Notes on Approximately Pure Submodules

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Abstract:

Let R be a commutative ring with identity 1 and M be a unitary left R-module. A submodule N of an R-module M is said to be approximately pure submodule of an R-module, if $N \cap IM = I\,N + J(R)M \cap\ (N\ \cap\ I\ M)$ for each ideal I of R. The main purpose of this paper is to study the properties of the following concepts: approximately pure essentialsubmodules, approximately pure closedsubmodules and relative approximately pure complement submodules. We prove that: when an R-module M is an approximately purely extending modules and N be Ap-puresubmodulein M, if M has the Ap-pure intersection property then N is Ap purely extending.

Key words: Approximately pure submodule, approximately pure essential submodule and approximately pure closedsubmodule.

Introduction:

In this paper we assume that R is commutative ring with identity and all R-modules are unitary left R-modules. A submodule N of an R-module M is called pure submodule if for every finitely generated ideal I of R, $N\ \cap\ I\ M = I\ N$ [1].Asubmodule K of an R-module M is said to be p-essential if for every pure submodule L of M, $K\ \cap\ L = 0$ implies $L = \{0\}$ [2]. Following [3], A submodule N of an R-module M is called approximately pure (simplyAp-pure) if $N\ \cap\ I\ M = I\ N + J(R)M \cap\ (N\ \cap\ I\ M)$ for each ideal I of R. It is clear that every pure submodule is approximately pure [3].

In this paper we introduce the concepts of approximately pure essential submodules approximately pure closed submodules and relative approximately pure complement submodule and we prove that: If N is relative approximately pure complement for some submodule K of M then N is an approximately pure closed submodule in M.

In [4] and [5] an R-module M is called purely extending module if every submodule is essential in pure submodule. We introduce the concept of approximately purely extending module.We prove that an R-module M is approximately pure extending if and only if every approximately pure closed submodule in M is approximately pure in M.

Main results:

In this section we introduce the concept of approximately pure essential submodules, approximately closed submodules relative approximately pure submodules and approximately purely extending modules. We give some properties of each concept. We start by the following definition:

Definition 1: A submodule K of R-module M is called approximately pure essential in M (simply Ap-p-essential) if for every approximately pure submodule L of M with $K \cap L\ \mathbf{t}\, J(R)M$ implies $L\ \mathbf{t}\, J(R)M$.

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M is called an Ap-p- essential extension of K.

It is clear that every P-essential is Ap-p-essential but the converse is not true in general for example: Let $K = \{\bar{0}, \bar{4}\}$ be a submodule of Z_8 as Z_8- module $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is Ap-pure submodule of Z_8 and $L = \{\bar{0}, \bar{4}\} \cap \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \subseteq$ J(R) M implies $L \subseteq \cap \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ = J (Z_8) =J(R) M. But $\{\bar{0}, \bar{4}\} \cap \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \neq \{\bar{0}\}$.

The following result is analogous to a similar result concerning P-essential submodule of a module.

Theorem 2: Let $K \leq N \leq M$ then:

① If K is Ap-p-essential in M, then N is Ap-p-essential in N and N is Ap-p-essential in M..

② If N is Ap-pure in M and J(R) M ∩ N=J(R) N and K is Ap-p-essential in M then K is Ap-p-essential in M.

③ If M has Ap-pure finite intersection property and if N is Ap-pure in M and J(R) M ∩N=J(R) N then K is Ap-p-essential in M if and only if K is Ap-p-essential in N and N is Ap-p-essential in M.

Proof:①We have to show that N is Ap-p-essential in M. Let L be Ap-pure submodule of M with N ∩L ⊆ J(R) M, since K≤ N, then K∩L ⊆N $\cap L \subseteq$ J(R) M thus K∩L ⊆ J(R) M. Since K is Ap-p-essential in M, then L⊆ J(R) M. Hence N is Ap-p-essential in M.

