Open Access
Baghdad Science Journal Vol.15(1)2018

DOI: http://dx.doi.org/10.21123/bsj.2018.15.1.0102

On Fully Stable Banach Algebra Modules and Fully Pesudo Stable Banach Algebra Modules

Samira Naji Kadhim

Received 20/10 /2017
Accepted 15 /1 /2018

Abstract:
The concept of fully pseudo stable Banach Algebra-module (Banach $A-$module) which is the generalization of fully stable Banach $A-$module has been introduced. In this paper we study some properties of fully stable Banach $A-$module and another characterization of fully pseudo stable Banach $A-$module has been given.

Keywords: fully stable Banach $A-$modules, the algebra of endomorphism of a fully stable Banach $A-$module, fully pseudo stable Banach $A-$module.

Introduction:

The theory of Banach algebras (BA) is an abstract mathematical theory. BA are started in the early twentieth century, when abstract concepts and structures were introduced, transforming both the mathematical language and practice. We give a historical account to develop the theory of BA.
1918: Riesz provides for the first time the axioms for a space with a norm 1920: Banach's thesis, first abstract study of normed spaces.
1929: von Neumann studies additional structure on a normed space. 1932: several issues occurred: Stone's "Linear transformations in Hilbert space and their applications to Analysis" is a major contribution to operator theory, Banach's book "Th'eorie des op'erationslin'eaires" and Wiener introduced the inequality $\|xy\| \leq \|x\|\|y\|$ without studying further consequences of it. 1936: The notion of abstract Banach algebra arises through Nagumo's "linear metric ring" and Yosida's "metrical complete ring".
1938: Mazur's theorem: every complex division algebra with a norm is isomorphic to$\mathbb{C}$ and every real one is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or the quaternions.
1939: Gelfand's thesis: foundations of the theory of commutative BA. End of 30's: The term normed ring is established in the Soviet school. 1941: publication of the results of Gelfand's thesis and proof of Wiener's 1943: Gelfand-Naimark representation theorems.

Department of Mathematics, College of Science for Women University of Baghdad, Iraq, Baghdad.
E-mail:picanto.korea@gmail.com

1945: Ambrose introduces the term Banach algebra.
1947: Segal proves the real analogue to the commutative Gelfand-Naimark representation theorem.
1956: Naimark's book "Normed Rings" is the first presentation of the whole new theory of BA, which was important to its development.
1960: Rickart's book "General theory of Banach algebras" is the reference book of all later studies of BA. 1970: C^*-algebras were revitalized by the introduction of topological methods by Brown, Douglas and Fillmore on extensions of C^*-algebras, Elliott's use of K-theory to classify approximately finite dimensional C^*- algebras and Kasparov's melding of the two in KK-theory. For these developments see [Davidson 1996]. For some recent interactions of the theory of BA with topics ranged from K-theory, over abstract harmonic analysis, to operator theory see [Lau, Runde 2004]" (1).
"Let A be an algebra which is a vector space A together with a multiplication, $A \times A \longrightarrow A\,; (a,b) \rightarrow ab$, it is associative and respects the vector space
$(ab)c = a(bc)$, $a(b+c) = ab + ac, (b + c)a = ba + ca \quad (a,b,c \in A)$
$(\alpha a)b = a(\alpha b) = \alpha(ab) (\alpha \in \mathbb{C}, a, b \ \in A)$ "(2).
"A maps from a left Banach $A-$module X into a left Banach $A-$module Y (A is not necessarily commutative), is a bounded linear operator called a multiplier if it satisfies: $T(a.x) = a.Tx$ for all $a \in A, x \in X$" (3)

"Banach algebra module X is called fully stable Banach A-module if for every submodule N of X and for each multiplier $\theta: N \longrightarrow X$ such that $\theta(N) \subseteq N$"(4).

Banach algebra module X is called fully pseudo stable ,if $f(N) \subseteq N$ for each submodule N of X and A −isometric multiplier f from N in to X.

