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Some Results on Pure Submodules Relative to Submodule

Shireen O. Dakheel ***Noor R. Adeb***

Mathematics Department, College of Science for Women, University of Baghdad

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Abstract:

Let R be a commutative ring with identity 1 and M be a unitary left **R**-module. A submodule N of an R-module M is said to be pure relative to submodule T of M (Simply T-pure) if for each ideal A of R, $\mathbf{N \cap AM = AN + T \cap (N \cap AM)}$. In this paper, the properties of the following concepts were studied: Pure essential submodules relative to submodule T of M (Simply T-pure essential),Pure closed submodules relative to submodule T of M (Simply T-pure closed) and relative pure complement submodule relative to submodule T of M (Simply T-pure complement) and T-purely extending. We prove that; Let M be a T-purely extending module and let N be a T-pure submodule of M. If M has the T-PIP, then N is T-purely extending.

Key words: T-pure submodule, T-pure essential submodule and T-pure closed submodule.

Introduction:

In this paper we assume that **R** is commutative ring with identity and all **R**-modules are unitary left **R**-module. A submodule **N** of an **R**-module **M** is called pure submodule if for every finitely generated ideal **I** of **R**, $\mathbf{N \cap IM = IN}$ [1]. A submodule **K** of an **R**-module **M** is said to be **P**-essential if for every pure submodule **L** of M, $\mathbf{K \cap L = 0}$ implies**L**={0} [2]. Following [3], A submodule **N** of an **R**-module **M** is called pure relative to submodule **T** of **M** (Simply **T**-pure) if $\mathbf{N \cap AM = AN + T \cap (N \cap AM)}$ for each ideal **A** of **R**. It is clear that every pure submodule is **T**-pure.

In this paper we introduce the concepts of **T**-pure essential submodules,**T**-pure closed subomdules and relative **T**-pure complement submodules and we prove that; Let A and C be submodules of an R-module M,then there exist T-p-closed submodule H in M which is T-pure such that C is T-pure closed in H.

In [4] an **R**-module **M** is called purely extending module if every submodule is essential in pure submodule. We introduce the concept of **T**-purelyextending module. We prove that; Let M be an R-module, then M is T-purely extending if and only if every T-p-closed submodule of M is T-direct summand of M.

1-Main results:

The notion of purity for abelian group was generalized to modules over arbitrary rings. In [2], the concept of P-essential was studied. In this section, the notion of T-p-essential submodules was introduced.

Definition 1.1. A submodule $\mathbf{K}$ of an $\mathbf{R}$-module $\mathbf{M}$ is called pure essential relative to submodule $\mathbf{T}$ of $\mathbf{M}$ (Simply $\mathbf{T}$-p-essential) if for every $\mathbf{T}$-pure submodule $\mathbf{L}$ with $\mathbf{K} \cap \mathbf{L} \subseteq \mathbf{T}$ implies $\mathbf{L} \subseteq \mathbf{T}$. M is called $\mathbf{T}$-p-essential extension of $\mathbf{K}$.

It is clear that every P-essential is $\mathbf{T}$-p-essential for every submodule but the converse may not be true in general, the submodule $\mathbf{Z}_4$ as $\mathbf{Z}_4$ -module. Let $\mathbf{K}=\{\bar{0},\bar{2}\}$,$\mathbf{L}=\bar{2}\,\mathbf{Z}_4$ is $(\bar{2}\mathbf{Z}_4)$-pure, $\mathbf{K} \cap \mathbf{L} \subseteq \bar{2}\,\mathbf{Z}_4$, thus $\mathbf{L} \subseteq \bar{2}\,\mathbf{Z}_4$. Hence $\mathbf{K}$ is $\mathbf{T}$-p-essential but not P-essential.

The following result is analogous to a similar concerning P-essential submodule of a module.

Theorem 1.2. Let $\mathbf{K} \subseteq \mathbf{N} \subseteq \mathbf{M}$ and let $\mathbf{T} \subseteq \mathbf{M}$ then:

1. If $\mathbf{K}$ is $\mathbf{T}$-p- essential in M, then $\mathbf{N}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$.

