2} \quad \ldots (9)$$

$$\left.\begin{aligned} h_{2,0}(t) &= \frac{\sqrt{2}}{\sqrt{\pi}} \\ h_{2,1}(t) &= \frac{\sqrt{2}}{2\sqrt{\pi}}(8t-6) \\ h_{2,2}(t) &= \frac{\sqrt{2}}{8\sqrt{\pi}}(64t^2 - 96\mathrm{t} + 34) \end{aligned}\right\} \frac{1}{2} \leq t < 1 \quad \ldots(10)$$

By integrating the above six functions from 0 to t and using eq(6), we obtain:

$\int_0^t h_{1,0}(t)dt = \left(\frac{1}{4}\right)h_{1,0}(t) + \left(\frac{1}{4}\right)h_{1,1}(t) + \left(\frac{1}{2}\right)h_{2,0}(t)$

$\int_0^t h_{1,1}(t)dt = \left(-\frac{1}{16}\right)h_{1,0}(t) + \left(\frac{1}{4}\right)h_{1,2}(t)$

$\int_0^t h_{1,2}(t)dt = \left(-\frac{1}{48}\right)h_{1,0}(t) + \left(-\frac{1}{6}\right)h_{2,0}(t)$

$\int_0^t h_{2,0}(t)dt = \left(\frac{1}{4}\right)h_{2,0}(t) + \left(\frac{1}{4}\right)h_{2,1}(t)$

$\int_0^t h_{2,1}(t)dt = \left(-\frac{1}{16}\right)h_{2,0}(t) + \left(\frac{1}{4}\right)h_{2,2}(t)$

$\int_0^t h_{2,2}(t)dt = \left(-\frac{1}{48}\right)h_{2,0}(t)$

In general Operational matrix of integration of Hermite wavelets becoms:

$$P=\begin{bmatrix} Q & \vdots & S \\ \dots & \vdots & \dots \\ O & \vdots & Q \end{bmatrix}$$ since

$$Q=\frac{1}{2^k}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & 0 \\ \frac{-1}{8} & 0 & \frac{1}{2} & 0 & \dots & 0 \\ \frac{-1}{24} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{(m+1)2^{(m+1)}} & 0 & 0 & 0 & \dots & \frac{1}{2} \end{bmatrix}$$

, $$S=\frac{1}{2^k}\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \frac{-1}{3} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{m+1}}{(m+1)} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and $$O=\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Operation matrix of product and there properties:

Product Operational matrix, which is important for solving our problem and many other problems such that Volterra integral equation , Fredholm integral equation and Integro defferential eqs [5-8] .

Let $h(t)h^T(t)C \simeq \widetilde{C}\, h(t)$ … (11)

where $\widetilde{C}$ is $2^k(M) \times 2^k(M)$ product Operational matrix for k=1,2,… n=1,2,…, 2^k m=0,1,2,…(M-1)

where $h_{i,0}h_{i,j} = h_{i,j}\sqrt{\frac{2}{\pi}}$

$i = 1,2, \dots, n$

$j = 0,1, \dots, m$

$$h_{i,1}h_{i,1} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\,(h_{i,0} + h_{i,2})$$

$h_{i,j}h_{l,f} = 0 \qquad if \qquad i \neq l$

$h_{i,j}h_{l,f} = h_{i,i} \qquad if \qquad j \leq f$

Thus we get

$$h(t)h^T(t) = \begin{bmatrix} h_{1,0}h_{1,0} & h_{1,0}h_{1,1} & h_{1,0}h_{1,2} & \cdots & h_{1,0}h_{2,0} & h_{1,0}h_{2,1} & h_{1,0}h_{2,2} \\ h_{1,1}h_{1,0} & h_{1,1}h_{1,1} & h_{1,1}h_{1,2} & \cdots & h_{1,1}h_{2,0} & h_{1,1}h_{2,1} & h_{1,1}h_{2,2} \\ h_{1,2}h_{1,0} & h_{1,2}h_{1,1} & h_{1,2}h_{1,2} & \cdots & h_{1,2}h_{2,0} & h_{1,2}h_{2,1} & h_{1,2}h_{2,2} \\ & \vdots\ \vdots\ \vdots & & \vdots & & \vdots\ \vdots\ \vdots & \\ h_{2,0}h_{1,0} & h_{2,0}h_{1,1} & h_{2,0}h_{1,2} & \cdots & h_{2,0}h_{2,0} & h_{2,0}h_{2,1} & h_{2,0}h_{2,2} \\ h_{2,1}h_{1,0} & h_{2,1}h_{1,1} & h_{2,1}h_{1,2} & \cdots & h_{2,1}h_{2,0} & h_{2,1}h_{2,1} & h_{2,1}h_{2,2} \\ h_{2,2}h_{1,0} & h_{2,2}h_{1,1} & h_{2,2}h_{1,2} & \cdots & h_{2,2}h_{2,0} & h_{2,2}h_{2,1} & h_{2,2}h_{2,2} \end{bmatrix}$$

