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On Fully Stable Banach Algebra Modules Relative to an Ideal

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Abstract:

In this paper, the concept of fully stable Banach Algebra modules relative to an ideal has been introduced. Let A be an algebra, X is called fully stable Banach A-module relative to ideal K of A, if for every submodule Y of X and for each multiplier $\theta: Y \longrightarrow X$ such that $\theta(Y) \subseteq Y + KX$. Their properties and other characterizations for this concept have been studied.

Keywords: Banach Algebra modules, fully stable modules, fully stable Banach Algebra modules relative to ideal.

Introduction:

The theory of Banach algebras(BA) is an abstract mathematical theory. BA started in the early twentieth century, when abstract concepts and structures were introduced, both the mathematical language and practice were transformed. Let A be a non-empty set, A is called an algebra if (1) $(A,+,.)$ is a vector space over a field F, (2) $(A,+,\circ)$ is a ring and, (3) $(\alpha a) \circ b = \alpha\ (a \circ b) = a \circ (\alpha b)$ for every $\alpha \in F$, for every $a, b \in A$ [1]. In [2]S. Burris and H. P. Sankappanavar show that a ring R is an algebra $< R, +, \cdot, -, 0 >$ where $+$ and $\cdot$ are binary , $-$ is unary and 0 is nullary satisfying, $< R, +, -, 0 >$is an a belian group, $< R, . >$ is a semigroup and $x.(y+z) = (x.y) + (x.z)$ and $(x+y).z = (x.z) + (y+z)$. In [3] they gave the definition of Banach left module as follows: let A be an algebra, a Banach space E is called a Banach left A-module if E is a left module over algebra A, and $\|a.x\| \leq \|a\|\|x\| (a \in A, x \in E)$ [3]. In [4] a multiplier (homomorphism) mean, a map from a left Banach $A-$module X in to a left Banach $A-$module Y (A is not necessarily commutative) if it satisfies $T(a.x) = a.Tx$ for all $a \in A, x \in X$. A submodule N of an $R-$module M is called to be stable, if $f(N) \subseteq N$ for each $R-$homomorphism from N to M. In case each submodule of it is stable,M is called a fully stable module[5]. In [6], M. J. Mohammed Ali and M Ali gave the definition of fully stable Banach A-module as follows: a Banach algebra module M is called fully stable Banach A-module if for every submodule N of M and for each multiplier $\theta: N \longrightarrow M$ satisfy$\theta(N) \subseteq N$. In this paper, we introduce the concept of fully stable relative to ideal for Banach A-module.

A Banach algebra module M is called fully stable Banach A-module relative to ideal K of A if for every submodule N of M and for each multiplier $\theta: N \longrightarrow M$ such that $\theta(N) \subseteq N + KM$. Structure of fully stable Banach A- module relative to an ideal in term of their elements is considered, see (2.8). Studying Baer criterion gives another characterization of fully stable Banach A-module relative to ideal K of A, see proposition (2.2).

Main Results:

Definition 2.1: Let X be Banach $A-$ module, X is called fully stable Banach $A-$module relative to ideal K of A, if for every submodule N of X and for each multiplier $\theta: N \longrightarrow X$ satisfy $\theta(N) \subseteq N + KX$. It is clear that every fully stable Banach $A-$module is fully stable Banach $A-$module relative to an ideal.

In [7] for a nonempty subset M in a left Banach $A-$module$X$, the annihilater $ann_A(M)$ of M is $ann_A(M) = \{a \in A\,; a.x = 0\ for\ all\ x \in M\}$. In [6], Let X a Banach $A-$module, $N_x = \{n_x |\ n \in N,\ x \in X\}$ and $P_y = \{p_y | p \in P,\ y \in X\}$ $ann_A N_x = \{a \in A\,,\ a.n_x = 0,\ \forall\ n_x \in N_x\}$ and $ann_A P_y = \{a \in A,\ a.p_y = 0,\ \forall p_y \in P_y\}$. The following proposition we gave another characterization of fully stable banach A-modules relative to an ideal.