② Let L be an Ap-pure submodule of N with K∩L⊆ J(R) N, thus K∩L⊆ J(R) M, since N is submodule of M and K is Ap-p-essential in M therefore L⊆ J(R) M, therefore L=L∩N ⊆ J(R) M∩N, thus L⊆ J(R) N, hence K is Ap-p-essential in N. Now we have to show that N is Ap-p-essential in M. Let L be an Ap-pure submodule of M with N∩L⊆J(R) M, thus K∩L⊆ N∩L⊆ J(R) M therefore L⊆ J(R) M. Hence N is Ap-p-essential in M

③⇐ Suppose that K is Ap-p-essential in N and N is Ap-p-essential in M, we have to show that K is Ap-p-essential in M. Let L be Ap-pure submodule of M with K∩L⊆J(R) M. By assumption N∩L is Ap-pure in M, thus N∩L is Ap-pure in N by [remark 2.1 , 5] since N is Ap-p-essential in M, thus N∩L⊆ J(R) N⊆ J(R) M hence L⊆ J(R) M. Hence K is Ap-p-essential in M.

⇒It is clear.

Corollary 3: Let M be an R-module that has the pure finite intersection property. If H is Ap-pure submodule of N and J(R) N∩H=J(R) H. Then K∩H is Ap-p-essential in M if and only if H is Ap-p-essential in M and K is Ap-p-essential in M, for any submodule K of M.

Proof:⇒the proof follows by theorem 2.

⇐ Let L be an Ap-pure submodule of M with (K∩H) ∩L⊆ J(R) M, by assumption H∩K is Ap-pure in M. Since K is Ap-p-essential in M, then H∩L⊆J(R) M. So since H isAp-p-essential in M, hence L⊆ J(R) M, therefore K∩H is Ap-p-essential in M.

Remark 4: If A is Ap-p-essential in B and $\acute{A}$ is Ap-p-essential in$\acute{B}$, then $A \oplus \acute{A}$ is not Ap-p-essential in $B \oplus \acute{B}$ [see example 4.6 [2]].

It is well known that, every direct summed of an R-module M is pure [5], hence every direct summed is Ap-direct summed. We introduce the concept of Ap-direct summed.

Definition 5: A submodule N of an R-module M is Ap-direct summed if there exists a submodule K of M with M=N+K and N∩K⊆ J(R) M.

Remark 6:

① Every Ap-direct summed of an R-module M is Ap-pure submodule.② Let M be an R-module. If P is Ap-pure submodule of M and Q is any submodule of M then P∩Q is Ap-pure in Q.

In [2], A submodule N of an R-module M is called a pure closed submodule of M if M does not contain a proper p-essential extension of N.We introduce the concept of approximately pure closed submodule.

Definition 7: A submodule N of an R-module M is called approximately pure closed submodule (simply Ap-p-closed) of M if M does not contain aproper Ap-p-essential extension of N.

Proposition 8: Let J(R)M∩K=J(R) K for every Ap-pure submodule K of M, then every Ap-direct summed is Ap-p-closed.

Proof: Let M=A⊕B, we want to show that A is Ap-p-closed. Suppose that A is Ap-p-essential in K we have to show that A=K. B is Ap-pure in M and K is any submodule of M. then B∩K is Ap-pure in K by remark (6) since A∩B ⊆ J(R) M. Hence (A∩ $B) \cap K \,t\, J(R)M \cap K$. Thus A∩(B∩K)⊆ J(R) K⊆ J(R) M since A is Ap-p-essential in M therefore B∩K ⊆ J(R) M. Then K=A, hence A is Ap-p-closed submodule of M.

Theorem 9: Let A, B and C be submodule of an R-module M then there exists an Ap-pure closed submodule H in M which is Ap-pure and J(R) M∩H=J(R) H such that C is Ap-pure closed in H.

Proof: Let V= {K: K is an Ap-pure submodule of H and J(R) M∩K= J(R) K such that C is Ap-p-essential in K}.It is clear that V≠ ∅.By Zorn's lemma, V has a maximal element say H. To show that H is Ap-p-closed in M.Let D is a submodule of M such that H is Ap-pure essential in D. Since C is Ap-p-essential in H and H is Ap-p-essential in D. Then by theorem 2 C is Ap-p-essential in D, thus H=D.