The Endomorphism Algebra of Fully Stable Banach Algebra Modules:

Definition 2.1 :- (4)

"For a nonempty subset M in a left Banach A −module X, the annihilater $ann_A(M)$ of M is $ann_A(M) = \{a \in A\,; a.x = 0\ for\ all\ x \in M\}$ ".

Definition 2.2 :- (5)

"A left Banach A −module X is called n −generated for $n \in N$ if there exists $x_1, \dots, x_n \in X$ such that each $x \in X$ can represented as$x = \sum_{k=1} a_k.x_k$ for some $a_1, \dots, a_n \in A$ .A cyclic module is just a 1-generated one".

Notations:-

"Let X a Banach A −module

1) $N_x = \{n_x |\ n \in N,\ x \in X\}$

   $K_y = \{k_y |\ k \in K,\ \ y \in X\}$

2) $ann_A N_x = \{a \in A\ ,\ a.n_x = 0,\ \forall\ n_x \in N_x\}$

   $ann_A K_y = \{a \in A,\ a.k_y = 0,\ \forall\, k_y \in K_y\}$ " [4]

$End_A(X)$: the set of all $A-$ multiplier of X " [5]

Proposition 2.3 :- (4)

"X is fully stable Banach A −module if and only if for each $x, y\ \in X$ and N_x, K_y subsets of X, $y\ \notin\ N_x$ implies $ann_A(N_x) \nsubseteq ann_A(K_y)$" .

Colloraly 2.4:- (4)

"Let X be a fully stable Banach A −module .Then for each x, y in $X, ann_A(K_y) = ann_A(N_x)$ implies $N_x = K_y$"

Proposition 2.5:-

If X is a fully stable Banach A −module, then $End_A(X)$ is a commutative algebra.

Proof:-

For each two elements T, S in $End_A(X)$and each $x \in X$. Hence(N_x) is a stable submodule of X, hence there is two elements a, b in A such that $\mathrm{T}(n_x) = a.n_x$ and $\mathrm{S}(n_x) = b.n_x$

$(\mathrm{ToS})(n_x) = \mathrm{T}(\mathrm{S}(n_x)) = \mathrm{T}(\mathrm{b}n_x) = \mathrm{a}(\mathrm{b}.n_x)\ = (\mathrm{ab})n_x = (\mathrm{ba})n_x = \mathrm{b}(\mathrm{a}n_x)\ = \mathrm{b}(\mathrm{T}(n_x)) = \mathrm{S}(\mathrm{T}(n_x)) = (\mathrm{SoT})(n_x)$. Therefor $End_A(X)$ is a commutative.

The converse of proposition 2.5 have been studied in the following proposition.

Proposition 2.6:-

Let X be Banach A −module in which every cyclic submodule is direct summand of X, if $End_A(X)$ is a commutative, then X is a fully stable Banach A −module.

Proof:-

Let $N =< N_x >$ be any cyclic submodule of X and $T: < N_x > \longrightarrow X$ any $A-$ multiplier.

There exists a submodule (K_y) of X such that $X = N_x \oplus K_y$, T can be extended to a multiplier $S: X \longrightarrow X$ by putting $S(k_y) = 0$ for each $k_y \in K_y$. Define $R: X \longrightarrow X$ by $R(n_x, k_y) = n_x$ for each $n_x \in N_x$ and $k_y \in K_y$. Let $T(N_x) =\ z_x +\ I_p$ for some $z_x \in N_x$ and $I_p \in K_y$, now $\mathrm{RoS}(w) = \mathrm{RoS}(\,n_x + k_y) = \mathrm{R}(\mathrm{T}(n_x)) = \mathrm{R}(\ z_x +\ I_p) = Z_x,\ w \in X$

$\mathrm{SoR}(w) = \mathrm{SoR}(\,n_x + k_y) = \mathrm{S}(n_x) = \mathrm{T}(\,n_x) = z_x +\ I_p$ since $\mathrm{RoS} = \mathrm{SoR}$,then $I_p = 0$ thus $T(n_x) \in N_x$ therefore X is a fully stable Banach A −module.

