Some Probability Characteristics Functions of the Solution of a Stochastic Non-Linear Fredholm Integral Equation of the Second Kind

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Abstract:

In this research, some probability characteristics functions (probability density, characteristic, correlation and spectral density) are derived depending upon the smallest variance of the exact solution of supposing stochastic non-linear Fredholm integral equation of the second kind found by Adomian decomposition method (A.D.M)

Key words: A.D.M, stochastic non-linear Fredholm integral equation.

Introduction:

In the beginning of 1980's, a new method for solving linear and non-linear integral (differential) equations for various kinds has been proposed by G.Adomian, the so called Adomian decomposition method, [1]. After that, in recent years many researchers had been used this method to solve analytically (numerically) either a stochastic Fredholm integral equations or especial kinds of linear (non-linear) Fredholm integral equations [2,3,4]. Most of those researchers are just interested as a final goal in finding the numerical solution on some definite closed interval to study a unique comparison between the numerical solution of a given Fredholm integral equation and its exact solution.

In this paper, our goal is not only interesting in the solution of the supposing stochastic Fredholm integral equation but we concentrate ourself in the derivation of many probability characteristics of this solution (mean, variance, characteristic function, correlation function and spectral density function) that is by depending upon the smallest value of the variance of this solution as a basis for that.

So, we consider the following one-dimensional non-linear stochastic Fredholm integral equation of the second kind

$$Y(w,t) = X(w,t) + \int_a^b k(t,s;w)Y(s,t)ds \quad \ldots(1)$$

where;

(i) $w \in \Omega$, Ω is a sample space sapporting of the probability measure space (Ω, F, P).

(ii) Y(w,t) is the unknown stochastic process for the time t > 0.

(iii) X(w,t) is known stochastic process defined for the time t > 0.

(iv) K(t,s;w) is known stochastic kernel defined by t > 0 and $s \in S$, where S is a compact metric space, d is any metric defined as S.

(v) Y(s,t) is a scalar function defined for the time t > 0, $s \in S$.

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Preliminaries

In equation (1), we consider as a especial case, X(w,t) is a standard process with E[X(w,t)] = 0, var[X(w,t)] = t, t > 0, - ∞ < w < ∞, (i.e.) X(w,t) ~ N(0,t) and
$k(t,s;w) = e^{-(w+s)t}$, 0 < s ≤ 1.

So that (1) becomes,

$$Y(w,t)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{w^2}{2t}}+\int_0^1 e^{-(w+s)t}Y(s,t)ds \quad \ldots(2)$$

The exact solution of (2) by the Adomian decomposition is [4]

$$\phi(w,t)=Y_0(w,t)+\sum_{n=1}^{\infty}Y_n(w,t) \quad \ldots(3)$$

where,

$$Y_0(w,t)=X(w,t)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{w^2}{2t}} \quad \ldots(4)$$

and

$$\left.\begin{array}{l} Y_{m+1}(w,t)=\int_0^1 k(t,s;w)Y_m(s,t)ds, m=0,1,2,\ldots \\ \text{with} \\ Y_0(s,t)=X(s,t)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{s^2}{2t}} \end{array}\right\} \quad \ldots(5)$$

For m = 0:

$$\begin{aligned} Y_1(w,t) &= \int_0^1 e^{-(w+s)t}Y_0(s,t)ds \\ &= \int_0^1 e^{-(w+s)t}\frac{1}{\sqrt{2\pi t}}e^{-\frac{s^2}{2t}}ds \\ &= \frac{1}{\sqrt{2\pi t}}e^{-wt+\frac{t^3}{2}}\int_0^1 e^{-\frac{(s+t^2)^2}{2t}}ds \\ &= \frac{e^{-wt+\frac{t^3}{2}}}{\sqrt{2\pi t}}\int_{\frac{t^2}{\sqrt{t}}}^{\frac{1+t^2}{\sqrt{t}}} e^{-\frac{y^2}{2}}dy \end{aligned}$$

or,

$$Y_1(w,t)=\left[N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)\right]^* e^{-wt+\frac{t^3}{2}}, -\infty<w<\infty, t>0 \quad \ldots(6)$$

