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Indirect Method for Optimal Control Problem Using Boubaker Polynomial

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Abstract:

In this paper, a computational method for solving optimal problem is presented, using indirect method (spectral methodtechnique) which is based on Boubaker polynomial. By this method the state and the adjoint variables are approximated by Boubaker polynomial with unknown coefficients, thus an optimal control problem is transformed to algebraic equations which can be solved easily, and then the numerical value of the performance index is obtained. Also the operational matrices of differentiation and integration have been deduced for the same polynomial to help solving the problems easier.

A numerical example was given to show the applicability and efficiency of the method. Some characteristics of this polynomial which can be used for solving optimal control problems have been deduced and studied for any future work.

Keywords: Optimal control problem, Boubaker polynomial, indirect spectral method.

Introduction:

Control theory is a branch of optimization theory concerned with minimizing or maximizing a given performance index which satisfying the system state equations and constraints [1]. The main goal is to find an optimal open loop control $u^*(t)$ or an optimal feedback control $u^*(t, x)$ that satisfies the dynamical system and optimizes in some sense performance index. Analytical solutions of optimal control problems are not always available, so a numerical solution for solving optimal control problems is the most logical way to treat them. [2]

The linear quadratic control (LQP) is a special case of the general nonlinear optimal control problem (OCP), the (LQP) is stated as follows;

Minimize the quadratic continuous time

$$cost\ function\ J = \int_{t_0}^{t_f} (x^T Q x + u^T Ru)dt \quad ...(1)$$

subject to the linear system state equations;

$$\dot{x}\,(t) = Ax(t) + Bu(t) \quad ...(2)$$

A particular form of the (LQP) that arises in many control system problems, where A represents an $n \times n$ system matrix. B is $n \times m$ input matrix, $x(t)$represents an n ×

1 state vector, and u(t) is $m \times 1$ input vector . [3]

The solution is known to be
$u^*(t) = -R^{-1}B^T\lambda(t)x(t)$.
Where x(t) satisfies the following equation

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

With boundary conditions

$x(t_0) = x_0$

$\lambda(t_f) = 0.$

Indirect methods are generally based on a reduction of the control problem to a problem involving a differential equation such as the HJB(Hamilton-Jacobi-Bellman) or TPBV(Two Point Boundary Value)problem that is based on the principle of optimality which in most casesare very difficult to solve, So the idea is using the solution of the first order necessary conditions for optimality that are obtained from Pontryagin's minimum principle for problems without inequality constraints, then the optimality conditions can be formulated as a set of differential algebraic equations[4], and to reduce them to an algebraic equations in terms of the orthogonal functions and the operational matrix of differentiation (or integration) matrix associated with this function. In[3,5-9] the same method has been used with different kinds of polynomials (e.g.,Chebyshev, Laguerre, Bernstein).

The spectral method was used in this paper to find the solution for these equations by the aid of Boubaker polynomials as the basis function, presenting it as an efficient tool with spectral method technique for solving a linear quadratic problem.

1-Boubaker Polynomials

The Boubaker polynomials were established for the first by Boubaker et al. as a guide for solving heat equation inside physical model and then for other physical applications [10,11,12]. During resolution process an intermediate sequence raised an interesting recursive formula leading to a class of polynomial functions that performs difference with common class. Boubaker polynomial is introduced by the following equation; [2]

$$B_n(t) = \sum_{p=0}^{\zeta(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right] (-1)^p t^{n-2p}$$

where $\zeta(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}$.

$B_0(t) = 1, B_1(t) = t, B_2(t) = t^2 + 2, \ldots$

…(3)

2-Operational Matrix of Differentiation:

We have derived the powers in terms of Boubaker polynomials which will help us in solving our problems. Then we have in matrix form the powers of t as follows;

$$T = KB(t)$$

where $T = \left[t^0, t^1, t^2, t^3, \ldots, t^n \right]'$

and $B(t) = [B_0(t), B_1(t), B_2(t), \ldots, B_n(t)]'$

where

$$K_{n\times n}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & . & . & . & 0\\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & . & . & . & 0\\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & . & . & . & 0\\ -4 & 0 & 3 & 0 & 2 & 0 & 1 & 0 & . & . & . & 0\\ 0 & -1 & 0 & 5 & 0 & 3 & 0 & 1 & . & . & . & 0\\ . & . & . & . & . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & . & . & . & . & .\\ k_{n1} & . & . & . & . & . & . & . & . & . & . & 1 \end{bmatrix}$$

$$K=(k_{ij})=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i<j\\ k_{i,1}=-2k_{i-1,2} & \\ k_{i,j}=k_{i-1,j+1}+k_{i-1,j-1} & i>3\end{cases}$$

