On Fully Stable Banach Algebra Modules Relative to an Ideal

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Received 15/6/2017
Accepted 22/8/2017

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Abstract:
In this paper, the concept of fully stable Banach Algebra modules relative to an ideal has been introduced. Let A be an algebra, X is called fully stable Banach A-module relative to ideal K of A, if for every submodule Y of X and for each multiplier θ: Y → X such that θ(Y) ⊆ Y + KX. Their properties and other characterizations for this concept have been studied.

Keywords: Banach Algebra modules, fully stable modules, fully stable Banach Algebra modules relative to ideal.

Introduction:
The theory of Banach algebras(BA) is an abstract mathematical theory. BA started in the early twentieth century, when abstract concepts and structures were introduced, both the mathematical language and practice were transformed. Let A be a non-empty set, A is called an algebra if (1) (A,+) is a vector space over a field F, (2) (A,+,∗) is a ring and, (3) (aα) ∗ b = α (a ∗ b) = a ∗ (ab) for every α ∈ F, for every a, b ∈ A [1]. In [2]S. Burris and H. P. Sankappanavar show that a ring R is an algebra < R,+,−,0 > where + and · are binary, − is unary and 0 is nullary satisfying, < R,+,−,0 >is an aelian group, < R,> is a semigroup and x.(y + z) = (x.y) + (x.z) and (x + y).z = (x.z) + (y + z). In [3] they gave the definition of Banach left module as follows: let A be an algebra, a Banach space E is called a Banach left A-module if E is a left module over algebra A, and ∥a.x∥ ≤ ∥a∥.∥x∥ (a ∈ A, x ∈ E) [3]. In [4] a multiplier (homomorphism) mean, a map from a left Banach A−module X in to a left Banach A−module Y (A is not necessarily commutative) if it satisfies T(a.x) = a.Tx for all a ∈ A, x ∈ X. A submodule N of an R−module M is called to be stable, if f(N) ⊆ N for each R−homomorphism from N to M. In case each submodule of it is stable,M is called a fully stable module[5]. In [6], M. J. Mohammed Ali and M Ali gave the definition of fully stable Banach A-module as follows: a Banach algebra module M is called fully stable Banach A-module if for every submodule N of M and for each multiplier θ: N → M satisfyθ(N) ⊆ N. In this paper, we introduce the concept of fully stable relative to ideal for Banach A-module.

A Banach algebra module M is called fully stable Banach A−module relative to ideal K of A if for every submodule N of M and for each multiplier θ: N → M such that θ(N) ⊆ N + KM. Structure of fully stable Banach A- module relative to an ideal in term of their elements is considered, see (2.8). Studying Baer criterion gives another characterization of fully stable Banach A-module relative to ideal K of A, see proposition (2.2).

Main Results:
Definition 2.1: Let X be Banach A−module, X is called fully stable Banach A−module relative to ideal K of A, if for every submodule N of X and for each multiplier θ: N → X satisfy θ(N) ⊆ N + KX. It is clear that every fully stable Banach A−module is fully stable Banach A−module relative to an ideal.

In [7] for a nonempty subset M in a left Banach A−moduleX, the annihilator anna(M) of M is anna(M) = {a ∈ A; a.x = 0 for all x ∈ M}. In [6], Let X a Banach A−module, N x = {n x | n ∈ N, x ∈ X} and P y = {p y | p ∈ P, y ∈ X} annaN x = {a ∈ A , a.n x = 0, ∀ n x ∈ N x} and annaP y = {a ∈ A, a.p y = 0, ∀ p y ∈ P y}. The following proposition we gave another characterization of fully stable Banach A-modules relative to an ideal.

Proposition 2.2: X is fully stable Banach A−module if and only if for each x, y ∈ X and N x,K y subsets of X, y ∉ N x + KX implies anna(N y) ⊈ anna(P y).

Proof:- Suppose that X is fully stable Banach A−module relative to ideal K of A, there exists
Then Baer criterion holds for cyclic submodules of another characterization of fully stable Banach algebra

In the following proposition and its corollary there is an element 𝑥 ∈ 𝑋 such that 𝑎(𝑛, 𝑥) = 0, hence 𝜽 is well defined. It is clear that 𝜽 is an 𝑂 − multiplier. By the assumption, there exists an element 𝑓 ∈ 𝑌 such that 𝑎(𝑛, 𝑥) = 𝑥 ∈ 𝑋 for each 𝑋 ∈ 𝑋, which implies that, in particular, 𝑓(𝑛, 𝑥) = 𝑥 ∈ 𝑋 for each 𝑋 ∈ 𝑋;

therefore, 𝑎𝑛𝑛(𝑎(𝑛, 𝑥)) ∈ 𝑋 + 𝑋, hence 𝑎𝑛𝑛(𝑎(𝑛, 𝑥)) = 𝑋 + 𝑋. Conversely, assume that 𝑎𝑛𝑛(𝑎(𝑛, 𝑥)) = 𝑋 + 𝑋, then (𝑛, 𝑥) = 𝑥 ∈ 𝑋 for each 𝑋 ∈ 𝑋.