For m = 1:

$$\begin{aligned} Y_2(w,t) &= \int_0^1 e^{-(w+s)t}Y_1(s,t)ds \\ &= \left[N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)\right]\int_0^1 e^{-(w+s)t}e^{-st+\frac{t^3}{2}}ds \\ &= \left[N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)\right]\left(\frac{1-e^{-2t}}{2t}\right)e^{-wt+\frac{t^3}{2}}, -\infty<w<\infty, t>0 \end{aligned} \quad \ldots(7)$$

For m = 2:

$$\begin{aligned} Y_3(w,t) &= \int_0^1 e^{-(w+s)t}Y_2(s,t)ds \\ &= \left[N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)\right]\left(\frac{1-e^{-2t}}{2t}\right)\int_0^1 e^{-(w+s)t}e^{-st+\frac{t^3}{2}}ds \\ &= \left[N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)\right]\left(\frac{1-e^{-2t}}{2t}\right)^2 e^{-wt+\frac{t^3}{2}}, -\infty<w<\infty, t>0 \end{aligned} \quad \ldots(8)$$

and by repeating for m = 3, 4, 5, …, one can get

$$Y_k(w,t)=\left[N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)\right]\left(\frac{1-e^{-2t}}{2t}\right)^{k-1}e^{-wt+\frac{t^3}{2}},k=4,5,... \qquad ...(9)$$

Therefore, (3) can be written as

$$\phi(w,t)=Y_0(w,t)+\left[N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)\right]\sum_{n=0}^{\infty}\left(\frac{1-e^{-2t}}{2t}\right)^n e^{-wt+\frac{t^3}{2}}$$

$$=Y_0(w,t)+\left[N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)\right]\left(\frac{2t}{e^{-2t}+2t-1}\right)e^{-wt+\frac{t^3}{2}}$$

Finally, by substituting (4) into above function, the exact solution of (3) over the whole real line will be

$$\phi(w,t)=\alpha(t)e^{-\frac{w^2}{2t}}+\beta(t)e^{-wt},-\infty<w<\infty,t>0 \qquad ...(10)$$

where

$$\alpha(t)=\frac{1}{\sqrt{2\pi t}},\beta(t)=\left[N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)\right]\left(\frac{2t}{e^{-2t}+2t-1}\right)e^{\frac{t^3}{2}}$$

Furthermore, the function φ(w,t) can also be considered as a solution over the interval $-1 \le w \le 1$ which permit to derive the probability characteristics of the solution over this interval and over any division of its ten equal subintervals each of length 0.2 starting from (– 1 to + 1) beside choosing many suitable different values for $t \in T$ such that

T={0.1, 0.4, 0.7, 1, 1.3, 1.6, 1.9, 2.2, 2.5, 2.8}

Calculations of α(t), β(t) and φ(w,t) are tabulated respectively in tables (1) and (2).

Probability Characteristics:

Moments, Variances

The moments and variances of the exact solution (10) are derived over $-1 \le w \le 1$ and for t which indicating in last section, (i.e.) for $t \in T$

1st moment:

$$E[\phi(w,t)]=\alpha(t)\int_{-1}^{1}w\,e^{-\frac{w^2}{2t}}dw+\beta(t)\int_{-1}^{1}w\,e^{-wt}dw$$

$$=\beta(t)\left[\frac{(1-t)e^{-t}-(1+t)e^{t}}{t^2}\right],t\in T \qquad ...(11)$$

2nd moment:

$$E[\phi(w^2,t)]=\alpha(t)\int_{-1}^{1}w^2\,e^{-\frac{w^2}{2t}}dw+\beta(t)\int_{-1}^{1}w^2\,e^{-wt}dw \qquad ...(12)$$

$$=\alpha(t)\left\{t\sqrt{2\pi t}\left[2N\left(\frac{1}{\sqrt{t}}\right)-1\right]-2te^{-\frac{1}{2t}}\right\}+\beta(t)\left[\left(\frac{1}{t}-\frac{2}{t^2}+\frac{2}{t^3}\right)e^{t}-\left(\frac{1}{t}+\frac{2}{t^2}+\frac{2}{t^3}\right)e^{-t}\right],t\in T$$

where α(t), β(t) are defined in (10).

while, the variance of the exact solution can be obtained from the first and second moments. Table (3) shows that, the smallest variance of (10) is when (t = 1.6). Hence, the solution of the supposing Fredholm integral equation (1) over the interval $-1 \le w \le 1$ that will be adopted in this paper takes the following form,

$$\phi(w,1.6)=\alpha(1.6)e^{-\frac{w^2}{2(1.6)}}+\beta(1.6)e^{-1.6w}$$

$$=0.3153e^{-\frac{w^2}{3.2}}+0.1067e^{-1.6w},-1\le w\le 1 \qquad ...(13)$$

Moreover this function has the following two properties

- $\phi(w,1.6)>0$
- $\int_{-1}^{1}\phi(w,1.6)dw \cong 1$

which means that φ(w,1.6) is a p.d.f. of the stochastic process Y(w,t) which is

(*) N(·) is a standard normal distribution.

defined in (1) just when $-1 \le w \le 1$, t = 1.6. (fig. (1) represents the curve of $\phi(w,1.6)$).