Proposition 2.2: X is fully stable Banach $A-$module if and only if for each $x, y \in X$ and N_x, K_y subsets of X, $y \notin N_x + KX$ implies $ann_A(N_x) \nsubseteq ann_A(P_y)$

Proof:- Suppose that X is fully stable Banach $A-$module relative to ideal K of A, there exists

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$x, y \in X$ such that $y \notin N_x + KX$ and $ann_A(N_x) \subseteq ann_A(P_y)$. Define $\theta: < N_x > \longrightarrow X$ by $\theta(a.n_x) = a.p_y$, for all $a \in A$, if $a.n_x = 0$ then $a \in ann_A(N_x) \subseteq ann_A(P_y)$. This implies that $a.p_y = 0$, hence θ is well defined, clear θ is a multiplier ,because X is fully stable relative to an ideal , there exists an element $t \in A$ such that $\theta(m_x) = tm_x + w$, $for\ each\ m_x \in N_x, w \in KX$. In particular, $p_y = \theta(n_x) = tn_x + w \in N_x + KX$. Which is a contradiction. Thus X is fully stable Banach module relative to an ideal. Conversely, assume that there is a subset N_x of X and a multiplier $\theta: < N_x > \longrightarrow X$ such that $\theta(N_x) \nsubseteq N_x + KX$ then there exists an element $m_x \in N_x$ such that $\theta(m_x) \notin N_x + KX$. Let $\in ann_A(N_x)$; therefore, $sn_x = 0$, $s\theta(m_x) = \theta(sm_x) = \theta(stn_x) = \theta(tsn_x) = \theta(0) = 0$. Hence $ann_A(N_x) \subseteq ann_A(\theta(m_x))$, which is a contradiction.

Corollary 2.3: Let X be a fully stable Banach A −module relative to an ideal K of A .Then for each x, y in $X, ann_A(P_y) = ann_A(N_x)$implies $N_x + KX = P_y + KX$

Proof:- Assume that there are two elements x, y in X such that $ann_A(N_x) = ann_A(P_y)$ and $N_x + KX \neq P_y + KX$. Then without loss of generality there is an element z_x in N_x not in P_y .By using proposition (2.2), we have $ann_A(P_y) \nsubseteq ann_A(Z_x)$but $ann_A(N_x) \subseteq ann_A(Z_x)$, hence $ann_A(P_y) \nsubseteq ann_A(N_x)$, which is a contradiction.

Definition 2.4: A Banach A −module X is said to satisfy Baer criterion relative to an ideal K of A, if each submodule of X satisfies Baer criterion, that is for every submodule N of X and $A-$ multiplier $\theta: N \longrightarrow X$, there exists an element a in A such that $\theta(n) - an \in KX$ for all $n \in N$.

Recall that a left Banach A −module X is n −generated for $n \in N$ if there exists $x_1, \dots, x_n \in X$ such that each $x \in X$ can represented as $x = \sum_{k=1} a_k . x_k$for some $a_1, \dots, a_n \in A$. A cyclic module is just a 1-generated [8].

In the following proposition and its corollary another characterization of fully stable Banach A −module relative to ideal is given.

Proposition 2.5: Let X be a Banach A −module .Then Baer criterion holds for cyclic submodules of X if and only if $ann_X(ann_A(N_x)) = N_x + KX\ for\ each\ x \in X$.

Proof:- Assume that Baer criterion holds. Let $y \in ann_X(ann_A(N_x))$. Define $\theta: < N_x > \longrightarrow X$ by $\theta(a.n_x) = a.p_y$, for all $a \in A$. Let $a_1 . n_x = a_2 . n_x$, thus $(a_1 - a_2)n_x = 0$, $a_1 - a_2 \in ann_A(N_x)$, hence $(a_1 - a_2) \in ann_A(P_y)$; therefore, $(a_1 - a_2)p_y = 0$, then $a_1 p_y = a_2 p_y$, hence θ is well defined. It is clear that θ is an $A-$ multiplier. By the assumption, there exists an element $t \in A$ such that $\theta(m_x) - tm_x \in KX\ for\ each\ m_x \in N_x$ which implies that, in particular, $p_y - tn_x = \theta(n_x) - tn_x \in KX$; therefore, $ann_X(ann_A(N_x)) \subseteq N_x + KX$, hence $ann_X(ann_A(N_x)) = N_x + KX$. Conversely, assume that $ann_X(ann_A(N_x)) = N_x + KX$, for each $N_x \subseteq X$. Then for each $A-$ multip $\theta: N_x \to X$, and s $\in ann_A(N_x)$, we have $s\theta(n_x) = \theta(sn_x) = 0$. Thus, $\theta(n_x) \in ann_X(ann_A(N_x)) = N_x + KX$, then $\theta(n_x) - t\, n_x \in KX$ for some $t \in A$, thus Baer criterion holds.

