On the Connection between the Dynamical System and the Ellis Compactification with Transitive Pointed System

Lieth A. Majed

Received 28/12/2016
Accepted 12/6/2017

Abstract:
In this paper, some relations between the flows and the Enveloping Semi-group were studied. It allows to associate some properties on the topological compactification to any pointed flows. These relations enable us to study a number of the properties of the principles of flows corresponding with using algebraic properties. Also in this paper proofs to some theorems of these relations are given.

Keywords: Dynamical system, semi-topological semi-group, isomorphisim of H-systems, monoidal compactification, transitive pointed systems, inclusive transitive pointed systems.

Introduction:
Historically the Enveloping semi-group which is a closure of the set of continuous functions on a compact space X was used to study the dynamical system as given by R.Ellis [1]. In 1991 A. Lisan characterized what is called transitive flows which is related with the right topological monidal compactification of the system [2]. In 1992 J. D. Lawson showed that many typical of lows can be recognized by restrictions on the minimal ideals of the enveloping semi-group[3]. In 2013, N. Hindman studied the intersection between the stone-cech compactification with the additive and multiplication operations to present conditions on system that makes the closure of the minimal ideal of stone-cech compactification a right ideal in the case of multiplication operation [4]. Recently in 2014 L. Jones pointed out some close relation between the theory of right topological semi-group and the theory of systems on compact spaces.

The Enveloping semi-group has been established to be an essential tool in the abstract theory of topological dynamical systems. We study this with respect to enveloping semi-group. We are looking to what we called inclusive compactification which is topologically and algebraically isomorphic to the enveloping semi-group, study this by using semi-group property of the universal compactification and give some properties related to these concepts.

Preliminaries:
Let H be a semi-group given with Hausdorff space. For m, n ∈ H we define \( f_n(m) = mn \) with product mn [5]. The function \( f_n \) is a right continuous. The semigroup H is called the right topology. If all \( f_n \) are continuous then the semigroup is semi topological [6]. Given H be a semi-topological semi-group, an H-system or H-flow is a triple \( (H, X, g) \), where X is a compact Hausdorff space and \( g: H \times X \to X \) is separately continuous[5]. Let \( g(h, x) = hx \), and g is defined an action if \( (hr)x = h(rx) \) for all \( h, r \in H \) and \( x \in X \) [7].

Suppose X and Y be two H-systems, a continuous function \( T: X \to Y \) satisfying \( T(hx) = hT(x) \) for all \( h \in H, x \in X \) is called a homomorphism. \( g: X \to Y \) is an isomorphism H-systems if it is onto homeomorphism[8]. We define the orbit of y for \( y \in X \) to be the set \( Hy = \{ hy : h \in H \} \) and we will denote \( U(y) \) to be the topological closure of \( Hy \). If there is a \( y \in X \) s.t \( \{ y \} \cup U(y) = X \) we call it the system pointed transitive[4].

Definition(2.1):[3]
A monoidal right topological compactification of H (where H has an identity, otherwise one can attach an identity to H) is a pair \( (T, f) \) s.t
(a) T is Hausdorff compact right topological semi-group which contains an identity.
(b) f is a homomorphism continuous which takes H into T.


development
(c) Let \( e \) be the identity element of \( T \). \( f(H) \cup \{ e \} \) is dense in \( T \).

(d) \( f \) takes the identity of \( H \) to the identity of \( T \).

(e) \( f(h) \subseteq \{ x \in T : g_x(y) = xy \text{ is continuous} \} \).

By standard compactification arguments there exists an inclusive right topological compactification \((CH,j)\) of \( H \) which is distinguished by the universal property that if \( f:H \rightarrow T \) is a right topological compactification of \( H \), then there exist a unique continuous homomorphism \( a:F:CH \rightarrow T \) s.t \( f = F \circ j \).

**Theorem 2.2:** Let \((X,q)\) be a pointed transitive \( H \)-system and \((Y,p)\) be a pointed \( H \)-system. Then there is at most one homomorphism from \((X,q)\) to \((Y,p)\).

**Proof:** See Proposition 1-1 in[4].

**Remark 2.3:** Let \((X,q)\) and \((Y,p)\) be two pointed transitive systems, if \( h:(X,q) \rightarrow (Y,p) \) and \( r:(Y,p) \rightarrow (X,q) \) are homomorphism, then \( h \) and \( r \) are inverse homeomorphism and therefore \( H \)-system is an isomorphism[9].