Characteristic Function of ϕ(w,1.6):

The characteristic function of $\phi(w,1.6)$ can be found as following, [5]

$$f(u;w,1.6) = E[e^{iuw}]$$
$$= \int_{-1}^{1} e^{iuw}\phi(w,1.6)dw$$
$$= \int_{-1}^{1} e^{iuw}\left[\alpha(1.6)e^{-\frac{w^2}{3.2}} + \beta(1.6)e^{-1.6w}\right]dw$$
$$= 0.3153e^{-\frac{1.6u^2}{2}}\int_{-1}^{1} e^{-\frac{(w-1.6u)^2}{3.2}}dw + 0.1067\int_{-1}^{1} e^{-(1.6-iu)w}dw$$

$$f(u;w,1.6) = 0.3153\sqrt{1.6}\,e^{-0.8u^2}\int_{\frac{-1-1.6iu}{\sqrt{1.6}}}^{\frac{1-1.6iu}{\sqrt{1.6}}} e^{-\frac{y^2}{2}}dy + 0.1067\int_{-1}^{1} e^{-(1.6-iu)w}dw$$

$$= 0.3988e^{-0.8u^2}\int_{\frac{-1-1.6iu}{\sqrt{1.6}}}^{\frac{1-1.6iu}{\sqrt{1.6}}}\left[1 - \frac{\left(\frac{y^2}{2}\right)}{1!} + \frac{\left(\frac{y^2}{2}\right)^2}{2!} - \frac{\left(\frac{y^2}{2}\right)^3}{3!} + \ldots\right]dy + 0.1067\int_{-1}^{1} e^{-(1.6-iu)w}dw$$

$$= 0.3988e^{-0.8u^2}\int_{\frac{-1-1.6iu}{\sqrt{1.6}}}^{\frac{1-1.6iu}{\sqrt{1.6}}}\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{(n!)(2)^n}dy + 0.1067\int_{-1}^{1} e^{-(1.6-iu)w}dw$$

$$= 0.3988e^{-0.8u^2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{(n!)(2)^n}\Bigg|_{\frac{-1-1.6iu}{\sqrt{1.6}}}^{\frac{1-1.6iu}{\sqrt{1.6}}}\right] - \frac{0.1067}{(1.6-iu)}e^{-(1.6-iu)w}\Bigg|_{-1}^{1}$$

or

$$f(u;w,1.6) = 0.3988e^{-0.8u^2}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n[(1-1.6iu)^{2n+1} - (-1-1.6iu)^{2n+1}]}{(n!)(2n+1)(3.2)^n}\right\} + 0.1067\left[\frac{e^{(1.6-iu)} - e^{(1.6-iu)}}{(1.6-iu)}\right], u \ge 0 \quad \ldots(14)$$

Correlation Function of ϕ(w,1.6):

For any $t_1 > t = 1.6$, $\tau = t_1 - 1.6 > 0$, the correlation function of $\phi(w,1.6)$ with the function $\phi(w,1.6+\tau)$ depends only on the difference $|\tau| = |-\tau| = |t_1 - 1.6|$ and can be found as following, [6],

B(τ) = E(ϕ(w,1.6) ϕ(w,1.6+τ))
= E(ϕ(w,1.6+τ) ϕ(w,1.6))
= B(–τ)
=E(ϕ(w,1.6)ϕ(w,1.6+τ))–E(ϕ(w,1.6) E(ϕ(w,1.6+τ))
=E(ϕ^2(w,1.6))+E(ϕ(w,1.6)ϕ(w,1.6+τ))– E(ϕ^2(w,1.6)) – E(ϕ(w,1.6)) E(ϕ(w,1.6+τ))
=E(ϕ^2(w,1.6))+E(ϕ(w,1.6)E(ϕ(w,1.6+τ))– (ϕ(w,1.6)) – E(ϕ(w,1.6)) E(ϕ(w,1.6+τ))
=E(ϕ^2(w,1.6)+E(ϕ(w,1.6)[E(ϕ(w,1.6+τ))– E(ϕ (w,1.6))] – E(ϕ(w,1.6)) E(ϕ(w,1.6+τ))
= E(ϕ^2(w,1.6)) – [E(ϕ(w,1.6)]^2