Corollary2.6: X is fully stable Banach $A-$ module relative to ideal K of A if and only if $ann_X(ann_A(N_x)) = N_x + KX\ for\ each\ x \in X$.

Recall that an R-module M is multiplication module if each submodule of M is of the form IM for some ideal I of R [9].

Definition 2.7: Banach A −module X is called multiplication A −module if each 1-generated submodule of X is of the form KX for some ideal K of A.

In the following proposition we discuses the relation between full stability and full stability relative to ideal for Banach algebra modules.

Proposition2.8: Let X be a multiplication Banach A −module. Then X is fully stable Banach A −module if and only if it is fully stable Banach A −module relative to ideal K of A.

Proof: Let N be any 1-generated submodule of X and f : N ⟶ X be any R-homomorphism. If N = (0), then it is clear that X is fully stable. Let N ≠ (0), and since X is multiplication A −module, then N = KX, for some non-zero ideal K of A. By hypothesis f(N) ⊆ N + KX = N + N = N. Hence, X is fully stable A −module.

Let A be a unital Banach algebra, let Q be an A-module and let α > 1. We shall then say that Q is α-injective if, whenever *i:M ⟶N* and *φ*:M ⟶Q are A −module homomorphisms such that *i* is an isometry (A −module isomorphism is an isometry A −multiplier) and ‖ *φ* ‖ ≤ 1, there exists *θ* A −module homomorphism *θ:N* ⟶ Q, such that *θ o i* = *φ* and ‖ *θ* ‖ ≤ α. We shall say that Q is uniformly injective if it is α-injective for all α > 1. We shall say that Q is injective if it is α -injective for some α [10].

In the following definition the concept of quasi α-injective has been introduced

Definition 2.9: Let A be a unital Banach algebra and let α >1. A −module X is called quasi α-injective if, *φ* : *N* ⟶ *X* is A −module homomorphisms such that ‖ *φ* ‖ ≤ 1, there exists

A −module homomorphism $\theta: X \longrightarrow X$, such that $\theta\ o\ i = \varphi$ and $\| \theta \| \leq \alpha$ where i is an isometry from submodule N of X. We shall say that X is quasi injective if it is quasi α - injective for some α

Definition 2.10: Let A be a unital Banach algebra and let $\alpha > 1$. A −module X is called quasi α-injective relative to an ideal K of A if, $\varphi : N \longrightarrow X$ is A −module homomorphism such that $\| \varphi \| \leq 1$, there exists A −module homomorphism $\theta: X \longrightarrow X$, such that $(\theta\ o\ i)(n) - \varphi(n) \in KX$ and $\| \theta \| \leq \alpha$ where i is an isometry from submodule N of X to X. We shall say that X is quasi injective relative to ideal if it is quasi α – injective relative to ideal for some α

The relation between quasi α-injective Banach A −module relative to ideal and fully stable Banach A −module relative to an ideal K of A has been given in the following proposition.

Proposition 2.11: If X is fully stable Banach A −module relative to ideal then X is quasi injective Banach A −module relative to ideal..

Proof: Let N be submodule of X, let α> 1 and f : N → X be any A −module homomorphism such that $\| f \| \leq 1$. Since X is a fully stable Banach A-module relative to K, then $f(N) \subseteq N + KX$, thus there exist t ∈ A such that $f(n) = t^* n + w$. Define g: X → X by g (x) = tx, it is clear that g is a well defined and A −module homomorphism. Now $f(x) - g(x) = t^*x + w - t^*x = w \in KX$ and for each y ∈ N, $(f\ o\ i)(y) - g(y) = f(y) - g(y) \in KX$, where i is isometry, and $\| g \| \leq \alpha$ for some α.

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مقاسات بناخ الاجبرا تامة الاستقرارية بالنسبة الى مثالي

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الخلاصة:

مفهوم مقاسات بناخ الاجبرا تامة الاستقرارية بالنسبة الى مثالي قد تم تعريفها في هذا البحث. ليكن A بناخ الاجبرا، مقاس بناخ الاجبرا M يسمى مقاس بناخ الاجبرا تام الاستقرارية بالنسبة الى مثالي K في A، اذا كان لكل مقاس جزئي N في M ولكل $\theta: N \longrightarrow M$ multiplier بحيث $\theta(N) \subseteq N + KM$. تم دراسة صفات وتشخيصات اخرى للمفهوم.

الكلمات المفتاحية: مقاسات بناخ الجبرا، مقاسات تامة الاستقرارية، مقاسات بناخ الجبرا التامة الاستقرارية بالنسبة الى مثالي.