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The Importance and Interaction Indices of Bi-Capacities Based on Ternary-Element Sets

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Abstract:

Grabisch and Labreuche have recently proposed a generalization of capacities, called the bi-capacities. Recently, a new approach for studying bi-capacities through introducing a notion of ternary-element sets proposed by the author. In this paper, we propose many results such as bipolar Mobius transform, importance index, and interaction index of bi-capacities based on our approach.

Key words: Bi-capacities; the bipolar Mobius transform; importance index; interaction index**.**

Introduction:

The concept of bi-capacity has recently been proposed by Grabisch and Labreuche [1] as a generalization of capacity [2] (fuzzy measure [3, 4, 5,6, 7] or non-additive measure [8]) in the context of decision making, who consider the case where scores are expressed on a bipolar scale, i.e. having a central neutral level, usually 0. Grabisch and Labreuche [1] have laid down the basis for the main concepts around bi-capacities, among them the Mobius representation, importance index, and interaction index. Other remarkable works on bi-capacities include the one of Fujimoto and Murofushi [9], who defined the Mobius transform of bi-capacities under the name of bipolar Mobius transform in order to avoid the complicated expression of the Choquet integral in terms of the Mobius transform given in [10].

In [11], the author has proposed a new approach for studying bi-capacities through introducing a notion of ternary-element sets. This approach is alternative approach from that defined by Grabisch and Labreuche [1] and allows a simple way to prove new results on bi-capacities as it was done for capacities. Consequently, we propose in this paper many results such as bipolar Mobius transform, importance index, and interaction index based on our approach.

The structure of the paper is as follows: in the next section we recall the definition of bi-capacities based on ternary-element sets. Section 3 presents the bipolar Mobius transform based on our approach. In sections 4, we propose importance and interaction indices of bi-capacities based on ternary element sets. The paper ends with some conclusions. Throughout the paper, the universal set $X = \{1,2 \cdots, n\}$ denotes a finite set of n elements (states of nature, criteria, individuals, etc), and we will consider R or $[-1, 1]$ with 0 as neutral level that will be considered as prototypical bipolar scales.

Bi-Capacities Baed on Ter-Element Sets

In this section, we begin by recalling the basic concepts of ternary-elementset (or simply ter-element set) and the equivalent definition of bi-capacities based on ternary element sets (for more details, see [11]). We consider every element $i \in X$ that has either a positive effect (i.e., i is positively important criterion of weighted evaluation not only alone but also is interactive with others), or a negative effect (i.e., i is negatively important criterion), or has no effect (i.e., i is criterion at neutral level). Hence, we represent the element i as i^+whenever i is positively important, as i^- whenever i is negatively important, and as $i^\emptyset$ whenever i is neutral, and we call this element a ternary-element (or simply ter-element). The ternary-element set (or simply ter-element set) is the set which contains only out of $i^+, i^-, and\ i^\emptyset$ for all $i,\ i \in \{1,2 \cdots, n\}$.

Thus, in our model we consider the set of all possible combinations of ternary elements of n criteria given by $T(X) = \{\{\tau_1, \dots, \tau_2\} \mid \forall\, \tau_i \ \in \{i^+, i^-, i^\emptyset\},\ i = 1, \dots, n\}$ which corresponds to $Q(X)$ in the notation of classical bi-capacities ([1]).

We have $T\,(X)$ can be identified with $\{-1,0,1\}^n$, hence $|T\,(X)| = \ 3^n$. Also, simply remarked that for any ter-element set $A\ \in\ T\,(X)$, A is equivalent to a ternary alternative $(\tau_1, \dots, \tau_n)$ with $\tau_i = 1\ if\ i^+ \in A\ ,\ \tau_i = 0\ if\ i^\emptyset\ \in A, and\ \tau_i = -1\ if\ i^- \in A,\ \forall\ \ i = 1,2 \cdots, n$.

We introduce the order relation $\sqsubseteq$ between ter-element sets of $T\,(X)$ as follows.

Definition 1 Let $T\,(X)$ be the set of all ter-element sets and $A, B \in\ T\,(X)$.

Then, $A\ \sqsubseteq\ B\ iff\ i\ \in\ X,$

$$``if\ i^+\ \ A\ implies\ i^+\ \ \ B\ "\ ,and\ ``if\ i^\emptyset \in\ A\ implies\ i^+\ or\ i^\emptyset \in B\ " \dots \dots (1)$$

Note that, $X^- = \{1^-, 2^- \cdots, n^-\} \sqsubseteq A$ and $X^+ = \{1^+, 2^+ \cdots, n^+\} \sqsupseteq A\ ,\ \forall\, A \in\ T(X).$

The following definition is equivalent definition of bi-capacities based on notion of ter-element sets.

