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Baghdad Science Journal Vol.12(3)2015

A New Three Step Iterative Method without Second Derivative for Solving Nonlinear Equations

Raghad I. Sabri

Applied Science Department , University of Technology.

Received 30, April, 2014
Accepted 1, June, 2014

Abstract:

In this paper , an efficient new procedure is proposed to modify third –order iterative method obtained by Rostom and Fuad [Saeed. R. K. and Khthr. F.W. New third –order iterative method for solving nonlinear equations. J. Appl. Sci .7(2011): 916-921] , using three steps based on Newton equation , finite difference method and linear interpolation. Analysis of convergence is given to show the efficiency and the performance of the new method for solving nonlinear equations. The efficiency of the new method is demonstrated by numerical examples.

Key words: nonlinear equations, Newton method, iterative method, three- step method, order of convergence.

Introduction:

Iterative methods play a crucial role in approximating the solution of nonlinear equation $(x) = 0$.
One of the most famous iterative methods is the classical Newton method, which is a well known basic method and possesses quadratic order of convergence.
Several different methods were developed by number of researchers for the computation the root of the nonlinear equations , these numerical methods have been suggested using Taylor series , Homotopy analysis method ,Hermit interpolation method and other techniques , see [1-11] .

In this paper , a new three-step iterative method was suggested without second derivative by considering a suitable approximate of $f''(x_n)$ [12] using Taylor series to find a simple root of nonlinear equation. The new method is sixth-order convergent was proved. Several examples are presented showing the accuracy of the proposed method .

Construction of new method and convergence criteria:

To illustrate the procedure of constructing our new iterative method , the following iterative method was considered:

Algorithm 1.1: For a given $x_\circ$, compute the approximate solution $x_{n+1}$by the iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which is known as Newton method and has quadratic convergence .
Rostom and Fuad [13] have obtained the following third order iterative method.

Algorithm 1.2: For a given $x_\circ$, compute the approximate solution $x_{n+1}$by the iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)} + \frac{f^2(x_n)f''(x_n)}{2f'^2(x_n)}$$
$$- \frac{\left(2\, f(x_n)\, f'^2(x_n) + 2f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right) f'^2(x_n) - f^2(x_n)f''(x_n)\right)^2 f''(x_n)}{8\, f'^7(x_n)}$$

Use Taylor's series to approximate $f''(x_n)$ in (Algorithm 1.2) with

$$f''(x_n) \approx \frac{2f(y_n)f'(x_n)^2}{f(x_n)^2} \quad \dots \ (1)$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $f'(x_n) \neq 0 \dots (2)$

Substituting (1) into (Algorithm 1.2) , then the following iterative scheme was constructed:

Algorithm 1.3: For a given $x_\circ$, compute the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$
$$x_{n+1} = x_n - [\frac{f(x_n)+f(y_n)}{f'(x_n)}] \quad \dots \quad (3)$$

To obtain a new three-step method of order six , a Newton method was used to rise the order of convergence , we have

Algorithm 1.4: For a given $x_\circ$, compute the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$
$$z_n = x_n - [\frac{f(x_n)+f(y_n)}{f'(x_n)}] \quad \dots (4)$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \quad \dots (5)$$

Using the linear interpolation on two points $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$

$$f'(x_n) \approx \frac{x - x_n}{y_n - x_n} f'(y_n) + \frac{x - y_n}{x_n - y_n} f'(x_n) \quad \dots \ (6)$$

to obtain the approximation

$$f'(z_n) \approx \frac{z_n - x_n}{y_n - x_n} f'(y_n) + \frac{z_n - y_n}{x_n - y_n} f'(x_n) \quad \dots (7)$$

$$f'(z_n) = \frac{f'(y_n)[f(x_n)+f(y_n)] - f'(x_n)f(y_n)}{f(x_n)} \quad \dots (8)$$

Combining (5) and (8) , the following new iterative method free from second derivative was suggested for solving nonlinear equation.

Algorithm 1.5: For a given $x_\circ$, compute the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$
$$z_n = x_n - [\frac{f(x_n)+f(y_n)}{f'(x_n)}].$$
$$x_{n+1} = z_n - \frac{f(z_n)f(x_n)}{f'(y_n)[f(x_n)+f(y_n)] - f'(x_n)f(y_n)} \quad \dots (9)$$

For the new method defined by (Algorithm 1.5), We have the following analysis of convergence .