Hence and by table (3)

$B(\tau) = var(\phi(w,1.6))=0.2887, -1 \le w \le 1, \tau > 0$ …(15)

Spectral Density Function of ϕ(w,1.6):

The spectral density function of $\phi(w,1.6)$ for known B(τ) (15) can be found by khinchine's formula as following, [6],

$$f_\phi(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\tau)e^{-i\lambda\tau}d\tau, |\lambda| < n\pi, n = 1,2,\ldots \quad \ldots(16)$$

$$f_\phi(\lambda) = \frac{0.2887}{2\pi}\int_{-\infty}^{\infty}(\cos\lambda\tau - i\sin\lambda\tau)d\tau$$
$$= \frac{0.2887}{\pi}\int_{-\infty}^{\infty}\cos\lambda\tau\, d\tau$$

and for $\tau = t_1 - 1.6 > 0$

$$f_{\phi}(\lambda)=\frac{0.2887}{\pi}\int_{0}^{t_1-1.6}\cos\lambda\tau\, d\tau$$
$$=\frac{0.2887}{\pi}\,\frac{\sin(\lambda(t_1-1.6))}{\lambda}$$
$$=0.0919\frac{\sin(\lambda(t_1-1.6))}{\lambda},\ |\lambda|\le n\pi, n=1,2;\ t_1>1.6$$

Also,

$$f_{\phi}(-\lambda)=0.0919\,\frac{\sin(-\lambda(t_1-1.6))}{-\lambda}=f_{\phi}(\lambda)$$

which means that $f_{\phi}(\lambda)$ is an even function, [6] and represents the average power in the solution (function) $\phi(w,1.6)$ at the angular frequency λ. (Fig.(2) represents the curve of $f_{\phi}(\lambda)$ for $0\le\lambda\le 2\pi$ when t_1=2.6, 3.6 and 11.6).

Conclusions:

1.For t > 3, the values of $\beta(t)$ defined in the solution (10) tend rapidly to zero while those for $\alpha(t)$ tend slowly.

2.The solution $\phi(w,t)$ is not a p.d.f. of (10) for all $t \in T$ except for t = 1.6.

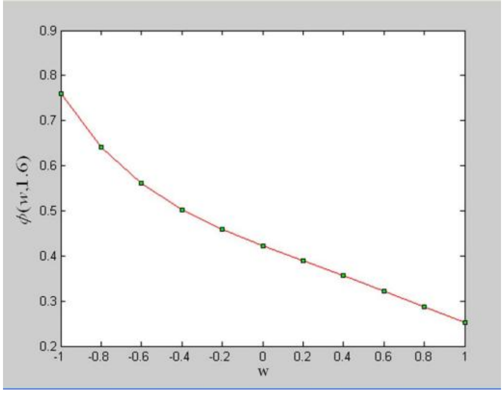

Fig. (1): The curve of $\phi(w,1.6)$, $-1\le w\le 1$

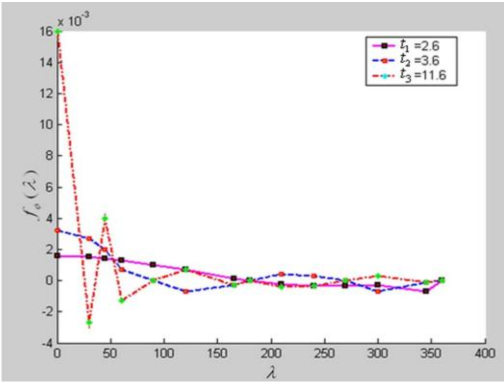

Fig. (2): The curve of $f_{\phi}(\lambda)$ with three different values of t_1, $0<\lambda\le 2\pi$

