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## On the Representations of M-Groups

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### Abstract:

The main object of this paper is to study the representations of monomial groups and characters technique for representations of monomial groups. We refer to monomial groups by M-groups. Moreover we investigate the relation of monomial groups and solvable groups.

Many applications have been given the symbol G *e.g.* group of order 297 is an M-group and solvable. For any group G, the factor group  $G/G'$  ( $G'$  is the derived subgroup of G) is an M-group in particular if  $G = S_n, SL(4, R)$ .

**Key words:** Representation Theory, Monomial Group, Solvable Group, Representation of Monomial Group.

### Introduction:

The theory of groups really begins with Galois (1811-1923) who demonstrates that polynomials are best understood by examining certain groups of permutations of their roots. Since that time, groups have arisen in almost every branch of mathematics. [1] The essential body of representation theory has been constructed by Richard Brauer (1901-1977). His processors; Frobenius Burnside and Schur, give the grand task to which character theory could make a central contribution, that is, the complete classification of finite simple groups [2].

In studying monomial group it is important to know as much as possible about the character of its subgroups. A

finite group is called monomial group (M-group) if every irreducible complex character is induced from a linear character of a subgroup of G. No non-linear character theoretic description of the class of M-groups has been found.

The process of imbedding a group in a larger group of some prescribed type has been one of the most useful tools in the investigation of properties of groups. The three principal types of representation of groups, each with its particular field of usefulness, are the following:

1. Permutation groups.
2. Monomial groups.

3. Linear or matrix representations of groups.

These three types of representation correspond to an imbedding of the group in the following groups:

1. The symmetric group.
2. The complete monomial group.
3. The full linear group, [3]

A finite group is termed **monomial** (or sometimes, a **M-group** or  $M_1$ -group) with respect to a field  $K$  (whose characteristic does not divide the group order) if it satisfies the following equivalent conditions:

1. Every irreducible representation of the group over  $K$  is induced from a one-dimensional representation of a subgroup, i.e., a linear character.
2. Every finite-dimensional linear representation of the group over  $K$  is a monomial linear representation: it is a direct sum of representations induced from one-dimensional representations of subgroups, [4].

Recently, Lakshmibai 2011 [5], Musili 2011 [6] and Seshadri 2012 [7] give detailed descriptions of the early development of standard monomial theory.

Bravi, Chirivi, Gendini and Maffei 2015 [8] study the standard monomial theory for wonderful varieties.

Hajim in 2007 [3] gave new result on representation of monomial groups and character technique for representation of monomial group.

Here we give more results on representation of monomial groups and discuss more properties of monomial groups and solvable groups.

## Representation and Solvable Groups:

The main aim of this section is to introduce the most basic definitions and facts in ordinary representation theory and solvable group.

### Definition 1: [2]

A group homomorphism  $\pi: G \longrightarrow GL(V)$  where  $G$  is a group and  $V$  is a vector space which is called a representation of  $G$  in  $V$ .

### Proposition 2: [2]

There is a bijective correspondence between the set of linear actions of  $G$  on vector space  $V$  and the set of homomorphisms from  $G$  to  $GL(V)$ .

### Example 3:

Consider the cyclic group  $C_5 = \{1, x, x^2, x^3, x^4\}$ . And let  $\pi: C_5 \longrightarrow GL(V)$ , the 5<sup>th</sup> root of unity  $\{1, x, x^2, x^3, x^4\}$  are:

$$\begin{aligned}\pi_1(x) &= 1 \\ \pi_2(x) &= e^{2\pi i/5} \\ \pi_3(x^2) &= (e^{2\pi i/5})^2 \\ \pi_4(x^3) &= (e^{2\pi i/5})^3 \\ \pi_5(x^4) &= (e^{2\pi i/5})^4.\end{aligned}$$

### Definition4: [9]

A group  $G$  is said to be a solvable group, if there exists a finite chain of subgroups  $G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \dots \supseteq H_k = \{e\}$  such that  $H_i \trianglelefteq H_{i-1}$  and the quotient group  $H_{i-1}/H_i$  is abelian, where  $\trianglelefteq$  is the symbol of normal subgroup.