Theorem 1: let f be the scalar function sufficiently smooth in the real open domain D, and r is a simple zero of f.If $x_\circ$ is sufficiently close to r , then the method defined by (Algorithm 1.5) has six- order convergence , satisfied the following error equation

$$e_{n+1} = [c_2^3\,(188 + 8c_2^2 - 194\, c_3) + c_2^2 c_4(62c_2 - 62) + c_2^4(4 + 8c_2) + c_5(13c_2 - 13c_3)]e_n^6 + O(e_n^7)$$

where $c_j = \frac{f^{(j)}(r)}{j!\, f'(r)}$, $j = 2,3,\dots$

proof:

let $e_n = x_n - r$ be the error at nth iteration and r is a simple zero of f . Assume that $d_n = y_n - r$ where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

Using Taylor's expansion and taking into account $(r) = 0$, we have

$f(x_n) = f'(r)e_n + \frac{1}{2!}f''(r)(e_n)^2 + \frac{1}{3!}f'''(r)(e_n)^3 + \frac{1}{4!}f^{(iv)}(r)(e_n)^4 + \frac{1}{5!}f^{(v)}(r)(e_n)^5 + \frac{1}{6!}f^{vi)}(r)(e_n)^6 + o({e_n}^7)$ ... (10)

$f(x_n) = f'(r)\{e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + o({e_n}^7)\}$. ... (11)

$f'(x_n) = f'(r)\{1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + o(e_n^7)\}$ ..(12)

Dividing (11) by (12) gives

$\frac{f(x_n)}{f'(x_n)} = \{e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + (8c_2^4 + 10c_2c_4 + 6c_3^2 - 4c_5 - 20c_3c_2^2)e_n^5 + (13c_2c_5 + 52c_3c_2^3 + 17c_4c_3 - 28c_4c_2^2 - 16c_2^5)e_n^6 + o(e_n^7)\}$... (13)

Thus ,we have

$d_n = e_n - \frac{f(x_n)}{f'(x_n)}$.

$= \{r + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + (4c_5 - 10c_2c_4 - 6c_3^2 - 8c_2^4 + 20c_3c_2^2)e_n^5 + (16c_2^5 - 17c_3c_4 + 28c_2^2c_4 + 33c_2c_3^2 - 52c_2^3c_3 - 13c_2c_5 + 5c_6)e_n^6 + o(e_n^7)\}$... (14)

Furthermore , Taylor expansion of $f(y_n)$ about r is given as follow:

$f(y_n) = f(r) + f'(r)d_n + \frac{1}{2!}f''(r)(d_n)^2 + \frac{1}{3!}f'''(r)(d_n)^3 + \frac{1}{4!}f^{(iv)}(r)(d_n)^4 + \frac{1}{5!}f^{(v)}(r)(d_n)^5 + \ldots\}$ ...(15)

Then we have

$f(y_n) = f'(r)\{c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + (4c_5 - 12c_2^4 - 10c_2c_4 + 24c_2^2c_3 - 6c_3^2)e_n^5 + (5c_6 - 73c_2^3c_3 + 34c_2^2c_4 + 37c_3^2c_2 + 28c_2^5 - 17c_3c_4 - 13c_2c_5)e_n^6 + o(e_n^7)\}$ ... (16)

And

$f'(y_n) = f'(r)\{1 + 2c_2^2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + (6c_2c_4 - 11c_3c_2^2 + 8c_2^4)e_n^4 + (8c_2c_3 + 28c_3c_2^3 - 20c_4c_2^2 - 16c_2^5)e_n^5 + (32c_2^6 + 56c_4c_2^3 - 26c_5c_2^2 - 16c_2c_3c_4 + 72c_3^3 - 64c_3c_2^4)e_n^6 + o(e_n^7)\}$ ... (17)

Substituting (11),(16),(17) into z_n in (9) , to get

$u_n = e_n - \frac{f(x_n)+f(y_n)}{f'(x_n)}$.

$=\{r + 2c_2^2e_n^3 - (9c_2^3 - 7c_2c_3)e_n^4 - (44c_2^2c_3 - 30c_2^4 - 10c_2c_4 - 6c_3^2)e_n^5 - (62c_2^2c_4 - 188c_2^3c_3 + 70c_3^2c_2 + 88c_2^5 - 17c_3c_4 - 13c_2c_5)e_n^6 + o(e_n^7)\}$ ... (18)

where

$u_n = z_n - r$.

Now , expand $f(z_n)$ about r to obtain

$f(z_n) = f'(r)\{2c_2^2e_n^3 - (9c_2^3 - 7c_2c_3)e_n^4 - (44c_2^2c_3 - 30c_2^4 - 10c_2c_4 - 6c_3^2)e_n^5 - (62c_2^2c_4 - 188c_2^3c_3 + 4c_2^4 + 70c_3^2c_2 + 88c_2^5 - 17c_3c_4 - 13c_2c_5)e_n^6 + o(e_n^7)\}$ ...(19)

According to (11),(12),(16),(17) and (19) we have

$\frac{f(z_n)f(x_n)}{f'(y_n)[f(x_n)+f(y_n)]-f'(x_n)f(y_n)} = \{2c_2^2e_n^3 + (7c_2c_3 - 9c_2^3)e_n^3 + (30c_2^4 - 44c_2^2c_3 + 10c_2c_4 + 6c_3^2)e_n^5 + (194c_2^3c_3 - 62c_2^2c_4 - 70c_2c_3^2 - 96c_2^5 + 17c_3c_4 + 13c_3c_5 - 4c_2^4)e_n^6 + o(e_n^7)\}$ .... (20)