Corollary 2.3: Let 𝑋 be a fully stable Banach 𝐴 − module relative to an ideal 𝐾 of 𝐴. Then for each 𝑥, 𝑦 in 𝑋, 𝑎𝑛𝑛(𝑃) = 𝑎𝑛𝑛(𝑋) satisfies Baer criterion, that is for every submodule 𝑁 of 𝑋 satisfies Baer criterion, that is for every submodule 𝑁 of 𝑋 and 𝐴 − multiplier 𝜽: 𝑁 → 𝑋, there exists an element 𝑎 in 𝐴 such that

Recall that a left Banach 𝐴 − module 𝑋 is n − generated for 𝑛 ∈ 𝑁 if there exists 𝑥₁, ..., 𝑥ₙ ∈ 𝑋 such that each 𝑥 ∈ 𝑋 can be represented as 𝑥 = ∑ 𝑎ₖ 𝑥ₖ for some 𝑎₁, ..., 𝑎ₙ ∈ 𝐴. A cyclic module is just a 1-generated [8].

In the following proposition and its corollary another characterization of fully stable Banach 𝐴 − module relative to ideal 𝐾 is given.

Proposition 2.5: Let 𝑋 be a Banach 𝐴 − module. Then Baer criterion holds for cyclic submodules of 𝑋 if and only if 𝑎𝑛𝑛(𝑋) = 𝑋 + 𝑋 for each 𝑋 ∈ 𝑋.

Proof: Assume that Baer criterion holds. Let 𝑦 ∈ 𝑎𝑛𝑛(𝑋). Define 𝜽: 𝑁 → 𝑋 by 𝜽(𝑛, 𝑥) = 𝑎𝑝, for all 𝑛 ∈ 𝐴. Let 𝑎₁, 𝑛₁ = 𝑎₂, 𝑛₂, 𝑒𝑛𝑡ℎ𝑒𝑟 𝐶, 𝑑 ∈ 𝑎𝑛𝑛(𝑛), hence (𝑎₁ − 𝑎₂) ∈ 𝑎𝑛𝑛(𝑃).
A –module homomorphism \( \theta : X \rightarrow X \), such that \( \theta \circ i = \varphi \) and \( \| \theta \| \leq \alpha \) where \( i \) is an isometry from submodule \( N \) of \( X \). We shall say that \( X \) is quasi injective if it is quasi \( \alpha \)-injective for some \( \alpha \).

**Definition 2.10:** Let \( A \) be a unital Banach algebra and let \( \alpha > 1 \). A –module \( X \) is called quasi \( \alpha \)-injective relative to an ideal \( K \) of \( A \) if, \( \varphi : N \rightarrow X \) is \( A \) –module homomorphism such that \( \| \varphi \| \leq 1 \), there exists \( A \) –module homomorphism \( \theta : X \rightarrow X \), such that \( (\theta \circ i)(n) = \varphi(n) \in KX \) and \( \| \theta \| \leq \alpha \) where \( i \) is an isometry from submodule \( N \) of \( X \) to \( X \). We shall say that \( X \) is quasi injective relative to ideal if it is quasi \( \alpha \) – injective relative to ideal for some \( \alpha \).

The relation between quasi \( \alpha \)-injective Banach \( A \) –module relative to ideal and fully stable Banach \( A \) –module relative to an ideal \( K \) of \( A \) has been given in the following proposition.

**Proposition 2.11:** If \( X \) is fully stable Banach \( A \) –module relative to ideal then \( X \) is quasi injective Banach \( A \) –module relative to ideal.

**Proof:** Let \( N \) be submodule of \( X \), let \( \alpha > 1 \) and \( f : N \rightarrow X \) be any \( A \) –module homomorphism such that \( \| f \| \leq 1 \). Since \( X \) is a fully stable Banach \( A \)-module relative to \( K \), then \( f(N) \subseteq N + KX \), thus there exist \( t \in A \) such that \( f(n) = tn + w \). Define \( g : X \rightarrow X \) by \( g(x) = tx \), it is clear that \( g \) is a well defined and \( A \) –module homomorphism. Now \( f(x) - g(x) = t x + w - tx = w \in KX \) and for each \( y \in N \), \( (f \circ i)(y) - g(y) = f(y) - g(y) \in KX \), where \( i \) is isometry, and \( \| g \| \leq \alpha \) for some \( \alpha \).

**References:**


