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Bounded Solutions of the Second Order Differential Equation $\ddot{x} + f(x)\dot{x} + g(x) = u(t)$

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Abstract:

In this paper we prove the boundedness of the solutions and their derivatives of the second order ordinary differential equation $\ddot{x} + f(x)\dot{x} + g(x) = u(t)$, under certain conditions on f,g and u. Our results are generalization of those given in [1].

Key words: Boundedness of solutions, Second order ordinary differential equations, Systems of differential equations, limit cycles.

Introduction:

Boundedness of solutions of differential equations is one of the mathematical properties that is needed in applications such as population growth, trophic function, capacity, voltage, differential and integral equations and chemical reactions etc. [2], [3]. The boundedness of solutions can be determined by finding conditions on the functions f, g and u, or by studying the stability of limit cycles, or determining the bounded region in the plane in which if the solution inters the region then it stays their for all t .

The general formula of the second order of differential equation is

$F(t, x, \dot{x}, \ddot{x})=0$ … (1)

and we interested in the boundedness of the solutions of equation (1).

Recall that a solution x(t) is called bounded on $[0, \infty)$ if there exists a positive constant M such that $|x(t)| \leq M$ for all $t \in [0, \infty)$ [4] .

A special case of equation (1) is the differential equation.

$\ddot{x} + f(x)\dot{x} + g(x) = u(t)$ … (2)

Equation (2) has a solution if f, g and u are continuous on an interval [a,b].

Also we know that the solution of equation (2) is at least $C^2[a,b]$ function. If the interval of definition of the solution is compact (closed and bounded) then the solution is bounded but we are interested in $[a, \infty)$, and here certainly we must return back to the classification of differential equations in such away that every equation has a special properties that make the solution bounded.

Some criterias for boundedness

1- For differential equations with constant coefficients

$\sum_{i=0}^{n} a_i y^{(i)} = 0$ … (3)

the general solution for (3) is bounded solution if the roots of the characteristic equation

$\sum_{i=0}^{n} a_i \lambda^{(i)} = 0$ … (4)

are as follows :

(a) The real parts of the repeated eigenvalues are negative.

(b) The real parts of the simple roots are not positive.

2- Using the concept of stability of limit cycles for the boundedness solutions [5].

3- Determining the regions (called global attractive regions) in which if the solution inters it then it stays there for all t [6].

4- Finding the condition(s) on the coefficients and the nonhomogeneous term such that the solutions are bounded.

5- A number of papers on the boundedness of solutions of second order were offered such as [5], [6], [7], [8], [9], [10], [11], [12].

Main Results:

In this section we impose conditions on f, g and u such that the solution x(t) and its derivative $\dot{x}(t)$ are bounded .

Theorem:- Consider the differential equation

$\ddot{x} + f(x)\dot{x} + g(x) = u(t) \quad \ldots (5)$

where u(t) is a bounded function. Assume that the functions f(x) and g(x) are continuous and satisfy the following conditions:

(a) f(x) and g(x) are positive functions for all x.

(b) $\int_0^{\infty} |u(t)|\, dt < \infty$.

(c) $\int_0^{\pm\infty} g(x)\, dx = \infty$.

then the solution x(t) of equation (5) and it is derivative $\dot{x}(t)$ are bounded .

Proof:-

By standard existence theorem [4] equation (5) has at least one solution on [0,∞).

In order to get the result we multiply both sides of equation (5) by $\dot{x}$

$\ddot{x}\,\dot{x} + f(x)\dot{x}^2 + g(x)\dot{x} = u(t)\,\dot{x}$

Now integration both sides from 0 to t gives.