**Remark 2.4:** Let \((H,X,D)\) be an \( H \)-system. For each \( h \in H \), let \( D^*\) denote the continuous mapping \( x \mapsto hx \), we fram the Ellis semigroup by capturing the closure of \( H(D) = \{ D^* : h \in H \} \) in the Cartesian product \( X \times X \) given with the product topology.

### Main results:

In this part, we present the new definitions and theorems related with the study and the relations between the dynamical system the enveloping semigroup.

**Definition 3.1:**[2]

Consider \((T_1,d_1)\) and \((T_2,d_2)\) be two monoidal compactifications of \( H \). We define the C-homomorphism \( K:T_1 \rightarrow T_2 \) which is a continuous homomorphism identity preserving such that \( K \circ d_1 = d_2 \).

**Lemma 3.2:** Let \((T_1,d_1)\) and \((T_2,d_2)\) be monoidal compactification then \( K \) is a homomorphism of pointed \( H \)-system iff \( K:(T_1,d_1) \rightarrow (T_2,d_2) \) is a C-homomorphism.

**Proof:** If \( K \) is a homomorphism pointed of \( H \)-systems, then for \( h \in H, n \in T_1 \) we get \( K(d_1(h)n) = K(hn) = hK(n) = d_2(h)K(n) \) ...(1). Take \( n = 1 \), we have \( K \circ d_1 = d_2 \) and thus \( K \circ d_1(h) = d_2(h) \) for all \( h \in H \). Let \( n, m \in T_1 \).

Since \( m \in T_1 \), then there is a net \( \{ d_1(h_a) \} \) converge to \( mn \). On the other hand, \( d_1(h_a)n = d_2(h_a)K(n) \) by (1) = \( K(d_2(h_a))K(n) \).

But by continuity of \( K \) we get \( K(d_1(h_a)) \) convergence to \( K(m) \) and so \( K(d_1(h_a))K(n) \).

Conversely, by definition (2), \( K \circ d_1 = d_2 \). Therefore \( K(nx) = K(d_1(h)n) = K(d_1(h))K(x) = d_2(h)K(x) \). Hence \( K \) is a homomorphism of pointed \( H \)-system.

**Remark 3.3:** \( T \) can be considered as an \( H \)-system by defining \( T(h,t) = j(h)t \) where \( H \) is topological semi-group and \((T,j)\) be a monoidal compactification.

**Lemma 3.4:** Consider \((H,X,D)\) as an \( H \)-system then

(a) \( f \mapsto f(q): V(X) \rightarrow X \) where \( q \in X \), is a homomorphism of \( H \)-systems, and a homomorphism of pointed \( H \)-systems, from \( (V(X),I_2) \), where \( I_2 \) is the identity function.

(b) If \((T,j)\) is a monoidal compactification of \( H \) defined on part (a) from \((V(T),1_7)\) to the flow \((T,1)\) is an isomorphic, where \( I \) is the identity of \( T \).

**Proof:** For part (a). Let \( R:V(X) \rightarrow X \) which is defined by \( R(f) = f(q) \) for \( q \in X \). Then \( R(hf) = (hf)(q) = sf(q) = hR(f) \) for all \( h \in H \) and \( f \in V(X) \).

Also \( R(Id_q) = I_2(q) = q \). Thus \( R \) is homomorphism pointed \( H \)-system. For part (b) is left for the reader.

**Remark 3.5:** Let \((H,X)\) and \((H,Y)\) be two systems, if \( K \) is surjective homomorphism from \( X \) to \( Y \) map then \( R:V(X) \rightarrow V(Y) \) is unique homomorphism which is defined by \( R(a) = K(a) \), where \( x \in X \) satisfies \( K(x) = y \).

**Theorem 3.6:** Let \( R:V(X) \rightarrow Y \) be the flow homomorphism \( h \mapsto hq \) Then \( R \) is an isomorphism of flows iff \((X,q)\) is isomorphic to the flow coming from some monidal compaction \((T,j)\).

**Proof:** Clearly \((X,q)\) must be a pointed \( H \)-system. Let \( R \) be an isomorphic of flows. Thus \((X,q)\) is an isomorphic to the Ellis compactification. On the other hand, assume that \((X,q)\) is isomorphic for some monidal compactification \((T,j)\). We can define an isomorphism \( Z:V(X) \rightarrow V(Y) \) as in remark (4). Therefore \((X,q)\) is isomorphic to its Ellis compactification \((V(X),I_2)\).