### Example 5: [1]

The symmetric group  $S_4$  is solvable.

### Definition 6: [1]

A group  $G$  is called nilpotent if it has a central series, that is a normal series  $1 = G_0 \subseteq G_1 \subseteq \dots \subseteq G_n = G$  such that  $G_{i+1}/G_i$  is contained in the center of  $G/G_i$  for all  $i$  (a finite group all of whose subgroup are normal is nilpotent).

## Characters and M-Groups:

Character theory was developed by Frobenius in 1896. It provides a powerful tool for proving theorems about finite groups. We use character

techniques to obtain more information and facts about M-groups.

**Definition 1:** [10]

Let  $\chi$  be a character of  $G$ . Then  $\chi$  is monomial if  $\chi = \lambda^G$  where  $\lambda$  is a linear character of some subgroup of  $G$ .

**Remark 2:** [11]

Let  $G$  be any group, we denote by  $\text{Irr}(G)$  for the set of all irreducible character of  $G$ .

**Definition 3:** [11]

A group  $G$  is an M-group (monomial group) if every  $\chi \in \text{Irr}(G)$  is monomial character.

**Theorem 4:** [1]

Every abelian group is a nilpotent group.

**Theorem (Taketa) 5:** [10]

Every M-group is a solvable group.

**Theorem (Taketa) 6 :** [9]

Every nilpotent group is an M-group.

**Proposition 7:** [2]

A finite M-group  $G$  of order  $n$  has exactly  $n$  distinct characters.

**Example 8:**

The Klein 4-group is given by  $V = \langle x \rangle \times \langle y \rangle$  where  $x^2 = y^2 = 1, xy = yx$ . In order to obtain the character of  $V$ , we have to compute

$$\begin{aligned} \chi_{[1,0]}((1,1)) &= e^{2\pi i(1 \cdot 0 + 0 \cdot 0)/2} = 1 \\ \chi_{[1,0]}((x,1)) &= e^{2\pi i(1 \cdot 1 + 0 \cdot 0)/2} = -1 \\ \chi_{[1,0]}((1,y)) &= e^{2\pi i(1 \cdot 0 + 0 \cdot 1)/2} = 1 \\ \chi_{[1,0]}((x,y)) &= e^{2\pi i(1 \cdot 1 + 0 \cdot 1)/2} = -1 \\ \chi_{[0,1]}((1,1)) &= e^{2\pi i(0 \cdot 0 + 1 \cdot 0)/2} = 1 \\ \chi_{[0,1]}((x,1)) &= e^{2\pi i(0 \cdot 1 + 1 \cdot 0)/2} = 1 \\ \chi_{[0,1]}((1,y)) &= e^{2\pi i(0 \cdot 0 + 1 \cdot 1)/2} = -1 \\ \chi_{[0,1]}((x,y)) &= e^{2\pi i(0 \cdot 1 + 1 \cdot 1)/2} = -1 \\ \chi_{[1,1]}((1,1)) &= e^{2\pi i(1 \cdot 0 + 1 \cdot 0)/2} = 1 \\ \chi_{[1,1]}((x,1)) &= e^{2\pi i(1 \cdot 1 + 1 \cdot 0)/2} = -1 \\ \chi_{[1,1]}((1,y)) &= e^{2\pi i(1 \cdot 0 + 1 \cdot 1)/2} = -1 \end{aligned}$$

$$\chi_{[1,1]}((x,y)) = e^{2\pi i(1 \cdot 1 + 1 \cdot 1)/2} = 1$$

Table (1) shows the results:

**Table (1) Character Table of  $G = \langle x \rangle \times \langle y \rangle$**

$C_\alpha$	(1,1)	(x,1)	(1,y)	(x,y)
$ C_\alpha $	1	1	1	1
$ C_G(C_\alpha) $	4	4	4	4
$\chi_{[0,0]}$	1	1	1	1
$\chi_{[1,0]}$	1	-1	1	-1
$\chi_{[0,1]}$	1	1	-1	-1
$\chi_{[1,1]}$	1	-1	-1	1

**Main Results and Applications:**

Now, we give the main results and applications of monomial groups and solvable groups.