2. If $\mathbf{N}$ is $\mathbf{T}$-pure in M and $\mathbf{T} \subseteq \mathbf{N}$ and K is $\mathbf{T}$-p-essential in $\mathbf{M}$, then K is $\mathbf{T}$-p-essential in $\mathbf{N}$ and $\mathbf{N}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$.

3. If $\mathbf{M}$ has $\mathbf{T}$-pure finite intersection property and if $\mathbf{N}$ is $\mathbf{T}$-pure in $\mathbf{M}$, then $\mathbf{K}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$ if and only if $\mathbf{K}$ is $\mathbf{T}$-p-essential in $\mathbf{N}$ and $\mathbf{N}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$.

Proof: 1. we have to show that $\mathbf{N}$ **is** $\mathbf{T}$-p-essential in M. Let L be T-pure submodule of M with $\mathbf{N} \cap \mathbf{L} \subseteq \mathbf{T}$, since $\mathbf{K} \subseteq \mathbf{N}$, then $\mathbf{K} \cap \mathbf{L} \subseteq \mathbf{N} \cap \mathbf{L} \subseteq \mathbf{T}$,thus $\mathbf{K} \cap \mathbf{L} \subseteq \mathbf{T}$. Since K is T-p-essential in M then $\mathbf{L} \subseteq \mathbf{T}$. Hence N is T-p-essential in $\mathbf{M}$.

2. Let L be T-pure submodule of N with $\mathbf{K} \cap \mathbf{L} \subseteq \mathbf{T}$, since $\mathbf{N}$ is submodule of $\mathbf{M}$ and $\mathbf{K}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$, therefore $\mathbf{L} \subseteq \mathbf{T}$, therefore $\mathbf{L} = \mathbf{L} \cap \mathbf{N} \subseteq \mathbf{T} \cap \mathbf{N}$, thus $\mathbf{L} \subseteq \mathbf{T}$, hence K is $\mathbf{T}$-p-essential in $\mathbf{N}$. Now we have to show that $\mathbf{N}$ is $\mathbf{T}$-P-essential in $\mathbf{M}$. Let $\mathbf{L}$ be $\mathbf{T}$-pure submodule of $\mathbf{M}$ with $\mathbf{N} \cap \mathbf{L} \subseteq \mathbf{T}$, thus $\mathbf{K} \cap \mathbf{L} \subseteq \mathbf{N} \cap \mathbf{L} \subseteq \mathbf{T}$ so $\mathbf{L} \subseteq \mathbf{T}$. Hence $\mathbf{N}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$.

3.$\Longrightarrow$It is clear.

$\Longleftarrow$ Suppose that $\mathbf{K}$ is $\mathbf{T}$-p-essential in $\mathbf{N}$ and $\mathbf{N}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$, we have to show that $\mathbf{K}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$. Let $\mathbf{L}$ be $\mathbf{T}$-pure submodule of $\mathbf{M}$ with $\mathbf{K} \cap \mathbf{L} \subseteq \mathbf{T}$. By assumption $\mathbf{N} \cap \mathbf{L}$ is $\mathbf{T}$-pure in $\mathbf{M}$, thus $\mathbf{N} \cap \mathbf{L}$ is $\mathbf{T}$-pure in $\mathbf{N}$ by [remark 5] since $\mathbf{K}$ is $\mathbf{T}$-p-essential in N and $\mathbf{K} \cap (\mathbf{N} \cap \mathbf{L}) \subseteq \mathbf{T}$ thus $\mathbf{N} \cap \mathbf{L} \subseteq \mathbf{T}$ and also since $\mathbf{N}$ is $\mathbf{T}$-p-essential hence $\mathbf{L} \subseteq \mathbf{T}$. Thus $\mathbf{K}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$.

Corollary 1.3. Let M be an $\mathbf{R}$-module that has $\mathbf{T}$-pure finite intersection property .If $\mathbf{H}$ is $\mathbf{T}$-pure in $\mathbf{M}$, then $\mathbf{H} \cap \mathbf{K}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$ if and only if $\mathbf{H}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$ and $\mathbf{K}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$ for any submodule $\mathbf{K}$ of $\mathbf{M}$.