Expanding each product by wavelet basis we have matrix D

$$D = \sqrt{\frac{2}{\pi}}\begin{bmatrix} h_{1,0} & h_{1,1} & h_{1,2} & \cdots & 0 & 0 & 0 \\ h_{1,1} & \frac{1}{2}(h_{1,0}+h_{1,2}) & h_{1,1} & \cdots & 0 & 0 & 0 \\ h_{1,2} & h_{1,1} & h_{1,0} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_{2,0} & h_{2,1} & h_{2,2} \\ 0 & 0 & 0 & \cdots & h_{2,1} & \frac{1}{2}(h_{2,0}+h_{2,2}) & h_{2,1} \\ 0 & 0 & 0 & \cdots & h_{2,2} & h_{2,1} & h_{2,0} \end{bmatrix}$$

By using the vector C, the $\tilde{C}$ is

$$\tilde{C} = \sqrt{\frac{2}{\pi}}\begin{bmatrix} \tilde{C}_1 & 0 \\ 0 & \tilde{C}_2 \end{bmatrix} \qquad \tilde{C}_i = \sqrt{\frac{2}{\pi}}\begin{bmatrix} C_{i,0} & C_{i,1} & C_{i,2} \\ C_{i,1} & \frac{1}{2}(C_{i,0}+C_{i,2}) & C_{i,1} \\ C_{i,2} & C_{i,1} & C_{i,0} \end{bmatrix}$$

and integrating matrix D from 0 to 1 we obtained new matrix denoted by matrix R

where

$$R = \int_0^1 h(t)h^T(t)dt \qquad (12)$$

if we take k=0, M=3 we obtain $R_{6\times 6}$

$$R = \frac{1}{\pi}\begin{bmatrix} 1 & 0 & \frac{-1}{12} & \vdots & 0 & 0 & 0 \\ 0 & \frac{11}{24} & 0 & \vdots & 0 & 0 & 0 \\ \frac{-1}{12} & 0 & 1 & & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & & 1 & 0 & \frac{-1}{12} \\ 0 & 0 & 0 & \vdots & 0 & \frac{11}{24} & 0 \\ 0 & 0 & 0 & & \frac{-1}{12} & 0 & 1 \end{bmatrix}$$

Problem Statement:

In large number of problems a rising in analysis, mechanics and geometry it is necessary to determine the maximal and minimal of acertain functional. Such problems are called Variational problems .

Consider the following Variational problem

$$J[x(t)] = \int_0^1 F(t, x(t), \dot{x}(t), \dots, x^n(t)dt \qquad \dots (13)$$

with $x(0) = a_0$, $\dot{x}(0) = a_1, \dots\dots, x^{n-1}(0) = a_{n-1}$

$x(1)$, $\dot{x}(1), \dots\dots, x^{n-1}(1)$ unspecified.

This problem with moving boundary conditions, and find the extremum of eq(13), above problem is reduced by using our method in to a set of algebric equations by following procetuer.

$x^n(t) = C^T h_{n,m}(t)$
$x^{n-1}(t) = C^T P h_{n,m}(t) + x^{n-1}(0)$
$\vdots$
$\vdots$
$x(t) = C^T P^n h_{n,m}(t) + \frac{x^{n-1}(0)t^{n-1}}{n-1} + \frac{x^{n-2}(0)t^{n-2}}{n-2} + \cdots + x(0)$

Numerical examples:

Example(1)

$J[x(t)] = \int_0^1 \dot{x}^2(t) + t\dot{x}(t)dt$ … (14)

with boundary conditions:

$x(0) = 0,\ \ x(1)$ unspecified (15)

We solve this problem by using Hermite wavelets. First we assume

$\dot{x}(t) = C^T h_{n,m}(t)$ … (16)

