Hypercyclictty and Countable Hypercyclicity for Adjoint of $\theta-$Operators

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Abstract

Let H be an infinite dimensional separable complex Hilbert space and let $T \in B(H)$, where $B(H)$ is the Banach algebra of all bounded linear operators on H. In this paper we prove the following results.

If $T \in B(H)$ is a $\theta-$operator, then

1. T^* is a hypercyclic operator if and only if $\sigma(T|_M)\bigcap D \neq \phi$ and $\sigma(T|_M)\bigcap(C\setminus\overline{D}) \neq \phi$ for every hyperinvariant subspace M of T.

2. If T is a pure, then T^* is a countably hypercyclic operator if and only if $\sigma(T|_M)\bigcap(C\setminus\overline{D}) \neq \phi$ and $\sigma(T)\bigcap D \neq \phi$ for every hyperinvariant subspace M of T.

3. T^* has a bounded set with dense orbit if and only if for every hyperinvariant subspace M of T, $\sigma(T|_M)\bigcap(C\setminus\overline{D}) \neq \phi$.

Keywords: $\theta-$operator, hypercyclic, countably hypercyclic, single valued extension property (SVEP), Bishop's property (β), decomposition property (δ).

1. Introduction

Let H be an infinite dimensional separable complex Hilbert space, and $B(H)$ be the set of all bounded linear operators on H, we denote as usual the spectrum, the point spectrum and the approximate point spectrum of T by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$. Following **[1]**, the $Lat(T)$, where $T \in B(H)$, denoted the collection of all $T-$ invariant closed linear subspaces of H. If $T \in B(H)$ and $M \in Lat(T)$, then $T|_M \in B(M)$ is the restriction of T to M.

An operator $T \in B(H)$ is called *$\theta-$operator* if T^*T commutes with $T+T^*$, **[2]**. Recall that $T \in B(H)$ is called *normaloid* if $r(T)=\|T\|$, where $r(T)=\sup\{|\lambda|: \lambda \in \sigma(T)\}$, **[3]**.

It is well known **[4]** that
$\theta-operator \Rightarrow$ normaloid

An operator $T \in B(H)$ is called *hyponormal* if $\|T^*x\| \leq \|Tx\|$ for all $x \in H$. Campbell and Gellar **[5]** gave an example of a $\theta-operator$ which is not hyponormal, also Al-Sultan **[6]** gave an example of an operator which is hyponormal but it is not $\theta-$operator.

If $T \in B(H)$ and $x \in H$, then the orbit of x under T is $Orb(T,x)=\{x, Tx, T^2x, \ldots\}$, **[7]**. If

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$E \subseteq H$, then the orbit of E under T is
$Orb(T,E) = \bigcup\{E, T(E), T^2(E), \ldots\} = \bigcup_{x\in E} Orb(T,x)$, **([8], [9]).**

An operator $T \in B(H)$ is called *hypercyclic* if there is a vector $x \in H$ with dense orbit $\{x, Tx, T^2x, \ldots\}$, **[7].**

Following **([8], [9])**, we say that an operator $T \in B(H)$ is *countably hypercyclic* if there exists a bounded, countable, separated set E with dense orbit. Recall that a set $E \subseteq H$ is *separated* if there exists an $\varepsilon > 0$ such that $\| x - y \| \geq \varepsilon$ for all $x, y \in E$ with $x \neq y$.