Now , substituting (18) and (20) into (9) leads to

$x_{n+1} = r + (188c_2^3 - 62c_2^2c_4 + 8c_2^5 - 194c_2^3c_3 + 13c_2c_5 - 13c_3c_5 + 62c_2^3c_4 + 4c_2^4)e_n^6 + o(e_n^7)$ ... (21)

Using the relation $e_n = x_n$- r in (21) , we get

$$e_{n+1} = [c_2^3\,(188 + 8c_2^2 - 194\,c_3) + c_2^2 c_4(62c_2 - 62) + c_2^4(4 + 8c_2) + c_5(13c_2 - 13c_3)]\,e_n^6 + o(e_n^7) \quad \ldots (22)$$

From eq.(22) ,we conclude that the new iterative method defined by (9) in (Algorithm 1.5) has a sixth-order convergence.

2- Numerical implementations and Discussion:

In this section, some test problems are presented to compare the efficiency of the proposed method (TSM) .
The following table (1) given the test functions and their solution r , found up to the 10th decimal places.

Tabel(1)

Functions	Solution
$f_1(x) = x^2 - \exp(x) - 3x + 2$	r =0.257530285
$f_2(x) = x^3 - 4x^2 - 10$	r = 1.365230013
$f_3(x) = (x+2)\exp(x) - 1$	r =-0.442854401
$f_4(x) = (x-1)^3 - 1$	r = 2
$f_5(x) = x\exp(x^2) - \sin^2(x) + 3\cos(x) + 5$	r = -1.207647827
$f_6(x) = \exp(x^2 + 7x - 30) - 1$	r = 3

Let $|f(x_{n+1})| < \varepsilon$ be the stopping criteria for computer programs , where $\varepsilon = 10^{-15}$. All numerical computations are worked out in matlab 7.8 . Table(2) represents comparison between the new method defined by (Algorithm 1.5) and other well known methods that are presented in (Homeier (HM) [14], Chun (CM) [15] , Rostom (N1)[13]) and classical Newton method(NM).

Tabel(2) Numerical examples for different functions

function with starting point	iteration	solution
$f_1(x), x_\circ = 2$		
Our method(Algorithm1.5)	3	0.257530285
NM	6	0.257530285
HM	5	0.257530285
CM	4	0.257530285
N1	4	0.257530285
$f_2(x), x_\circ = 3$		
Our method(Algorithm1.5)	3	1.365230013
NM	6	1.365230013
HM	4	1.365230013
CM	4	1.365230013
N1	4	1.365230013
$f_3(x), x_\circ = 2$		
Our method(Algorithm1.5)	3	-0.442854401
NM	8	-0.442854401
HM	5	-0.442854401
CM	6	-0.442854401
N1	6	-0.442854401
$f_4(x), x_\circ = 3.5$		
Our method(Algorithm1.5)	6	2
NM	6	2
HM	4	2
CM	5	2
N1	5	2
$f_5(x), x_\circ = -2$		
Our method(Algorithm1.5)	3	-1.207647827
NM	9	-1.207647827
HM	5	-1.207647827
CM	5	-1.207647827
N1	6	-1.207647827
$f_6(x), x_\circ = 3.5$		
Our method(Algorithm1.5)	4	3
NM	13	3
HM	7	3
CM	7	3
N1	9	3

Discussion:

In this paper, a new three –step iterative method without second derivative was suggested for finding roots of nonlinear equations. Analysis of the convergence is performed to show that the convergence order of the new method is six. Numerical results indicate that the new method takes a less number of iteration than the other methods.

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طريقة تكرارية جديدة ذات ثلاث خطوات لاتحتوي على مشتقة ثانية لحل المعادلات اللاخطية

رغد ابراهيم صبري
الجامعة التكنولوجية، قسم العلوم التطبيقية.

الخلاصة:
في هذا البحث ، تم عرض خوارزمية(طريقة) جديدة كفوءة لتطوير الطريقة التكرارية من الرتبة الثالثة المقدمة من قبل الباحثين رستم وفؤاد ، باستخدام ثلاث خطوات مستندة على معادلة نيوتن ، طريقة الفروق المحددة (المنتهية) والاندراج الخطي . تم توضيح تحليل الاقتراب لبيان أداء وكفاءة الطريقة الجديدة لحل المعادلات اللاخطية. كفاءة الطريقة الجديدة تم توضيحها عن طريق الامثلة العددية.

الكلمات المفتاحية: المعادلات اللاخطية ، طريقة نيوتن، الطريقة التكرارية، الطريقة ذات ثلاث خطوات ، رتبة التقارب .