**Definition 3.7:**

A pointed transitive \( H \)-system \((X,p)\) is said to be inclusive if there is a homomorphism from \((X,p)\) to any other pointed \( H \)-system.

**Definition 3.8:**[9]

Let \((H,+)\) be a semigroup, for any \( A \subseteq H \) and \( x \in H \) we define \( x + A = \{ h \in H : x + h \in A \} \). Given any two ultrafilters \( p, q \in CH \) we define their sum by \( p + q = \{ A \subseteq H / \{ x \in H / -x + A \in q \} \in p \} \).

In the next Theorem, we will see how the inclusive right topological compactification of \( H \) led to the inclusive pointed transitive \( H \)-flow \((CH,\emptyset)\).
Theorem (3.9): Let \((CH, j)\) be the inclusive right topological compactification of \(H\). Let we define 
\[ K: H \times CH \rightarrow CH \]
by 
\[ k(n, q) = j(n)q \]
where the right hand side is added in \(CH\). Then \((H, CH, K)\) is an \(H\)-system, and the pointed \(H\)-system \((CH, \varepsilon)\) where \(\varepsilon\) is the base point.

Proof: Note that by definition of filter addition above we have 
\[ K(n, q) = j(n) + q = f_{j(n)}(q) \]
where \(f_{j(n)}\) is right continuous. Thus \(K\) is separately continuous. Since \(j\) is a homomorphism, 
\[ K(n_1 + n_2, q) = j(n_1 + n_2)q = j(n_1) + j(n_2)x = j(n_1)j(n_2)x = j(n_1)n_2q = K(n_1, n_2q). \]
Therefore, \(K\) is defined as an action of \(H\) on \(CH\). Since 
\[ j(\varepsilon) = \varepsilon \]
then 
\[ K(\varepsilon, x) = \varepsilon = \varepsilon x = \varepsilon x = x \]
acts as an identity on \(CH\). Also 
\[ K(H \times \{\varepsilon\}) \cup \{\varepsilon\} = j(H)\varepsilon \cup \{\varepsilon\} = j(H) \cup \{\varepsilon\} \]
dense in \(CH\), implies the action is a point transitive. To show \((CH, \varepsilon)\) is inclusive. Let \((Y, p)\) be a pointed \(H\)-system. Thus, the extended action \(K: CH \times Y \rightarrow Y\) is right continuous action of \(CH\) on \(Y\). Define 
\[ \tau: CH \rightarrow Y \] \(by \) 
\[ \tau(t) = tp, \]
where the right hand side is the extended action. Clearly \(g\) preserves distinguished points. Finally, for \(\kappa \in H\) and since \(j\) is a homomorphism then 
\[ \tau(\kappa x) = \tau(j(\kappa)x) = \tau(j(\kappa)(r)) \]
where \(j(r) = x\)
\[ = \tau(j(\kappa + r)) = j((\kappa + r)q) = j(\kappa)(r)q = \kappa\tau(r) = \kappa\tau(j(r)) = \kappa\tau(x). \]
Thus \(g\) is a homomorphism of \(H\)-system and the proof is complete.

References:

العلاقة بين النظام الديناميكى مع تراص ايليس والنظام الديناميكى المتعدى الحاوي على نقطة اساسية

ليث عبد الطيف مجيد

قسم الرياضيات، كلية العلوم، جامعة ديالى، ديالى، العراق.

الخلاصة: في هذا البحث تم دراسة بعض العلاقات بين الأنظمة الديناميكية وبين مايسمى النظام الديناميكى المتعدى الحاوي على نقطة اساسية. هذه الدراسة سوف تنسمي لدراسة جوانب كثيرة من نظرية التدفقات والأنظمة الديناميكية من خلال منظور الخصائص الجزئية. نلاحظ أنه في هذا البحث تم تضمين العلاقة بينها سوف تنجي لنا ربط الخصائص الموجودة بين الفضاءات التي تم تضمينها تكون فضاءات متراصة وبين النظام الديناميكى الذي يحتوي على نقطة اساسية. هذه الدراسة سوف تكون جزءاً من نظريات التدفقات والأنظمة الديناميكية من خلال منظور الخصائص الجزئية. }

الكلمات المفتاحية: النظام الديناميكى، شبه التبولوجي شبه الزمرة، تماثل الأنظمة، نظام النقطة المتعدى، النقطةامية ذات قاعدة شاملة.