**Proposition 1:** [9]

If  $(H,*)$ ,  $(K,*)$  are subgroups of the group  $(G,*)$  such that  $H*K = K*H$ , then  $(H*K,*)$  is a subgroup of  $(G,*)$ .

**Proposition 2:**

If  $(H_1*K_1,*)$  and  $(H_2*K_2,*)$  are two abelian subgroups of a group  $(G,*)$  then  $(H_1*K_1) \times (H_2*K_2)$  and  $(H_2*K_2) \times (H_1*K_1)$  are isomorphic and  $(H_1*K_1) \times (H_2*K_2)$  is M-group hence it is solvable group.

**Proof:**

Since  $(H_1*K_1,*)$  and  $(H_2*K_2,*)$  are abelian groups, then  $(H_1*K_1) \times (H_2*K_2)$  and  $(H_2*K_2) \times (H_1*K_1)$  are abelian groups, so  $(H_1*K_1) \times (H_2*K_2)$  and  $(H_2*K_2) \times (H_1*K_1)$  are nilpotent by Theorem (3.4), therefore  $(H_1*K_1) \times (H_2*K_2)$  and  $(H_2*K_2) \times (H_1*K_1)$  are M-group by Theorem (3.6). Hence  $(H_1*K_1) \times (H_2*K_2)$  and  $(H_2*K_2) \times (H_1*K_1)$  are solvable group by Theorem (5).

**Proposition 3:** [9]

Let  $H_1, H_2, \dots, H_{n-1}, H_n$  be a subgroups of the group  $G$ , then  $H_1*H_2*\dots*H_{n-1}*H_n$  and  $H_n*H_{n-1}*H_2*H_1$  are isomorphic and  $H_1*H_2*\dots*H_{n-1}*H_n$  is abelian only if  $H_1, H_2, \dots, H_{n-1}, H_n$  are abelian.

**Proof:**

It is a simple substantiation that  $H_1 * H_2 * \dots * H_{n-1} * H_n$  and  $H_n * H_{n-1} * H_2 * H_1$  are isomorphic under the mapping  $(h_1, h_2, \dots, h_{n-1}, h_n) \longrightarrow (h_n, h_{n-1}, h_2, h_1)$  where  $h_1 \in H_1, h_2 \in H_2, \dots, h_{n-1} \in H_{n-1}, h_n \in H_n$ . Let  $(h_1, h_2, \dots, h_{n-1}, h_n), (h'_1, h'_2, \dots, h'_{n-1}, h'_n)$  be elements of  $H_1 * H_2 * \dots * H_{n-1} * H_n$ . Then

$$(h_1, h_2, \dots, h_{n-1}, h_n) * (h'_1, h'_2, \dots, h'_{n-1}, h'_n) = ((h_1 * h'_1) * (h_2 * h'_2), \dots, (h_{n-1} * h'_{n-1}), (h_n * h'_n))$$

if and only if  $h_1 * h'_1 = h'_1 * h_1, h_2 * h'_2 = h'_2 * h_2, \dots, h_{n-1} * h'_{n-1} = h'_{n-1} * h_{n-1}$  and  $h_n * h'_n = h'_n * h_n$  for all  $h_i, h'_i \in H_i, i = 1, 2, \dots, n$ . So  $H_1 * H_2 * \dots * H_{n-1} * H_n$  is abelian if and only if  $H_1, H_2, \dots, H_{n-1}, H_n$  are abelian.