$\int_0^t \ddot{x}(s)\,\dot{x}(s)ds + \int_0^t f(x(s))\dot{x}^2(s)ds + \int_0^t g(x(s))\dot{x}(s)ds = \int_o^t u(s)\dot{x}(s)ds \ldots(6)$

Note that

$\int_0^t \ddot{x}(s)\,\dot{x}(s)ds = \frac{\dot{x}^2(t)}{2} - \frac{\dot{x}^2(0)}{2}$,

in third term of left hand sides of equation (6) if we let u= x(s) then du=$\dot{x}(s)$ds, s=0 and s=t implie u=x(0) and u=x(t) respectively, therefore

$$\int_o^t g(x(s))\dot{x}(s)ds = \int_{x(0)}^{x(t)} g(u)du,$$

Substituting the last two equalities in (6) to get

$\frac{\dot{x}^2(t)}{2} - \frac{\dot{x}^2(0)}{2} + \int_0^t f(x(s))\dot{x}^2(s)ds + \int_{x(0)}^{x(t)} g(u)du = \int_o^t u(s)\dot{x}(s)ds$

$\frac{\dot{x}^2(t)}{2} + \int_0^t f(x(s))\dot{x}^2(s)ds + \int_{x(0)}^{x(t)} g(u)du = \frac{\dot{x}^2(0)}{2} + \int_o^t u(s)\dot{x}(s)ds \ldots(7)$

it is clear that

$\frac{\dot{x}^2(t)}{2} + \int_0^t f(x(s))\dot{x}^2(s)ds + \int_{x(0)}^{x(t)} g(u)du \le \frac{\dot{x}^2(0)}{2} + \int_0^t |u(s)\dot{x}(s)|\, ds \ldots(8)$

Now if x(t) is unbounded function, then for large values |x| we have that the left hand side of (8) is positive since g and f are positive functions.

Now, if we applied the mean value theorem to the second term in the right hand side of (8) we get

$\frac{\dot{x}^2(t)}{2} + \int_0^t f(x(s))\dot{x}^2(s)ds + \int_{x(0)}^{x(t)} g(u)du \le \frac{\dot{x}^2(0)}{2} + |\dot{x}(\bar{t})| \int_0^{\infty} |u(t)|\, dt$

let

$\int_0^{\infty} |u(t)|\, dt$ =k, where $\bar{t} \in (0, t)$,

then,

$\frac{\dot{x}^2(t)}{2} + \int_0^t f(x(s))\dot{x}^2(s)ds + \int_{x(0)}^{x(t)} g(u)du \le \frac{\dot{x}^2(0)}{2} + k\,|\dot{x}(\bar{t})| \ldots (9)$

From our hypothesis the left hand side of equation (9) goes to ∞ as x(t) approaches ∞, while the right hand side is bounded , which is impossible .

Now dividing equation (9) by $\dot{x}(t)$ (assume that $\dot{x}(t) > 0$ a similar argument works for $\dot{x}(t) < 0$ except the inequality is reversed) in order to obtain

$$\frac{\dot{x}(t)}{2}+\frac{1}{\dot{x}(t)}\left[\int_0^t f(x(s))\dot{x}^2(s)ds + \int_{x(0)}^{x(t)} g(u)du\right] \le \frac{1}{\dot{x}(t)}\left[\frac{\dot{x}^2(0)}{2} + k\,|\dot{x}(\bar{t})|\right] \quad (10)$$

Similarly, if $|\dot{x}(t)|$ goes to ∞, then the left hand side is unbounded, while the right hand side bounded which is impossible also. From these conclutions, the solution $x(t)$ and its derivative $\dot{x}(t)$ must be bounded. The solution can be extended on all of $[0, \infty)$ by using a standard argument see [14].

Conclusions:

(1) In this paper we replace the two conditions (b) and (c) in [1] by condition (b) in our theorem which is weaker than them.

(2) The derivative $\dot{x}(t)$ is bounded function also, while this property was not discussed in [1].

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محدودية حلول المعادلة التفاضلية من الرتبة الثانية $\ddot{x} + f(x)\dot{x} + g(x) = u(t)$

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الخلاصة:
تم في هذا البحث إثبات أن حلول المعادلة التفاضلية غير الخطية من المرتبة الثانية :- $\ddot{x} + f(x)\dot{x} + g(x) = u(t)$ مع مشتقاتها محددة تحت شروط معينة على الدوال f و g و u النتائج المستحصلة هي تعميم للنتائج في المصدر الاول.

الكلمات المفتاحية:حلول تفاضلية، معادلات تفاضلية، انظمة معادلات متكاملة، دراسات غائبة.