A recursive relation is given by;

$$B_m(t)=tB_{m-1}(t)-B_{m-2}(t),\quad \text{for } m>2.$$

Using the recursive relation for $\frac{d}{dt}B_m(t)$, then we have $\dot{B}_0(t)=0$

$$\dot{B}_1(t)=B_0(t)$$

$$\dot{B}_2(t)=2B_1(t)$$

$$\dot{B}_3(t)=3B_2(t)-5B_0(t)$$

$$\dot{B}_4(t)=4B_3(t)-4B_1(t)$$

$$\dot{B}_5(t)=5B_4(t)-3B_2(t)+13B_0(t)$$

the recursive relation is,

$$\dot{B}_m(t)=t\dot{B}_{m-1}(t)+B_{m-1}(t)-\dot{B}_{m-2}(t).$$

The differentiation operational matrix for Boubaker polynomials, which is orthogonal polynomials, was deduced as follows;

$$\dot{B}(t)=bB(t)$$

$$\begin{bmatrix}\dot{B}_0(t)\\ \dot{B}_1(t)\\ \dot{B}_2(t)\\ \dot{B}_3(t)\\ \dot{B}_4(t)\\ \dot{B}_5(t)\\ \dot{B}_6(t)\\ \dot{B}_7(t)\\ \dot{B}_8(t)\\ \dot{B}_9(t)\\ \dot{B}_{10}(t)\\ .\\ .\\ .\\ \dot{B}_m(t)\end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ -5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ 0 & -4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ 13 & 0 & -3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ 0 & 14 & 0 & -2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0\\ -41 & 0 & 15 & 0 & -1 & 0 & 7 & 0 & 0 & 0 & 0 & . & . & . & 0\\ 0 & -40 & 0 & 16 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & . & . & . & 0\\ 121 & 0 & -39 & 0 & 17 & 0 & 1 & 0 & 9 & 0 & 0 & . & . & . & 0\\ 0 & 122 & 0 & -38 & 0 & 18 & 0 & 2 & 0 & 10 & 0 & . & . & . & 0\\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\ b_{m,1} & . & . & . & . & . & . & . & . & . & . & . & . & . & b_{m,n} \end{bmatrix}*\begin{bmatrix}B_0(t)\\ B_1(t)\\ B_2(t)\\ B_3(t)\\ B_4(t)\\ B_5(t)\\ B_6(t)\\ B_7(t)\\ B_8(t)\\ B_9(t)\\ B_{10}(t)\\ .\\ .\\ .\\ B_m(t)\end{bmatrix}\quad \ldots(4)$$

Such that;

$b_{2,1} = 1$

$$\left.\begin{array}{l}
(1)\ b_{i,j} = 0 \quad \text{if } i \le j,\ \text{where } i = 1,2,3,\ldots,\quad j = 1,2,3,\ldots \\
(2)\ b_{i,j} = 0 \quad \text{if } i > j,\ \text{s.t; } i - j \text{ is even.} \\
(3)\ b_{i,1} = -[3 \times b_{(i-2),1} + 2],\ \text{where } i > 2 \text{ and } i\text{'s is even.} \\
(4)\ b_{2n+k,k+1} = b_{2n,1} + k \quad \text{where } k = j - 1,\ 2n = i - j + 1.\ k \text{ and } n \text{ are integers.}
\end{array}\right\} \ldots(5)$$

3- Operational Matrix of Integration

The integrationof the vector B(t) for Boubaker polynomials can be obtained as follows,

$$\int_0^t B(\theta)\,d\theta = G\,B(t)$$

where G is operational matrix for integration.

$$G = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{5}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{4}{8} & 0 & \frac{4}{8} & 0 & \frac{2}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{24}{15} & 0 & \frac{3}{15} & 0 & \frac{3}{15} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{44}{24} & 0 & -\frac{24}{24} & 0 & \frac{2}{24} & 0 & \frac{4}{24} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{72}{35} & 0 & -\frac{24}{35} & 0 & \frac{1}{35} & 0 & \frac{5}{35} & 0 & 0 & 0 & 0 \\
-\frac{180}{48} & 0 & \frac{72}{48} & 0 & -\frac{24}{48} & 0 & 0 & 0 & \frac{6}{48} & 0 & 0 & 0 \\
0 & -\frac{216}{63} & 0 & \frac{72}{63} & 0 & -\frac{24}{63} & 0 & -\frac{1}{63} & 0 & \frac{7}{63} & 0 & 0 \\
\frac{604}{80} & 0 & -\frac{216}{80} & 0 & \frac{72}{80} & 0 & -\frac{24}{80} & 0 & -\frac{2}{80} & 0 & \frac{8}{80} & 0 \\
0 & \frac{648}{99} & 0 & -\frac{216}{99} & 0 & \frac{72}{99} & 0 & -\frac{24}{99} & 0 & -\frac{3}{99} & 0 & \frac{9}{99}
\end{bmatrix}$$

such that;

$g_{1,2} = 1;\ g_{2,1} = -1;\ g_{2,3} = 1/2;\ g_{3,2} = 5,$

Denominator of all terms of matrix is $i(i - 2)$ for all $i > 2$

The following g's represent only the numerator.