Definition 2 Let $T\,(X)$ be the set of all ter-element sets. A set function $v:\ T\,(X) \to\ [-1, 1]$, is called bi-capacity based on the ter-element sets if it satisfies the following requirements:

(i) $v(X^-) = v(\{1^-, 2^- \cdots, n^-\}) = -1,\ v(X^\emptyset) = v(\{1^\emptyset, 2^\emptyset \cdots, n^\emptyset\}) = 0,\ and\ \ v(X^+) = v(\{1^+, 2^+ \cdots, n^+\}) = 1.$

(ii) $\forall\, A, B \in\ T(X)$, $A \sqsubseteq B\ implies\ v(A) \le v(B).$

Bi-capacities are functions defined on the structure of the underlying partially ordered set [12]. There are several orders on the structure $Q(X)$ that have been introduced by Grabisch and Labreuche [1] and Bilbao et al. [13]. Here, we introduce an order on the structure $T\,(X)$ different from the order $\sqsubseteq$ described in definition 1.

We consider the following definition of an order on $T\,(X)$ which is equivalent to Bilbao order on bi-cooperative game [13]. For convenience, we denote by $\subseteq$ the order relation defined on $T\,(X)$ as in the classical order relation, and we will use the order $\subseteq$ on $T\,(X)$ to establish our next results of this research.

Definition 3 Let $T\ (X)$ be the set of all ter-element sets and $A, B \in T\ (X)$. Then, $A \subseteq B\ iff\ \forall\, i \in X,$
$``if\ i^+ \in A\ implies\ i^+ \in B\ "$
, and
$``if\ i^+ \in A\ implies\ i^+ \in B"$ (2)

- the number of positively important elements i^+ of the ter-element set $A \in T\ (X)$, denoted by a^+, is defined as $a^+ = \sum_{i=1}^{n} \chi_A(i^+)$

where,

$$\chi_A(i^+) = \begin{cases} 1 & if\ i^+ \in A \\ 0 & if\ i^+ \notin A. \end{cases}$$

- the number of negatively important elements i^- of the ter-element set $A \in T\ (X)$, denoted by a^-, is defined as $a^- = \sum_{i=1}^{n} \chi_A(i^-)$ where,

$$\chi_A(i^-) = \begin{cases} 1 & if\ i^- \in A \\ 0 & if\ i^- \notin A. \end{cases}$$

- the cardinality of the ter-element set $A \in T\ (X)$, is

$$a = |A| = a^+ + a^- \qquad (3)$$

Bipolar Mobius Transforms of Bi-Capacities Based on Ter-Element Sets

The Mobius transform is important concept for capacities since the Mobius transform represents the coordinates of capacities in the basis of unanimity game. Moreover, the Choquet integral has a very simple expression when the Mobius transform is used. In [1], Grabisch M. and Labreuche Ch. have defined the Mobius transform for bi-capacity. Another equivalent representation of bi-capacities has been proposed by Fujimoto and Murofushi [9] who called the bipolar Mobius transform of a bi-capacity v.

In this section, we define equivalent expression of bipolar Mobius transform for bi-capacities based on ter-element sets. We have the order (see, Definition 3) $A, B \in T\ (X)$ with $B \subseteq A$. The bipolar Mobius transform of this order as follows:

Definition 4 To any bi-capacity based on ter-element set v on $T\ (X)$, another function $b_v : T\ (X) \to R$ can be associated by

Furthermore, in this order:

$$v(A) = \sum_{B \subseteq A} b_v(B), \quad \forall\, A \in T\ (X) \qquad \ldots\ldots\ (4)$$

The function b_v is called the bipolar Mobius transform based on ter-element set of v, and is given by the following proposition:

Proposition 1 Let $v : T\ (X) \to R$ be a bi-capacity based on ter-element set and $b_v : T\ (X) \to R$ the bipolar Mobius transform of v. Then,

$$b_v(A) = \sum_{B \subseteq A} (-1)^{a-b}\, v(B),$$

$$\forall\, A \in T\ (X) \qquad (5)$$

Proof:

From definition 4, we have

$$v(B) = \sum_{C \subseteq B} b_v(C), \qquad \forall\, B \in T\ (X)$$

Then,

$$\sum_{B \subseteq A} (-1)^{a-b}\, v(B)$$
$$= (-1)^a \sum_{B \subseteq A} (-1)^b \quad v(B)[\sum_{C \subseteq B} b_v(C)]$$

where the order $\subseteq$ is defined by Equation (2), and the cardinality of the ter-element sets is defined by Equation (3).

$$= (-1)^a \sum_{C \subseteq A} b_v(C) \quad v(B)[\sum_{B, C \subseteq B \subseteq A} (-1)^b]$$

$$= (-1)^a \sum_{C=A} b_v(C) \left[\sum_{B,C\subseteq B\subseteq A} (-1)^b\right] + (-1)^a \sum_{C\subset A} b_v(C) \left[\sum_{B,C\subseteq B\subseteq A} (-1)^b\right]$$
$$= (-1)^a\, b_v(A)(-1)^a + (-1)^a \sum_{C\subset A} b_v(C) \quad (0)$$
$$= b_v(A) \qquad \blacksquare$$