**Example 4:**

Let  $G$  be group of order 297 then it is M-group and solvable group where  $297 = 3^3 \times 11$ ,  $G$  has a 3-sylow subgroup  $H$ . It is of order 27, and a 11-sylow subgroup  $K$  of order 11. Let  $H$  have  $n$  conjugates. Then  $n = 1 + 3r$  ( $r \geq 0$ ) and divides 297 and so we conclude easily that as the factors of 297 are 1, 3, 9, 11, 27, 33, 99, 297,  $n = 1$  and hence  $H$  is normal in  $G$ . Similarly,  $K$  is normal in  $G$ , we have  $G = HK$ . Since  $|HK| = |HK|/|H \cap K| =$

$|H||K| = 297$ , thus  $G$  is isomorphic to the direct product  $H * K$ . But  $H$  is abelian and  $K$  is cyclic and thus by Proposition (3)  $HK$  is an abelian subgroup. Hence  $G$  is abelian then  $G$  is nilpotent by Theorem (4). Now  $G$  is M-group by Theorem (6).

**Proposition 5:** [9]

Let  $H_1, H_2, \dots, H_n$  and  $K_1, K_2, \dots, K_n$  be subgroups of the group  $G$ , then  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  and  $(H_n * K_n) \times \dots \times (H_2 * K_2) \times (H_1 * K_1)$  are isomorphic and  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  is abelian if & only if  $(H_1 * K_1), (H_2 * K_2), \dots, (H_n * K_n)$  are abelian.

**Proof:**

It is an easy confirmation that  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  and  $(H_n * K_n) \times \dots \times (H_2 * K_2) \times (H_1 * K_1)$  are isomorphic under the mapping  $(h_1 * k_1, h_2 * k_2, \dots, h_n * k_n) \longrightarrow (h_n * k_n, \dots, h_2 * k_2, h_1 * k_1)$  where  $h_i \in H_i, k_i \in K_i, i = 1, 2, \dots, n$ . Allow  $(h_1 * k_1, h_2 * k_2, \dots, h_n * k_n), (h'_1 * k'_1, h'_2 * k'_2, \dots, h'_n * k'_n)$  be elements of  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$ . Thus  $(h_1 * k_1, h_2 * k_2, \dots, h_n * k_n) = (h'_1 * k'_1, h'_2 * k'_2, \dots, h'_n * k'_n) =$

$$((h_1 * k_1)(h'_1 * k'_1), (h_2 * k_2)(h'_2 * k'_2), \dots, (h_n * k_n)(h'_n * k'_n))$$

if and only if

$$(h_1 * k_1)(h'_1 * k'_1) = (h'_1 * k'_1)(h_1 * k_1), (h_2 * k_2)(h'_2 * k'_2) = (h'_2 * k'_2)(h_2 * k_2),$$

$$\& (h_n * k_n)(h'_n * k'_n) = (h'_n * k'_n)(h_n * k_n)$$

for all  $h_i * k_i, h'_i * k'_i \in H_i * K_i, i = 1, 2, \dots, n$ . So,

$(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  is abelian if and only if  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  are abelian.

**Proposition 6 :**

If  $G = \langle Z_1 \rangle \times \langle Z_2 \rangle \times \dots \times \langle Z_m \rangle$  is an abelian group where  $Z_i$  is a group of order  $n_i$  and  $|G| = n_1 n_2 \dots n_i$ ,  $X = Z_1^{a_1} Z_2^{a_2} \dots Z_m^{a_m}$ ,  $0 \leq a_i \leq n_i, i = 1, 2, \dots, m$  then  $\chi_{[r]}(x) = e^{2\pi i \sum_{i=1}^m a_i r_i / n_i}$  is a monomial character and  $G$  is M-group.