(1) $g_{i,j} = 0$, if $i < j$ & $i - j$ is even .

(2) $g_{i,j} = i - 2$ if $j = i + 1$.

(3) $g_{i,2} = (-1)^{\lfloor \frac{i}{3} \rfloor}(3)^{(\frac{i-3}{2})} * 8$, if i is odd & $i \ge 5$.

(4) $g_{i,1} = g_{i+1,2} + 4(-1)^{\frac{i}{2}}(i + 1)$.

(5) $g_{i+k,2+k} = g_{i,2}$, if $i > 4$ & k is a positive integer.

(6) $g_{2n+k,k+1} = g_{2n+1,2} - k + 1$ where $k = j - 1$ and $2n = i - j + 1$, k & n as positive integers.

4-Spectral method Technique

Spectral method is used to solve finite Linear quadratic optimal control problems with the aid of classical polynomials usually like, Hermite, Laguerre polynomials… etc, as the basis functions [4]. In this work, Boubaker polynomials have been used with the following procedure,

-Writing the necessary conditions to determine the optimal solution of the problem equations (1) and (2), which are the followings,

$$\dot{x} = Ax - \frac{1}{2}BR^{-1}B^T\lambda \ldots (6)$$

$$\dot{\lambda} = 2Qx - A^T\lambda \ldots (7)$$

$$u = -\frac{1}{2}R^{-1}B^T\lambda \ldots (8)$$

with the initial conditions $x(0)=x_0$, and the final conditions $\lambda(t_f)=0$.

- Choosing a set of state and adjoint variables and approximating them using a basis function to approximate $x_j^N(t) and \lambda_j^N(t)$ and substituting in (7-9), we get

$$x_j(t) \approx x_j^N(t) = \sum_{i=0}^{N} a_{ij} B_i(t) \qquad \ldots(9)$$

$$\lambda_j(t) \approx \lambda_j^N(t) = \sum_{i=0}^{N} b_{ij} B_i(t) \qquad \ldots(10)$$

where $a_{ij} and\ b_{ij}; i = 1,2,\ldots,N\ ,\ j = 1,2,\ldots,q$.

and B_i are Boubaker polynomials.

The remaining 2(n-q) state and adjoint variables are obtained from the system state and adjoint equations.

- Form the 2q (N×N) system of algebraic equations as follows;

Differentiate the basis functions $B_i(t)$, i=1,2,…,N then introducing Boubaker polynomials differentiation operational matrix D_B to yield

$$\dot{B}(t) = D_B B(t) \ldots (11)$$

where the matrix D_B is given by equation (5).

Note that $x_j^N(t)$ and $\lambda_j^N(t)$, $j = 1,2,\ldots,n$ can be written as

$$x = \alpha B(t)$$

$$\lambda = \beta B(t)$$

Differentiating with respect to t yield

$$\dot{x} = \alpha D_B B(t)$$

$$\dot{\lambda} = \beta D_B B(t)$$

- Solve the above resulting square of equations using Gauss elimination procedure with pivoting, to find the entries of α and β .

-Find the approximate value of the performance index J in equation (1).

To illustrate the procedure, the following numerical example is given.

5- Numerical ExampleConsider the problem

$$J = \int_0^1 u^2\, dt \ldots (12)$$

$\dot{x}_1 = x_2, x_1(0) = 1, \qquad x_1(1) = 0$

$\dot{x}_2 = u, x_2(0) = 1, \qquad \lambda_2(1) = 0$

The exact solution is J=12.