Proof: $\Longrightarrow$The proof follows by theorem (1.2).

$\Longleftarrow$Let $\mathbf{L}$ be $\mathbf{T}$-pure submodule of $\mathbf{M}$ with $\mathbf{K} \cap (\mathbf{H} \cap \mathbf{L}) \subseteq \mathbf{T}$, by assumption $\mathbf{H} \cap \mathbf{L}$ is $\mathbf{T}$ $-$pure in $\mathbf{M}$ and since $\mathbf{K}$ is $\mathbf{T}$-p-essential in M, then $\mathbf{H} \cap \mathbf{L} \subseteq \mathbf{T}$. So again since $\mathbf{H}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$ then $\mathbf{L} \subseteq \mathbf{T}$, therefore $\mathbf{K} \cap \mathbf{H}$ is $\mathbf{T}$-p-essential in $\mathbf{M}$.

Remark 1.4. If $\mathbf{A}$ is $\mathbf{T}$-p-essential in $\mathbf{B}$ and $\mathbf{A}'$ is T-p-essential in $\mathbf{B}'$, then $\mathbf{A} \oplus \mathbf{A}'$ is not T-p-essential in $\mathbf{B} \oplus \mathbf{B}'$, for example see example 4.6 in [2].

In [3], a submodule $\mathbf{N}$ of an $\mathbf{R}$-module $\mathbf{M}$ is said to be relative direct summand to a submodule $\mathbf{T}$ of $\mathbf{M}$ (Simply $\mathbf{T}$-direct summand) if there exist a submodule $\mathbf{K}$ of $\mathbf{M}$ with $\mathbf{M} = \mathbf{N} + \mathbf{K}$ and $\mathbf{N} \cap \mathbf{K} \subseteq \mathbf{T}$. It is clear that every direct summand is $\mathbf{T}$-direct summand.

Remark 1.5. 1. Every $\mathbf{T}$-direct summand of an $\mathbf{R}$-module $\mathbf{M}$ is $\mathbf{T}$-pure submodule.

2. Let $\mathbf{M}$ be an R-module and $\mathbf{T} \subseteq \mathbf{M}$. If $\mathbf{N}$ is $\mathbf{T}$-pure submodule of $\mathbf{M}$ and $\mathbf{K}$ is any submodule of $\mathbf{M}$, then $\mathbf{N} \cap \mathbf{K}$ is $\mathbf{T}$-pure submodule in $\mathbf{K}$.

3. Let $\mathbf{H} \subseteq \mathbf{M}, \mathbf{K} \subseteq \mathbf{M}$, then $\mathbf{H} \cap \mathbf{K}$ is **T**-p-essential in **M** if and only if **H** is **T**-p-essential in **M** and **K** is **T**-p-essential in **M**, where $\mathbf{H} \subseteq \mathbf{T}, \mathbf{K} \subseteq \mathbf{T}$.

4. If $\mathbf{K} \subseteq \mathbf{M}$ and H is T-pure in M, then $\mathbf{K} \cap \mathbf{H}$ is T-pure in M.

In [2], a submodule **N** of an **R**-module **M** is called a pure closed submodule of **M if M** dose not contain a proper p-essential extension of **N**. We introduce the concept of relative pure closed submodule to submodule.

Definition1.6. Let **M** be an **R**-module and let **T** be submodule of **M**. A submodule **N** of an **R**-module **M** is called relative pure closed submodule to submodule **T** of **M** (Simply **T**-p-closed) **of M if M** dose not contain a proper **T**-p-essential extension of **N**.

Proposition 1.7. Any **T**-direct summand of an **R**-module M is **T**-pure closed.