**Proof:**

$G = \langle Z_1 \rangle \times \langle Z_2 \rangle \times \dots \times \langle Z_m \rangle$  is an abelian group thus  $\langle Z_1 \rangle, \langle Z_2 \rangle, \dots, \langle Z_m \rangle$  are abelian since  $H \times K$  is an abelian group iff  $H, K$  are abelian group and  $\langle Z_1 \rangle \times \langle Z_2 \rangle \times \dots \times \langle Z_m \rangle$  isomorphic to  $\langle Z_m \rangle \times \dots \times \langle Z_2 \rangle \times \langle Z_1 \rangle$ . Every abelian group is M-group and the sum of character is character  $\chi_i \in \text{Irr}(G)$  (irreducible of  $G$ ),  $i = 1, 2, \dots, m$  thus write  $\chi_1 = \lambda_1^G$  and  $\chi_2 = \lambda_2^G, \dots, \chi_m = \lambda_m^G$  where  $\lambda_1 \in \text{Irr}(U_1)$  and  $\lambda_2 \in \text{Irr}(U_2), \dots, \lambda_m \in \text{Irr}(U_m)$  are linear characters. Let  $v \in \text{irr}(U_1 \cap U_2 \cap \dots \cap U_m)$  be defined by  $v = \lambda_{1(U_1 \cap U_2 \cap \dots \cap U_m)} \lambda_{2(U_1 \cap U_2 \cap \dots \cap U_m)} \dots \lambda_{m(U_1 \cap U_2 \cap \dots \cap U_m)}$  and note that  $v$  is consistent of  $\chi_{(U_1 \cap U_2 \cap \dots \cap U_m)}$ , so that  $\chi$  is a consistent of  $v^G$ . We have however  $v^G(1) = |G:U_1 \cap U_2 \cap \dots \cap U_m| \leq |G:U_1| |G:U_2| \dots |G:U_m| = \chi_1(1)\chi_2(1)\dots\chi_m(1) = \chi(1)$  and thus  $\chi = v^G$ .

**Lemma 7:**

If  $H_i, K_i, i = 1, 2, \dots, n$  are two subgroups of the group  $G$  and  $(H_1 * K_1), (H_2 * K_2), \dots, (H_n * K_n)$  are abelian subgroups, then  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  is M-group, hence a solvable group.

**Proof:**

Since  $(H_1 * K_1), (H_2 * K_2), \dots, (H_n * K_n)$  are abelian subgroups, thus  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  is an abelian group by Proposition (3). So  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  is nilpotent by Theorem (4). Therefore  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  is M-group by Theorem (6). Hence  $(H_1 * K_1) \times (H_2 * K_2) \times \dots \times (H_n * K_n)$  is solvable by Theorem (5).

**Definition 8:**

Let  $H$  and  $K$  are two subgroups of the group  $G$ , then the derived subgroup

or commutator subgroup of  $G$  denoted by  $[H * K, H * K]$  where  $[H * K, H * K] = \{ \prod [h_i * k_i, h_i * k_i], h_i * k_i \in H_i * K_i, i = 1, 2, \dots, n, \text{ where } \prod \text{ is a finite product with one or more factors} \}$ .

**Theorem 9:**

The group  $([H * K, H * K], *)$  is a normal subgroup of the group  $(G, *)$ .

**Proof:**

Let  $c$  an element of  $[H * K, H * K]$  and  $h * k$  is an element of  $H * K$ . To prove that  $(h * k) * c * (h * k)^{-1} \in [H * K, H * K]$ . Now:  $(h * k) * c * (h * k)^{-1} = [(h * k) * c * (h * k)^{-1} * c^{-1}] * c = [H * K, c] * c$ . Thus  $(h * k) * c * (h * k)^{-1} \in [H * K, H * K]$ . Then  $([H * K, H * K], *)$  is a normal subgroup of the group  $(G, *)$ .

**Theorem 10:**

Let  $G'$  be the normal subgroup of the group  $H * K$ . Then  $(H * K / G', \otimes)$  is abelian group iff  $[H * K, H * K] \subseteq G'$ .