Importance and Interaction Indices of Bi-Capacities Based on Ter-Element Sets

Derivation of Bi-Capacities Based on Ter-Element Sets

To introduce the interaction index based on ter-element set, we need to define the notion of the derivation of a bi-capacity based on ter-element set. The notion of derivation of a classical capacity is shown in [14] and [15]. Grabisch and Labreuche [1] extended the notion of derivation of capacities to that of classical bi-capacities. In this subsection, we define corresponding notions for bi-capacities based on ter-element sets. The derivation of a bi-capacity v based on ter-element set on $T(X)$ is given by the following definition:

Definition 5 Let v be a bi-capacity based on ter-element set on $T(X)$. For $B \in T(X)$, the B-derivative at point $A \cup (X^-\backslash(B \cap B^c))$ is given by

$$\Delta_B v(A \cup (X^-\backslash(B \cap B^c))) = \sum_{L\subseteq B} (-1)^{b^+-l^++l^-}\; v(A \cup L) \qquad ..(6)$$
$$\forall\, A \subseteq (X^+\backslash(B \cup B^c)).$$

The particular cases of this definition when $b = |B| = 1$ are the derivation of a bi-capacity with respect to a positively important criterion i^+:

$$\Delta_{i^+} v(A \cup \{\dots,(i-2)^-,(i-1)^-,i^\emptyset,(i+1)^-,(i+2)^-,\dots\}) = v\big(A \cup \{\dots,(i-2)^\emptyset,(i-1)^\emptyset,i^+,(i+1)^\emptyset,(i+2)^\emptyset,\dots\}\big) - v(A),$$

and the derivative of a bi-capacity with respect to a negatively important criterion i^-:

$$\Delta_{i^-} v(A \cup \{\dots,(i-2)^-,(i-1)^-,i^\emptyset,(i+1)^-,(i+2)^-,\dots\} = v(A) - v\big(A \cup \{\dots,(i-2)^\emptyset,(i-1)^\emptyset,i^-,(i+1)^\emptyset,(i+2)^\emptyset,\dots\}\big)$$

The following proposition gives the expression of the derivation in terms of the bipolar Mobius transform b_v:

Proposition 2 For any $B \in T(X)$, we have

$$\Delta_B v(A \cup (X^-\backslash(B \cap B^c))) = (-1)^{b^-} \sum_{L\subseteq A} b_v(B \cup L) \qquad .....(7)$$
$$\forall\, A \subseteq (X^+\backslash(B \cup B^c)).$$

Proof: From definition 5 and definition 4, for any $B \in T(X)$ and any $A \subseteq (X^+\backslash(B \cup B^c))$,

$$\Delta_B v(A \cup (X^-\backslash(B \cap B^c))) = \sum_{L\subseteq B} (-1)^{b^+-l^++l^-}\; v(A \cup L)$$
$$= \sum_{L\subseteq B} (-1)^{b^+-l^++l^-} \sum_{K\subseteq A\cup L} b_v(K) = \sum_{L\subseteq B} (-1)^{b^+-l^++l^-} \sum_{\substack{K_1\subseteq A\\ K_2\subseteq L}} b_v(K_1 \cup K_2)$$
$$= \sum_{K_1\subseteq A} \sum_{K_2\subseteq L} \sum_{L\subseteq B} (-1)^{b^+-l^++l^-} \sum_{\substack{K_1\subseteq A\\ K_2\subseteq L}} b_v(K_1 \cup K_2)$$

Since $b = b^+ + b^-$ and $l = l^+ + l^-$, then

$$(-1)^{b^+-l^++l^-} = (-1)^{2l^--b^-}(-1)^{b-l}.$$

Thus, $\Delta_B v(A \cup (X^-\backslash(B \cap B^c)))$

$$= (-1)^{2l^- - b^-} \sum_{K_1 \subseteq A} \sum_{K_2 \subseteq B} \sum_{L : K_2 \subseteq L \subseteq B} (-1)^{b-l} b_v(K_1 \cup K_2)$$

$$= (-1)^{2l^- - b^-} \sum_{K_1 \subseteq A} \sum_{K_2 \subseteq B} \sum_{l=0}^{b-k_2} \binom{b-k_2}{l} (-1)^l \ b_v(K_1 \cup K_2)$$

$$= (-1)^{2l^- - b^-} \sum_{K_1 \subseteq A} \sum_{K_2 \subseteq B} (1-1)^{b-k_2} \ b_v(K_1 \cup K_2)$$

$$= (-1)^{2l^- - b^-} \sum_{K_1 \subseteq A} b_v(K_1 \cup B)$$

$$= (-1)^{-b^-} \sum_{L \subseteq A} b_v(L \cup B). \qquad \blacksquare$$

Importance and Interaction Indices Based on Ter-Element Sets

The Shapley interaction index related to a capacity, among a combination of criteria has been introduced by Grabisch [16] as a natural generalization of the Shapley importance value [17]. Later, Grabisch and Labreuche [1] generalized the Shapley interaction index to a bi-capacity v on $Q(X)$.