Fully pseudo stable Banach A −module:

"Let M be an R-module. A submodule N of M is called pseudo stable if $\alpha(N) \subseteq N$ for each R-monomorphism $\alpha$from N into M. If each submodule of M is pseudo stable, then M is called fully pseudo stable. A ring R is called fully pseudo stable if it is fully pseudo stable R-module.It is clear that every stable submodule is pseudo-stable and hence every fully stable R-module is fully pseudo-stable"(6).

Definition 3.1:-

Banach $A-$ module, X is called fully pseudo stable Banach A −module if for each submodule N of X and for each isometric multiplier (A −module isomorphism is an isometry A −multiplier) $f: N \longrightarrow X$ such that $f(N)\ \subseteq\ N$.

It is clear that every fully stable Banach A −module is fully pseudo stable Banach A −module

Recall that" an $R-$ module M is uniform if any non –zero submodules of M has non-zero intersection" (7).

Proposition 3.2:-

Every uniform fully pseudo stable Banach A −module is fully stable Banach A −module.

Proof:-

Suppose that X is a fully pseudo stable Banach A −module , N be a submodule of X and A − multiplier $f: N \longrightarrow X$.
If $ker f = 0$, nothing to prove . Otherwise $ker f$ is non-zero.
But $\ker(f) \cap \ker(\ I_N + f) = 0$
Hence uniform property of X implies that $I_N + f(N) \subseteq N$ and hence$(N) \subseteq N$.

Proposition 3.3:-

Let X be a fully pseudo stable Banach A −module .Then for each x, y in X,
$ann_A(N_x) = ann_A(K_y)$ implies that $N_x = K_y$

Proof:-

Assume that $X$is fully pseudo stable Banach$A$ −module. $f: < N_x > \longrightarrow X$ is well defined by $f(a.n_x) = a.k_y$,for all $a \in A$.Then $f(a.n_x) \subseteq N_x$ and hence $(\ a.k_y) \subseteq N_x$
By symmetry $N_x \subseteq K_y$ then $N_x = K_y$
Recall that "a left Banach A −module X is n −generated for $n \in N$ if there exists $x_1, \dots, x_n \in X$ such that each $x \in X$ can represented as $x = \sum_{k=1} a_k . x_k$for some $a_1, \dots, a_n \in A$. A cyclic module is just a 1-generated" (8).

Proposition 3.4:-

Let X be a Banach A −module .Then the following statement are equivalent.
1-X is fully pseudo stable Banach$A$ −module.
2- distinct 1- generated submodules of X are not isomorphic.

Proof :-

Assume that X is fully pseudo stable Banach$A$ −module and X has two distinct 1-generated submodulesN_x and K_y .Without loss of generality if it is assumed that $N_x \nsubseteq K_y$ .There exists$n_x \in N_x$and $n_x \notin K_y$. Let. $f: < N_x > \longrightarrow < K_y >$ be an isometric , consider the following two A − isometric
Assume distinct 1- generated submodules of X are not isomorphic, and there exists submodulesN_x of X and A − isometric multiplier $f: < N_x > \longrightarrow X$ such that $f(N_x) \nsubseteq N_x$then N_x and $f(N_x)$ are distinct 1- generated submodules of X .. By assumption, then
$f(N_x)$ is not isomorphic to N_x which is an absurd.

Proposition 3.5:-

Let X be fully stable Banach$A$ −module, for each x, y in $X ann_A(N_x) = ann_A(K_y)$. implies $N_x = K_y$. Then distinct 1- generated submodules of X are not isomorphic.

Proof:-

Assume $ann_A(N_x) = ann_A(K_y)$and there are two distinct 1- generated submodules N_x and K_y such that $N_x \cong K_y$ .No loss of generality , there is an element z_x in N_x not in K_y . Let $\theta: < N_x > \longrightarrow < K_y >$ be an isomorphism, $z_x \neq \theta(z_x)$, then $ann_A(Z_x) = ann_A(\ \theta(z_x))$,but ,$z_x \neq \ \theta(z_x)$ which is a contradiction .
"Let A be a unital Banach algebra and let α > 1. A −module X is called quasi α-injective if, φ : $N \longrightarrow X$ is A −module homomorphisms such that $\| \varphi \| \leq 1$, there exists A −module homomorphism θ*: X* ⟶ *X*, such that $\theta\ o\ i = \varphi$ and $\| \theta \| \leq$ α where i is an isometry from submodule N of X. We shall say that X is quasi injective if it is quasi α - injective for some α " [9].
The concept of pseudo α-injective has been introduced in the following definition.