**Proof:**

Let  $(H * K / G', \otimes)$  be an abelian group, thus for all  $(h_1 * k_1), (h_2 * k_2)$  of  $H * K$   $(h_1 * k_1 * h_2 * k_2)^{-1} * G' = ((h_1 * k_1) * G') \otimes ((h_2 * k_2) * G') \otimes ((h_1 * k_1)^{-1} * G') \otimes ((h_2 * k_2)^{-1} * G')$   
 $= ((h_1 * k_1) * G') \otimes ((h_1 * k_1)^{-1} * G') \otimes ((h_2 * k_2) * G') \otimes ((h_2 * k_2)^{-1} * G')$   
 $= ((h_1 * k_1) * ((h_1 * k_1)^{-1} * (h_2 * k_2) * (h_2 * k_2)^{-1}) * G'$   
 $= e * G' = G'$   
 Thus  $[H * K, H * K] \subseteq G'$ .  
 The convex  
 Let  $[H * K, H * K] \subseteq G'$ , for all  $(h_1 * k_1) * G', (h_2 * k_2) * G'$  of  $H * K / G'$ . Thus  $((h_1 * k_1) * (h_2 * k_2)) * G' \otimes ((h_1 * k_1) * (h_2 * k_2))^{-1} * G'$   
 $= ((h_1 * k_1) * (h_2 * k_2)) * G' \otimes ((h_1 * k_1)^{-1} * (h_2 * k_2)^{-1}) * G'$   
 $= (h_1 * k_1) * (h_2 * k_2) * (h_1 * k_1)^{-1} * (h_2 * k_2)^{-1} * G'$   
 $= (h_1 * k_1) * (h_2 * k_2) * (h_1 * k_1)^{-1} * (h_2 * k_2)^{-1}$   
 $= [(h_1 * k_1), (h_2 * k_2)] \in G'$

$$(h_1 * k_1) * (h_2 * k_2) * (h_1 * k_1)^{-1} * (h_2 * k_2)^{-1} * G' = G'$$

$$((h_1 * k_1) * G') \otimes ((h_2 * k_2) * G') = G'$$

Thus

$$\begin{aligned} ((h_1 * k_1) * (h_2 * k_2)) * G' &= ((h_2 * k_2) * (h_1 * k_1)) * G' \\ ((h_1 * k_1) * G') \otimes ((h_2 * k_2) * G') &= ((h_2 * k_2) * G') \otimes ((h_1 * k_1) * G') \end{aligned}$$

**Theorem 11:**

Let  $(H * K)'$  be the normal subgroup of the group  $H * K$ . Then  $(H * K) / (H * K)'$  is an M-group if and only if  $[H * K, H * K] \subseteq (H * K)'$ .

**Proof:**

(1) Suppose that  $(H * K) / (H * K)'$  is an M-group to prove  $[H * K, H * K] \subseteq (H * K)'$ .

Let  $a, b \in H * K$ , to show  $[a, b] \in (H * K)'$  since  $a, b \in H * K$  so  $a \in (H * K)'$ ,  $b \in (H * K)'$

$$\begin{aligned} (a(H * K)') (b(H * K)') &= (b(H * K)') (a(H * K)') \\ (ab) (H * K)' &= (ba) (H * K)' \quad aba^{-1}b^{-1} \in (H * K)' \end{aligned}$$

$[a, b] \in (H * K)'$ , where  $a = h_1 * k_1$ ,  $b = h_1 * k_1$ ,  $h_1, h_2 \in H$ ,  $k_1, k_2 \in K$ .

Thus  $[H * K, H * K] \subseteq (H * K)'$ .

(2) Suppose that  $[H * K, H * K] \subseteq (H * K)'$

$$[a, b] \in (H * K)', \forall a, b \in H * K$$

$$aba^{-1}b^{-1} \in (H * K)'$$

$$(a, b)(ba)^{-1} \in (H * K)'$$

$$(a, b)(H * K)' = (ba)(H * K)'$$

$(H * K) / (H * K)'$  is commutative, by Theorem (8)

$(H * K) / (H * K)'$  is nilpotent by Theorem (4).