Proof: Let $\mathbf{M} = \mathbf{A} \oplus \mathbf{B}$, where **AandB** submodules of**M**. If**AisT**-p-essential in $\mathbf{K} \subseteq \mathbf{M}$, then by [remark 1.5 (2)] $\mathbf{K} \cap \mathbf{B}$ is**T − pureinK**. But $\mathbf{A} \cap (\mathbf{K} \cap \mathbf{B}) \subseteq \mathbf{T}$, but $\mathbf{K} \cap \mathbf{B} \subseteq \mathbf{T}$ and so $\mathbf{K} = \mathbf{A}$.

Proposition 1.8. Let $\mathbf{N} \subseteq \mathbf{T} \subseteq \mathbf{M}$, and **N is T**-p-closed in **M. If** $\mathbf{N} \subseteq \mathbf{K}$and **K** is **T**-p-essential in**M**, then $\frac{\mathbf{K}}{\mathbf{N}}$ is **T**-p-essential in $\frac{\mathbf{M}}{\mathbf{N}}$.

Proof: Let $\frac{\mathbf{L}}{\mathbf{N}}$ be T-pure in $\frac{\mathbf{M}}{\mathbf{N}}$ with $\frac{\mathbf{K}}{\mathbf{N}} \cap \frac{\mathbf{L}}{\mathbf{N}} \subseteq \frac{\mathbf{T}}{\mathbf{N}}$, then $\mathbf{K} \cap \mathbf{L} \subseteq \mathbf{T}$.

But K is T-p-essential in M, thus $\mathbf{L} \subseteq \mathbf{T}$ and $\mathbf{N} \subseteq \mathbf{T}$ and $\mathbf{N} \subseteq \mathbf{L}$, hence $\frac{\mathbf{L}}{\mathbf{N}} \subseteq \frac{\mathbf{T}}{\mathbf{N}}$.

Theorem 1.9. Let C be a T-p-essential submodules of an R-module M with C ⊆ T, then there exists T-p-closed submodule H in M which is T-pure such that C is T-pure closed in H.

Proof: let V= { K: K is T-pure submodule of M such that C is T-p-essential in K}. V≠∅,(since T is T-pure submodule of M, T ⊆ T and C is T-p-essential in M then by theorem(1.2) C is T-p-essential in T). By Zorn's Lemma, V has a maximal element say H. To show that H is T-p-closed in M, let L be a submodule of M such that H is T-p-essential in L. Since C is T-p-essential in H and H is T-p-essential in L, then by theorem(1.2) C is T-p-essential in L and thus H=L.

Let NandKbe submodules of anR − moduleMwithKpureinM, K is called pure relative complement ofNinMifK is maximal with the property $K \cap N = \{0\}$[2]. We introduce the concept of relative pure complement relative to submodule TofM (Simply T-p-complement).

Definition 1.10. Let NandK be two submodules of anR-module M with KisT-pure in M, Kis called relative T-p-complement of NinMifK is maximal with $K \cap N \subseteq T$.

Compare the following result with proposition (4.14) in [2].

Proposition 1.11. Every submodule of an **R**-module **M** has a relative **T**-p-complement in**M**

Proof: Let N be a given submodule of M and consider the set S= {K⊆M, K is T-pure in M with N⋂K⊆T}. It is clear that S≠∅ by [2], and every chain of S has an upper bound. By Zorn's Lemma, S has maximal element which means N has relative T-p-complement in M.

The following proposition gives the relation between T-p-closed submodule and relative T -p-complement submodule.

Proposition 1.12. Let N be a submodule of an R-module M and T ⊆ F, for every T-pure submodule F of M. If N is relative T-p-complement for some K of M, then N is T-p-closed in M.

Proof: Let L be T-pure submodule of M with N is T-p-essential. We have N⋂K ⊆ T, (N ⋂ K) ⋂ L ⊆ T ⋂ L,sinceL is T-pure in M, thenK ∩ Lis T-pure in L by remark (1.5) thus N ⋂ (K ⋂ L) ⊆ T ⋂ L=T ,hence L=N, hence N is T-p-closed in M.

In [4], an R-module is called purely extending module, if every submodule of M is essential in a pure submodule of M. We introduce the concept of relative purely extending module to submodule T of M (simply T-purely extending).