Definition 3.6: Let A be a unital Banach algebra and let α > 1. A −module X is called principally pseudo α-injective if, φ : $N \longrightarrow X$ is A −module monmorphisms such that $\| \varphi \| \leq 1$, there exists A −module homomorphism $\theta: X \longrightarrow X$, such that $\theta\ o\ i = \varphi$ and $\| \theta \| \leq$ α where i is an isometry from 1-generated submodule N of X. We shall say that X is principally pseudo injective if it is principally pseudo α - injective for some α.
Recall that" a Banach A −module X is multiplication A −module if each 1-generated submodule of X is of the form KX for some ideal K of A" (9).

Proposition 3.7: Let X be a multiplication A-module. Then X is fully pseudo stable Banach A −module if and only if X is principally pseudo-injective Banach A −module.
***Proof*:** Let $\alpha : N_x \rightarrow X$ be an isometry where x in X. Then principal pseudo injectivity of X implies that α is extendable to $\beta : X \rightarrow X$. There is an ideal $K$of A such that $N_x = XK$. Hence $\alpha(N_x) = \beta(N_x) = \beta(XK) = \beta(X)K \subseteq XK = N_x$. The conversely is clear.

References:

1. Petrakis S. Introduction to Banach Algebras and the Gelfand-Naimark theorems, specialsubject II and IIIAristotle, University of Thessaloniki Department of Mathemayics 2008.
2. Ramesh G. Banach Algebras, department of Mathematics I I T, Hyderabad, ODF Estate, Yeddumailaram A P, India. 2013; 502 – 205.
3. Bracic J. Simple Multipliers on Banach Modules. University of Ljubljana, Slovenia, Glasgow Mathematical Journal, May 2003; 45 (2):309-322.

4. Mohammed Ali MJ, Ali M. Fully Stable Banach Algebra modules, Mathematical Theory and Modeling, 2016; 6 (1):136-139.
5. Daws MDP. Banach algebras of operators, Ph. D, Thesis, The University of Leeds, School of Mathematics, Department of Pure Mathematics, 2004.
6. Abbas MS. On Fully Stable Modules, Ph. D, Thesis, University of Baghdad, Iraq, 1990.
7. Kasch F. Modules and Rings, Academic Press, Lordon, New York, 1982.
8. Cegarra AM. Projective Covers of Finitely Generated Banach Modules and the Structure of Some Banach Algebras, O. Yu. Aristov, Russia, 2006; 21(1):1-26.
9. Kadhim SN, Mohammed AMJ. On Fully Stable Banach Algebra Modules Relative to an Ideal, Baghdad Science Journal, 2017; 14(4):813-815.

حول مقاسات بناخ الاجبرا تامة الاستقرارية و مقاسات بناخ الاجبرا تامة الاستقرارية الكاذبة

سميرة ناجي كاظم

قسم الرياضيات ، جامعة بغداد، كلية العلوم للبنات، بغداد، العراق.

الخلاصة:

تم دراسة مفهوم مقاسات بناخ الاجبرا تامة الاستقرارية الكاذبة و التي هي تعميم لمفهوم مقاسات بناخ الاجبرا تامة الاستقرارية حيث تم اعطاء تشخيصات اخرى لمفهوم مقاسات بناخ الاجبرا تامة الاستقرارية الكاذبة و كذلك دراسة بعض خواص مقاسات بناخ الاجبرا تامة الاستقراية.

الكلمات المفتاحية : مقاسات بناخ الاجبرا تامة الاستقرارية ، تشاكلات الاجبرا لمقاسات بناخ الاجبرا تامة الاستقرارية، مقاسات بناخ الاجبرا تامة الاستقرارية الكاذبة.