Table(1): $\alpha(t)=\dfrac{1}{\sqrt{2\pi t}}$, $\beta(t)=\left[N\left(\dfrac{1+t^2}{\sqrt{t}}\right)-N\left(\dfrac{t^2}{\sqrt{t}}\right)\right]\left(\dfrac{2t}{e^{-2t}+2t-1}\right)e^{\frac{t^3}{2}}$

t	$\alpha(t)$	$N\left(\frac{1+t^2}{\sqrt{t}}\right)-N\left(\frac{t^2}{\sqrt{t}}\right)$	$\frac{2t}{e^{-2t}+2t-1}$	$e^{\frac{t^3}{2}}$	$\beta(t)$
0.1	1.2614	0.4873	10.6950	1.0005	5.2143
0.4	0.6307	0.3676	10.6776	1.0325	4.0458
0.7	0.4767	0.2450	3.2086	1.1871	0.9332
1	0.3989	0.1359	2.1652	1.6487	0.4851
1.3	0.3500	0.0603	1.7616	3.0000	0.3187
1.6	0.3153	0.0088	1.5527	7.7524	0.1067
1.9	0.2894	0.0024	1.4281	30.6812	0.1052
2.2	0.2689	0.0002	1.3464	205.2031	0.0552
2.5	0.2523	0.00005	1.2895	2471.3010	0.0381
2.8	0.2384	0.000005	1.2479	58454.2690	0.0289

Table (2): $\phi(w,t)=\alpha(t)e^{-\frac{w^2}{2t}}+\beta(t)e^{-wt}$

w	t =0.1	t =0.4	t =0.7	t = 1	t = 1.3	**t = 1.6**	t =1.9	t = 2.2	t = 2.5	t =2.8
– 1	5.7712	6.2225	2.1126	1.5605	1.4004	**0.7545**	0.6449	0.7062	0.2467	0.2241
– 0.8	5.7000	5.8550	1.9355	1.3692	1.1752	**0.6420**	0.5345	0.5533	0.2464	0.2267
– 0.6	5.7452	5.5453	1.7889	1.2171	0.9998	**0.5605**	0.4608	0.4544	0.2495	0.2315
– 0.4	5.9939	5.2641	1.6599	1.0919	0.8650	**0.5024**	0.4125	0.3924	0.2533	0.2363
– 0.2	6.4339	4.9828	1.5367	0.9835	0.7569	**0.4583**	0.3787	0.3522	0.2557	0.2393
0	6.4757	4.6765	1.4099	0.8840	0.6685	**0.4220**	0.3526	0.3241	0.2536	0.2399
0.2	6.2253	4.2510	1.2746	0.7882	0.5902	**0.3889**	0.3295	0.3020	0.2523	0.2376
0.4	5.5766	4.3346	1.1305	0.6934	0.5184	**0.3653**	0.3070	0.2822	0.2455	0.2322
0.6	5.1191	3.5847	0.9818	0.5994	0.4507	**0.3227**	0.2834	0.2625	0.2354	0.2238
0.8	4.8648	3.2213	0.8349	0.5075	0.3861	**0.2879**	0.2583	0.2420	0.2225	0.2128
1	4.7266	2.8927	0.6968	0.4204	0.3250	**0.2522**	0.2319	0.2203	0.2068	0.1995

Table (3):Mean, Variance of $\phi(w,t)$, $-1 < w < 1$

t	$E[\phi(w,t)]$	$E[\phi(w^2,t)]$	$Var[\phi(w,t)]$
0.1	– 0.3129	4.2905	4.1926
0.4	– 1.0954	6.3630	5.1841
0.7	– 0.4572	0.9271	0.7181
1	– 0.3569	0.6254	0.4980
1.3	– 0.4073	0.5153	0.3494
1.6	– 0.1457	0.3099	**0.2887**
1.9	– 0.1877	0.3267	0.2915
2.2	– 0.1276	1.4507	1.4344
2.5	– 0.1131	0.8881	0.8753
2.8	– 0.1099	1.2747	1.2626

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بعض دوال المزايا الاحتمالية لحل معادلة فريدهولم التكاملية والعشوائية وغير الخطية من النوع الثاني

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الخلاصة:

في هذا البحث، بعض دوال المزايا الاحتمالية (كثافة الاحتمالية، المميزة، الارتباط والكثافة الطيفية) تم اشتقاقها استناداً لأصغر تباين لحل معادلة افتراضية لفريدهولم التكاملية من النوع الثاني، الغير خطية والعشوائية والذي تم ايجاده بطريقة أدوميان التحليلية.