Exact trajectories are

$x_1(t) = t^3 - 3t^2 + t + 1$

$x_2(t) = 3t^2 - 6t + 1$

$u(t) = 6t - 6, \qquad 0 \le t \le 1.$

$$H = u^2 + \lambda_1 x_2 + \lambda_2 u\ .$$

Sufficient condition

$$\frac{\partial H}{\partial u} = 0 \rightarrow 2u + \lambda_2 = 0$$

$$u = -\frac{\lambda_2}{2}$$

Necessary conditions

$$\dot{\lambda} = -\frac{\partial H}{\partial x}, x = \frac{\partial H}{\partial \lambda},$$

$$\dot{x}_1 = x_2 \ldots (13)$$

$$\dot{x}_2 = -\frac{\lambda_2}{2} \ldots (14)$$

$$\dot{\lambda}_1 = 0 \ldots (15)$$

$$\dot{\lambda}_2 = -\lambda_1 \ldots (16)$$

With boundary conditions $x_1(0) = 1,\ x_1(1) = 0,\ x_2(0) = 1,\ \lambda_2(1) = 0,$

The state variable $x_1(t)$ and adjoint variable $\lambda_2(t)$ are approximated by third order Boubaker polynomials, then $x_2(t)$ can be found from equation (13) while $\lambda_1(t)$ is from equation (16).

$$x_1(t) \approx \sum_{i=0}^{3} a_i B_i,$$

$$x_1(t) \approx a_0 B_0(t) + a_1 B_1(t) + a_2 B_2(t) + a_3 B_3(t),$$

$$\lambda_2(t) \approx \sum_{i=0}^{3} b_i B_i,$$

$$\lambda_2(t) \approx b_0 B_0(t) + b_1 B_1(t) + b_2 B_2(t) + b_3 B_3(t),$$

Substituting in equations (13-14), using equation (11) we get

$$2a_2B_0(t) + 6a_3B_1(t) = -\frac{1}{2}b_0B_0(t) - \frac{1}{2}b_1B_1(t) - \frac{1}{2}b_2B_2(t) - \frac{1}{2}b_3B_3(t) \quad ...(17)$$

from equation(16)

$$\lambda_1 = -[(b_1 - 5b_3)B_0(t) + 2b_2B_1(t) + 3b_3B_2(t)]$$

Substituting in equation (15)

$$-2b_0B_0(t) - 6\,b_3B_1(t) = 0 \;...(18)$$

fromequations(17) and (18) with boundary conditions the following algebraic equations are obtained,

$$4a_2 + b_0 = 0 \quad ...(19)$$

$$12a_3 + b_1 = 0 \quad ...(20)$$

$$2b_2 = 0 \quad ...(21)$$

$$6b_3 = 0 \quad ...(22)$$

$$x_1(0) = 1 \rightarrow a_0 + 2a_2 = 1 \quad ...(23)$$

$$x_2(0) = 1 \rightarrow a_1 + a_3 = 1 \quad ...(24)$$

$$x_1(1) = 0 \rightarrow a_0 + a_1 + 3a_2 + 2a_3 = 0 \quad ...(25)$$

$$\lambda_2(1) = 0 \rightarrow b_0 + b_1 + 3b_2 + 2b_3 = 0 \quad ...(26)$$

The following state and control approximations are:

$$x_1(t) = 7B_0(t) + (0)B_1(t) - 3B_2(t) + (1)B_3(t),$$

$$x_2(t) = -5B_0(t) - 6B_1(t) + 3B_2(t),$$

$$u(t) = -6B_0(t) + 6B_1(t)$$

with the approximate value of J^*=12, which is a good approximation to the exact value.

Conclusions:

Boubaker polynomials can be used for transforming an optimal control problem to algebraic equation by the aid of indirect method (spectral method technique). The operational matrix of differentiation was derived and applied for solving the optimal conrol problem by reducing it into algebraic problem, also the operational matrix of integration was derived to be used in a future work, and then an example has been presented which showed the applicability of this method.

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طريقة غير مباشرة لمسألة سيطرة مثلى باستخدام متعددة حدود بوبكر

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الخلاصة:

في هذا البحث، تم حساب مسألة سيطرة مثلى باستخدام الطريقة غير المباشرة(تقنية طريقة الطيف) وبمتعددة الحدود المعروفة ببوبكر.بواسطة هذه الطريقة تم تقريب متغيرات الحالة والمتغيرات المرافقة باستخدام متعددة حدود بوبكر مع المعاملات المجهولة حيث تم تحويل مسألة السيطرة المثلى الى معادلات جبرية يمكن حلها بسهولة وايجاد القيمة العددية للكلفة المثلى ، بالاضافة الى استنتاج مصفوفة المشتقات ثم مصفوفة التكاملات لنفس متعددة الحدودالمستخدمة مع مثال تطبيقي لتوضيح امكانية وكفاءة هذه الطريقة.

الكلمات المفتاحية:مسألة سيطرة مثلى، متعددة حدود بوبكر، طريقة تقنية الطيف غير المباشرة.