In **[7]**, Feldman, Miller, and Miller proved that the cohyponormal operators (the adjoint of hyponormal operators) are hypercyclic if and only if $\sigma(T|_M) \bigcap D \neq \phi$ and $\sigma(T|_M) \bigcap (C \setminus \overline{D}) \neq \phi$ for every hyperinvariant subspace M of T. Recently Feldman **[8]** showed that there are countably hypercyclic operators which are not hypercyclic. Furthermore, Feldman showed that the pure cohyponormal operators are countably hypercyclic if and only if $\sigma(T|_M) \bigcap (C \setminus \overline{D}) \neq \phi$ and $\sigma(T) \bigcap D \neq \phi$ for every hyperinvariant subspace M of T. In this paper we give an example of a θ – operator which is not hypercyclic and prove that the adjoint of θ – operator is hypercyclic if and only if $\sigma(T|_M) \bigcap D \neq \phi$ and $\sigma(T|_M) \bigcap (C \setminus \overline{D}) \neq \phi$ for every hyperinvariant subspace M of T. We also give an example of a θ – operator which is not countably hypercyclic and prove that the adjoint of pure θ – operator is countably hypercyclic if and only if $\sigma(T|_M) \bigcap (C \setminus \overline{D}) \neq \phi$ and $\sigma(T) \bigcap D \neq \phi$ for every hyperinvariant subspace M of T. Finally we prove the adjoint of θ – operator has bounded set with dense orbit if and only if for every hyperinvariant subspace M of T, $\sigma(T|_M) \bigcap (C \setminus \overline{D}) \neq \phi$.

2. Preliminaries

An operator $T \in B(H)$ is said to have *single valued extension property* (SVEP) *at* λ_0 if for every open set $U \subseteq C$ containing λ_0 the only analytic solution $f : U \to H$ of the equation

$$(T - \lambda_0) f(\lambda) = 0 \qquad (\lambda_0 \in U)$$

is the zero function **[1]**. An operator T is said to have SVEP if T has SVEP at every $\lambda \in C$.

Given $T \in B(H)$, the *local resolvent set* $\rho_T(x)$ of T at the point $x \in H$ is defined as the union of all open subsets $U \subseteq C$ for which there is an analytic function $f : U \to H$ such that

$$(T - \lambda) f(\lambda) = x \qquad (\lambda \in U)$$

The *local spectrum* $\sigma_T(x)$ of T at x is then defined as $\sigma_T(x) = C \setminus \rho_T(T)$

For $T \in B(H)$, we define the *local* (resp. *glocal*) *spectral subspaces* of T as follows. Given a set $F \subseteq C$ (resp. a closed set $G \subseteq C$).
$H_T(F) = \{x \in H : \sigma_T(x) \subseteq F\}$
(resp.
$\mathrm{H}_T(F) = \{x \in H :$ there exists an analytic function $f : C \setminus G \to H$ such that $(T - \lambda) f(\lambda) = x$ for all $\lambda \in C \setminus G\}$).

Note that T has SVEP if and only if $H_T(F) = \mathrm{H}_T(F)$ for all closed sets $F \subseteq C$, **[1, Proposition (3.3.2)].**

If $U \subseteq C$ is an open set, then define $H_T(U) = \bigcup\{H_T(F) : F \subseteq U$ is compact$\}$. $H_T(U)$ contains all eigenvectors for T whose eigenvalues belong to U and that $H_T(U)$ is a hyperinvariant subspace for T, althought it is not necessarily closed, **[8]**.

An operator $T \in B(H)$ has *Dunford's property* (C) if the local spectral subspace $H_T(F)$ is closed for every closed set $F \subseteq C$. An operator $T \in B(H)$ is said to has *Bishop's property* (β) if for every sequence $f_n : U \to H$ such that $(T-\lambda)f_n(\lambda) \to 0$ uniformly on compact subsets in U, it follows that $f_n \to 0$ uniformly on compact subsets in U. It is well known **[1]** that

Bishop's property (β) $\Rightarrow$ Dunford's property (C) $\Rightarrow$ SVEP

Moreover, an operator $T \in B(H)$ has *decomposition property* (δ) if $H = H_T(\overline{U}) + H_T(\overline{V})$ for every open cover $\{U,V\}$ of C.

As shown in **[1]**, an operator $T \in B(H)$ has property (δ) iff it is the quotient of a decomposable operator. Moreover properties (β) and (δ) are dual to each other, in the sense that an operator $T \in B(H)$ has *property* (β) iff its adjoint has property (δ), and conversely, T has property (δ) iff its adjoint has property (β).