Definition 1.13. Let M be an R-module, M is called T-purely extending module if every submodule is T-p-essential in T-pure submodule of M.

The following theorem gives a characterization of T-purely extending module.

Theorem 1.14. Let M be an R-module, then M is T-purely extending if and only if every T-p-closed submodule of M is T-direct summand of M and every submodule is submodule of T.

Proof: Suppose M is an T-purely extending and let K be a T-p-closed submodule of M. Then there exists a T-pure submodule B of M such that K is T-p-essential in B. Conversely, let A be T-p-essential submodule of M, by theorem (1.9) there exists a T-p-closed submodule H in M such that A is T-p-essential in H. Since H is T-p-closed in M, then by our assumption H is T-pure in M and hence M is T-purely extending.

Remark 1.15. Every purely extending module M is T-purely extending.

Proof: Let A be a submodule of an R-module M. Since M is purely extending, then there exists a pure submodule B of M such that A is essential in B. Thus B is T-pure in M and hence M is T-purely extending.

Proposition 1.16. If an R-module M is T-purely extending and N is T-p-closed submodule of M, then $\frac{M}{N}$ is T-purely extending.

Proof: Let $\frac{K}{N}$ be a submodule of $\frac{M}{N}$. Since M is T-purely extending, then there exists a T-pure submodule A of M such that K is T-p-essential in A and since N ⊆ K and N is T-p-closed in M then by proposition (1.8) $\frac{K}{N}$ is T-p-essential in $\frac{A}{N}$. But A is T-pure in M, so by remark (1.5) $\frac{A}{N}$ is T-pure in $\frac{M}{N}$.

In [5], an R-module M has the relative pure to submodule T of M intersection property (Simply T-PIP) if the intersection of any two T-pure submodule is T-pure submodule.

Now, we give a condition which a pure submodule of T-purely extending module is T-purely extending.

Corollary 1.17. The homomorphic image of T-purely extending is T-purely extending if every submodule is T-p-closed.

Proposition 1.18. Let M be a T-purely extending module and let N be a T-pure submodule of M with N⊆T. If M has the T-PIP, then N is T-purely extending.

Proof: Let A be a T-p-closed submodule in N, then by theorem (1.9) there exists a T-p-closed submodule B in M such that A is T-p-essential in B. Since N is T-p-essential in N, then A=A ∩ N is T-p-essential in B ∩ N ⊆ N, but A is T-p-closed in N, therefore A=B ∩ N. Since M is T-purely extending and B is T-p-closed in M, then by theorem (1.13) B is T-pure in M. But N is T-pure in M and M has the T-PIP, so A=B∩ N is T-pure in M and hence A is T-pure in N. Thus N is T-purely extending.

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نتائج حول المقاسات الجزئية النقية بالنسبة الى مقاس جزئي

شيرين عودة دخيل **نور رياض أديب**

قسم الرياضيات –كلية العلوم للبنات-جامعة بغداد

الخلاصة:

لتكن $\mathbf{R}$ حلقة أبدالية ذات عنصر محايد 1 و ليكن M مقاساً أيسراً أحادياً على الحلقة $\mathbf{R}$.يسمى المقاس الجزئي N نقياً بالنسبة الى المقاس الجزئي $\mathbf{T}$ من $\mathbf{M}$ أذا كان:
$\mathbf{N \cap AM = AN + T \cap (N \cap AM)}$. لكل مثالي $\mathbf{A}$ في R.
في هذا البحث تم دراسة خواص المقاسات الجزئية الجوهرية النقية بالنسبة الى مقاس جزئي والمقاسات الجزئية المغلقة النقية بالنسبة الى مقاس جزئي والمقاسات الجزئية المكملة النقية بالنسبة الى مقاس جزئي و كذلك تم دراسة بعض خواص المقاسات التوسيعية النقية بالنسبة الى مقاس جزئي.

الكلمات المفتاحية: مقياس جزئي مخلص من النمط T، مقياس جزئي جوهري نقي من النمط T، مقياس جزئي مغلق نقي من النمط T.