In this subsection, we propose equivalent definition of the Shapley interaction index for bi-capacities based on ter-element sets. The definition of Shapley interaction index with respect to a bi-capacity v based on ter-element set on $T(X)$ is as follows.

Definition 6 Let v be a bi-capacity based on ter-element set on $T(X)$. For $B \in T(X)$ the interaction index with respect to B is defined by

$$I_B = \sum_{A \subseteq X^+ \setminus (B \cup B^c)} \frac{(n-b-a)!\, a!}{(n-b+1)!} \Delta_B v(A \cup (X^- \setminus (B \cap B^c))) \qquad \text{….(8)}$$

The following numerical example illustrates the definition of interaction index with respect to a bi-capacity v based on ter-element set on $T(X)$:

Example 1: We consider $X = \{1, 2, 3\}$ and $B = \{1^+, 2^-, 3^\emptyset\} \in T(X)$. Applying the formula (8) with $X^+ \setminus (B \cup B^c) = \{1^\emptyset, 2^\emptyset, 3^+\}$ and $X^- \setminus (B \cap B^c) = \{1^\emptyset, 2^\emptyset, 3^-\}$ we get:

$$I_{\{1^+,2^-,3^\emptyset\}} = \frac{1!\,0!}{2!} \Delta_{\{1^+,2^-,3^\emptyset\}} v(\{1^\emptyset, 2^\emptyset, 3^-\}) + \frac{0!\,1!}{2!} \Delta_{\{1^+,2^-,3^\emptyset\}} v(\{1^\emptyset, 2^\emptyset, 3^+\})$$

$$= \frac{1}{2}\left[-v(\{1^\emptyset, 2^\emptyset, 3^-\}) + v(\{1^+, 2^\emptyset, 3^-\}) - v(\{1^+, 2^-, 3^-\}) + v(\{1^\emptyset, 2^-, 3^-\})\right] + \frac{1}{2}\left[-v(\{1^\emptyset, 2^\emptyset, 3^+\}) + v(\{1^+, 2^\emptyset, 3^+\}) - v(\{1^+, 2^-, 3^+\}) + v(\{1^\emptyset, 2^-, 3^+\})\right].$$

The special case of the interaction index on singleton of ter-element set (i.e. when $b = |B| = 1$) is the importance index related to a bi-capacity of ter-element set $B \in T(X)$. Therefore, we define the importance index of i when it is positively important i^+ or when it is negatively important i^- as follows.

Definition 7 Let v be a bi-capacity based on ter-element set on $T(X)$. The important index of $\tau_i \in \{i^+, i^-\}$ is defined by

$$\phi_{t_i} = \sum_{A \subseteq \{\dots, (i-2)^+, (i-1)^+, i^\emptyset, (i+1)^+, (i+2)^+, \dots\}} \frac{(n-a-1)!\, a!}{n!} \times \Delta_{t_i} v\big(A \cup \{\dots, (i-2)^-, (i-1)^-, i^\emptyset, (i+1)^-, (i+2)^-, \dots\}\big). \qquad \text{…..(9)}$$

Conclusion:

In our approach [11], we defined a bi-capacity based on ter-element set, satisfying properties similar to the classical definition of bi-capacities [1]. According to this definition and introducing to another order relation equivalent to Bilbao order on bi-cooperative game [13], the bipolar Mobius transform, importance index, and interaction index are appropriately proposed in this paper. The proposed definitions are consistent as generalizations of the Mobius transform, importance index, and interaction index for capacities.

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دليل الأهمية و دليل التفاعل للسعات الثنائية على اساس المجموعات ثلاثية العنصر

جبار عباس غافل

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الخلاصة :

اقترح Grabisch و Labreuche مؤخرا تعميما للسعات ، سميت بالسعات الثنائية. حديثا، نهج جديد لدراسة السعات الثنائية من خلال تقديم مفهوم المجموعات ثلاثية العنصر اقترحت من قبل المؤلف. في هذا البحث، نقترح العديد من النتائج مثل تحويل موباس ثنائي القطب و دليل الأهمية و دليل التفاعل للسعات الثنائية على أساس نهجنا الجديد.

الكلمات المفتاحية: السعات الثنائية، تحويل موباس ثنائي القطب ، دليل الأهمية ، دليل التفاعل