Let N and K be submodule of an R-module with K pure in M, K is called pure relative complements of N in M if K is maximal with the property K∩N= {0} [2]. We introduce the concept of relative approximately pure complement submodule.

Definition 10: Let N and K be submodules of an R-module M with K is Ap-pure in M, K is called relative Ap-pure complement of N in M if K is maximal with K∩N ⊆ J(R) M. Compare the following results with proposition 4.14 in [2].

Proposition 11: Every submodule of M has a relative Ap-pure complement in M.

Proof: Let N be a given submodule of M and consider the set S= {K≤M, K is Ap-pure in M with N∩K⊆ J(R) M}.It is clear that $S \neq \emptyset$ and every chain of S has an upper bound. By Zorn's Lemma S has maximal element, which means that N has relative Ap-pure complement in M.

The following propositions give the relation between Ap-pure closed submodule and relative Ap-pure complement submodule.

Proposition 12 Let N be a submodule of an R-module M and J(R) M∩F= J(R) F for every Ap-pure submodule F of M. If N is relative Ap-pure complement for some submodule K of M, then N is Ap-pure closed submodule in M.

Proof: Let L be an Ap-pure submodule of M with N is Ap-p-essential in L. We have N∩K⊆ J(R) M, (N∩K) ∩L⊆ J(R) M∩L.since L is Ap-pure in M, then K∩L is Ap-pure in L by remark 6 thus N ∩(K∩L)⊆ J(R) L ⊆J(R) M, hence L=N, hence N is Ap-pure closed in M.

In [4], an R module M is called a purely extending module, if every submodule of M is essential in a pure submodule of M.

Definition 13: Let M be an R-module, M is called an approximately purely extending (simply Ap-purely extending) module if every submodule is Ap-p-essential in Ap-pure submodule of M.

The following theorem gives a characterization of Ap-purely extending module.

Theorem 14: Let M be an R-module, then M is purly extending if and only if every Ap-pure closed submodule in M is Ap-direct summand in M.

Proof: Suppose that M is Ap-purely extending and let K be Ap-pure closed submodule in M. Then there exists an Ap-pure submodule B of M such that K is Ap-p-essential in B. But K is Ap-pure closed in M so K=B.Conversely, let A be a submodule of M, by theorem 9, there exists an Ap-pure closed submodule H in M such that A is Ap-p-essential in H. Since H is Ap-pure closed in M, then by our assumption, H is Ap-pure in M and hence M is Ap purely extending.

Recall that an R-module M have the approximately intersection (simply Ap-PIP) if the intersection of any two Ap-pure submodule is again Ap-pure. [6].

Proposition 15: Let M be Ap-purely extending module and let N be Ap-pure submodule of M. If M has the Ap-PIP, then N is Ap-purely extending.

Proof: Let A be an Ap-pure closed submodule in N. By theorem 8, there exists an Ap-pure closed submodule B in M such that A is Ap-p-essential in B. Since N is Ap-p-essential in N, then $A=A\cap N$ is Ap-p-essential in $B\cap N$ [illegible] N. But A is Ap-pure closed in N, therefore $A=B\cap N$. Since M is Ap-purely extending and B is Ap-pure closed in M, then by theorem 12 B is Ap-pure in M. But N is Ap-pure in M and M has Ap-PIP so $A=B\cap N$ is Ap-pure in M; hence A is Ap-pure in N. Thus N is Ap-purely extending.

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ملاحظات حول المقاسات الجزئية النقية تقريبا

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الخلاصة:

لتكن R حلقة ابدالية ذات عنصر محايد 1 وليكن M مقاسا ايسرا احاديا على الحلقة R. يسمى المقاس الجزئي N نقيا تقريبا اذا كان $(N\cap IM)$ M $\cap$ $N\cap IM=IN+J(R)$لكل مثالي I في R.

الهدف الرئيسي في هذا البحث هو دراسة خواص المقاسات الجزئية الجوهرية النقية تقريبا والمقاسات الجزئية المغلقة النقية تقريبا والمقاسات الجزئية المكملة النقية تقريبا. وكذلك تم دراسة بعض خواص المقاسات التوسيعية النقية تقريبا.