$t = d^T h_{n,m}(t)$ …(17)

Substituting (16),(17) in (14) and we used matrix R then eq(14) becomes

$$J = C^T RC + d^T RC$$

According to transversality condition, we have:

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{t=1} = 0 \rightarrow \dot{x}(1) = -0.5$$

By using (16)

$$\dot{x}(1) = C^T h_{n,m}(1) = -0.5$$

Let $J^* = J + \lambda\left(C^T h_{n,m}(1) + 0.5\right)$

$\frac{\partial J^*}{\partial C^T} = 2RC + R^T d = 0$

$\frac{\partial J^*}{\partial \lambda} = C^T h_{n,m}(1) + 0.5 = 0$

(18)

For M=3 and k=1, we obtain:

$d = [0.3133 \quad 0.3133 \quad 0 \quad 0.9400 \quad 0.3133 \quad 0]^T$

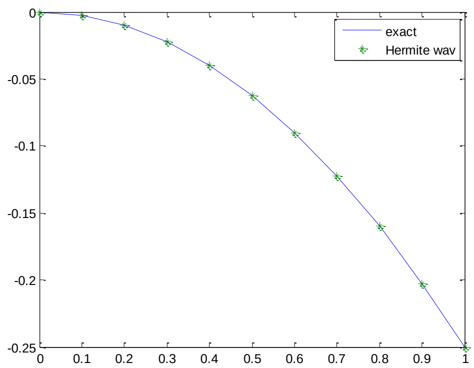
Thus

$C = [-0.15666427 \quad -0.15666427 \quad 0 \quad -0.46999280 \quad -0.15666427 \quad 0]^T$

The exact solution $\dot{x}(t) = -0.5t$ And $x(t) = -0.25t^2$. Table 1 shows the numerical results for this example with k=1, M=3 with error $=10^{-8}$ are compared with exact solution graphically in fig1.

Table 1:some numerical results for example 1

x	Exact solution	Approximat solution $h_{n,m}(x)$	$AbsouteError$ $\lvert exact - h_{nm}\rvert$
0	0.00000000	0.00000000	0.00000000
0.1	-0.00250000	-0.00250000	0.00000000
0.2	-0.01000000	-0.01000000	0.00000000
0.3	-0.02250000	-0.02250000	0.00000000
0.4	-0.04000000	-0.04000000	0.00000000
0.5	-0.06250000	-0.06250000	0.00000000
0.6	-0.09000000	-0.09000000	0.00000000
0.7	-0.12250000	-0.12250000	0.00000000
0.8	-0.16000000	-0.16000000	0.00000000
0.9	-0.20250000	-0.20250000	0.00000000
1	-0.25000000	-0.25000000	0.00000000
$M.S.E = 10^{-8}$ $L.S.E = 10^{-8}$			



Fig(1) error =10^-8 are compared with exact solution

Example(2):

$J[x(t)] = \int_0^1 \frac{\ddot{x}^2(t)}{2} + (4-4t)\dot{x}(t)dt$ … (19)

Subject to

$x(0) = 0\ ,\ \ \dot{x}(0) = 0,\ \ x(1),\ \dot{x}(1)$ unspecified … (20)

For solving this problem by using the Hermite wavelets, let:

$\ddot{x}(t) = C^T h_{n,m}(t)$ …(21)

From Euler Lagrang equation we have:

$$\frac{\partial F}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial F}{\partial \ddot{x}}\bigg|_{t=1} \to (4-4t) - \dddot{x}(t)|_{t=1} = 0 \to \quad \dddot{x}(1) = 0, \quad \frac{\partial F}{\partial \ddot{x}}\bigg|_{t=1} \to \ddot{x}(1) = 0 \qquad (22)$$

$$\dddot{x}(t) = C^T P h_{n,m}(t) + \ddot{x}(0) \qquad (23)$$

$$\ddot{x}(1) = C^T \int_0^1 h_{n,m}(\tau)d\tau + \ddot{x}(0)$$

Let

$$\int_0^1 h_{n,m}(\tau)d\tau = Q = \begin{bmatrix} 0.5 & 0 & 0 & & 0.5 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ -0.0417 & 0 & 0 & & -0.0417 & 0 & 0 \\ \cdots & & & \cdots & \cdots & & \\ 0.5 & 0 & 0 & & 0.5 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ -0.0417 & 0 & 0 & & -0.0417 & 0 & 0 \end{bmatrix}$$

$$\ddot{x}(0) = -C^{\tau} Q = C^T F h(t) \qquad (24)$$

Then

$$F = \begin{bmatrix} -0.5 & 0 & 0 & & -0.5 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0.0417 & 0 & 0 & & 0.0417 & 0 & 0 \\ \cdots & & & \cdots & \cdots & & \\ -0.5 & 0 & 0 & & -0.5 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0.0417 & 0 & 0 & & 0.0417 & 0 & 0 \end{bmatrix}$$