$(H * K) / (H * K)'$  is M-group by Theorem (6).

**Corollary 12:**

Let  $S_n$  be a symmetric group and  $H_1, H_2, \dots, H_n$  are normal subgroup of  $S_n$ . Then  $S_n / (H_1 \cap H_2 \cap \dots \cap H_n)$  is M-group.

**Proof:** Clear.

**Example 13:**

$S_4$  is the symmetric group of order 24, the subgroup

$H_{32} = A_4 = \{i, (123), (132), (124), (134), (234), (243), (142), (143), (12)(34), (13)(24), (14)(23)\}$  of order 12 is a normal subgroup of  $S_4$ , thus  $S_4/A_4$  is an M-group.

**Example 14:**

Let  $GL(4, R)$  be the general linear group of non-singular  $4 \times 4$  matrices over  $R$  and let  $SL(4, R)$  be the special linear group of  $4 \times 4$  matrices each having determinant 1. Then  $GL(4, R) / SL(4, R)$  is an M-group, hence it is a solvable group.

**Solution:**

In the beginning we prove that  $SL(4, R)$  is the derived group of  $GL(4, R)$ . We use the result that if  $x, y$  are  $n \times n$  matrices then  $\det(xy) = \det(x)\det(y)$ .

This relation asserts that mapping  $x \rightarrow \det(x)$  ( $x \in GL(4, R)$ ) is a homomorphism of  $GL(4, R)$  into  $R^*$ , the multiplicative group of non-zero real numbers.

The kernel of this homomorphism is precisely  $SL(4, R)$  which is a normal subgroup of  $GL(4, R)$ .

Furthermore, since  $R^*$  is abelian  $[GL(4, R)]' \subseteq SL(4, R)$ . The difficult part of our argument is to show that  $SL(4, R) \subseteq [GL(4, R)]'$  or equivalently, to show that every  $4 \times 4$  matrices of determinant 1 is a product of commutators, we consider some simple matrices of determinant 1 namely:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4 \times 4}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & s & 1 \end{pmatrix}_{4 \times 4}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & r \\ 0 & 0 & 0 & 1/t \end{pmatrix}_{4 \times 4} \dots (1)$$

(where  $r, s, t \in R, t \neq 0$ ).

The following matrix calculation, then, shows that these matrices are actually commutators:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -r \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -s & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & s & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & 2t \\ 0 & 0 & 2t^2 & t^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/(3t) & 2/(3t^2) \\ 0 & 0 & 2/(3t) & -1/(3t)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1/t \end{pmatrix}$$

The matrices of (1) also generate  $GL(4,R)$  since if  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \in SL(4,R)$  then

$a, d, b, c \in R$  and  $ad - bc = 1$  and if  $c \neq 0$  we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (a-1)/c \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (d-1)/c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And if  $c = 0$  we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b/a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus  $SL(4,R)$  is generated by commentators and so  $SL(4,R) \subseteq [GL(4,R)]'$ . Then  $SL(4,R)$  is the derived group of  $GL(4,r)$ . Now  $GL(4,R) / SL(4,R)$  is M-group by Theorem (6) and solvable by Theorem (5)

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## حول تمثيلات الزمر - M

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### الخلاصة:

الهدف الرئيسي من البحث هو دراسة حول تمثيلات الزمر الاحادية وتقنية المميزات (الشواخص) في تمثيلات الزمر الاحادية. رمزنا للزمر الاحادية بالزمر-M. ثم بحثنا العلاقة بين الزمر الاحادية والزمرة القابلة للحل.

أعطينا عدد من التطبيقات مثلاً اي زمرة من الرتبة 297 تكون زمرة احادية وقابلة للحل لأي زمرة G تكون زمرة القسمة  $G/G'$  زمرة احادية كحالة خاصة  $G = S_n, SL(4, R)$ .

الكلمات المفتاحية: نظرية التمثيل، زمرة أحادية، زمرة قابلة للحل، تمثيل الزمرة الاحادية.