Also

$$\dot{x}(t) = C^T (P+F) P h_{n,m}(t) \qquad (25)$$

Let

$$(4-4t) = d^T$$

Similarly example(1) is written J

For M=3 and k=1 we obtain:

$d = \left[3 \ \ \frac{\sqrt{\pi}}{\sqrt{2}} \ \ 0 \ \ \frac{\sqrt{\pi}}{\sqrt{2}} \ \ \frac{\sqrt{\pi}}{\sqrt{2}} \ \ 0\right]^T$, and

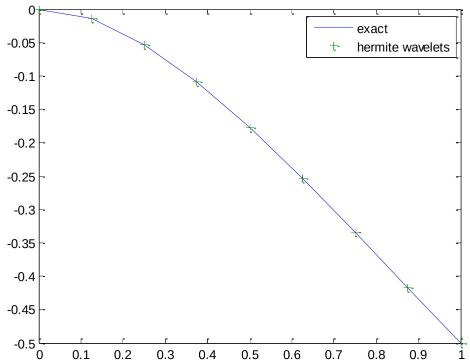
$C = [3.75994242 \ \ -1.25331413 \ \ 0 \ \ 1.25331413 \ \ -1.25331413 \ \ 0\,]^T$

Approximate solution will be achieved $x(t) = -\frac{t^4}{6} + \left(\frac{2}{3}\right)t^3 - t^2$ which is the exact solution.

Table (2), the values of x(t) using the Hermite wavelets, Chebyshev wavelets and Sine-Cosine , are compared with exact solution. The Hermite wavelets with error $=10^{-8}$ graphically in fig(2).

Table 2:some numerical results for example 2

x	Sine-cosin wavelets	Chebyshev wavelets	Exact solution	Approximat solution $h_{n,m}(x)$	$AbsouteError$ $\lvert exact - h_{nm}\rvert$
0	-0.0253	0.0020	0.00000000	0.00000000	0.00000000
0.125	-0.0155	-0.0161	-0.01436068	-0.01436068	0.00000000
0.25	-0.0119	-0.0523	-0.05273438	-0.05273438	0.00000000
0.375	-0.1069	-0.1066	-0.10876465	-0.10876465	0.00000000
0.5	-0.2513	-0.1765	-0.17708333	-0.17708333	0.00000000
0.625	-0.2494	-0.2537	-0.25329590	-0.25329590	0.00000000
0.750	-0.4119	-0.3335	-0.33398438	-0.33398438	0.00000000
0.875	-0.4116	-0.4158	-0.41670736	-0.41670736	0.00000000
1	-0.4946	-0.5001	-0.50000000	-0.50000000	0.00000000
$M.S.E = 10^{-8}$ $L.S.E = 10^{-8}$					



Fig(2) The Hermite wavelets with error $=10^{-8}$

Conclusion:

In this work, nonlinear Variational problems have solved by using Hermite wavelets in direct method. Comparison of the approximate solutions and the exact solution shows that the proposed method is very efficient tool.

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الطريقة المباشرة لحل مسائل التغاير الغير خطية باستخدام هرمت الموجية

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الخلاصة:

في هذا العمل، تم اولا بناء دوال هرمت الموجية في الفترة (1 ، 0] مع ضرب هذه الدوال ومصفوفة العمليات التي تكون ابعادها $2^kM\times2^kM$ ، وتم استخدام هذه المصفوفات في حل مسائل التغاير الغير خطية بالطريقة المباشرة مع تحويلها لنظام من المعادلات الجبرية الخطية. وفي النهاية بعض الأمثلة المعطاة التي تم حلها بالطريقة المعروضة والتي توصلنا من خلالها بانها الطريقة الأكفاء .

الكلمات المفتاحية:هرمت الموجية ، مصفوفة العمليات للتكامل، مصفوفة الضرب، مسائل التغاير الغير خطية.