Proposition 2.1. [1] *Suppose that the operator* $T \in B(H)$ *on the Hilbert space* H *has SVEP, and that* $F \subseteq C$ *is a closed set for which the space* $H_T(F)$ *is closed. Then*

$\sigma(T|_{H_T\{F\}}) \subseteq F \cap \sigma_T(x)$

The following result from Feldman, Miller and Miller **[7]**, gives the relation between parts of the spectrum and the local spectra of an operator with Dunford's property (C).

Proposition 2.2. [7] *If* $T \in B(H)$ *has Dunford's property* (C), *then* $\sigma_T(x) = \sigma(T|_{H_T(F)})$ *whenever* $F = \sigma_T(x)$ *for some nonzero* $x \in H$.

The following result from Feldman, Miller and Miller **[7]**, gives sufficient condition for an operator to be hypercyclic, we denote the interior and exterior of the unit circle by $D, C \setminus \overline{D}$ respectively.

Corollary 2.3. [7] *Let* H *be a complex Hilbert space and suppose that* $T \in B(H)$ *has the decomposition property* (δ). If $\sigma_{T^*}(x) \cap D \neq \phi$ and $\sigma_{T^*}(x) \cap (C \setminus \overline{D}) \neq \phi$ *for every nonzero* $x \in H$. *Then* T *is hypercyclic*.

The following result from Feldman **[8]**, gives sufficient condition for an operator to be countable hypercyclic.

Theorem 2.4. [8] (The Countably Hypercyclic Criterion) *Suppose that* $T \in B(H)$. *If there exists two subspaces Y and Z in* H, *where Y is infinite dimensional and* Z *is dense in* H *such that*

1. $T^n x \to 0$ *for every* $x \in Y$, and
2. *There exists functions* $B_n : Z \to H$ *such that* $T^n B_n = I|_Z$ *and* $B_n x \to 0$ *for every* $x \in Z$

Then T *is countably hypercyclic.*

Theorem 2.5. [8] *Suppose that* $T \in B(H)$ *If* $H_T(D)$ *is infinite dimensional and* $H_T(C \setminus \overline{D})$ *is dense, then* T *is countably hypercuclic.*

Proposition 2.6. [8]

a. *If* $T \in B(H)$ *and there is a bounded set* E *with* $Orb(T,E)$ *dense, then* $\sup \| T^n \| = \infty$.

b. *If there is a set* E *that is bounded away from zero and* $Orb(T,E)$ *is dense, then* T cannot *be expensive, that is there exists an* $x \in H$ *such that* $\| Tx \| < \| x \|$.

In what follows, $B(a,r)$ will denote the open ball at a with radius r, where for $a \in H$ and $r > 0$.

Remark 2.7.

a. Notice that if T is countably hypercuclic and $E = \{x_n\}$ a bounded separated sequence with dense orbit, then one may assume that $x_n \neq 0$ for all n, thus it follows that E is both bounded and bounded away from zero, **[8].**

b. If an operator T has a set with dense orbit, then any non–zero multiple of that set also has dense orbit. Thus T has a bounded set with dense orbit if and only if the unit ball has dense orbit if and only if $B(a,r)$ has dense orbit for any $r > 0$, **[8].**

3. Hypercyclicity

It is well known that the restriction of $\theta-$operator $T|_M$ is a $\theta-$operator for every $M \in Lat(T)$, and if T is a $\theta-$operator and invertiable, then T^{-1} is a $\theta-$operator, **[7]**. Recall that an operator $T \in B(H)$ is dominant if $(T-\lambda)H \subset (T-\lambda)^* H$ for all scalars λ, Y. kato show that every $\theta-$operator is dominant, **[10]**.

Before proving one of important results in this paper, we need the following.

Definition 3.1. [11] An operator $T \in B(H)$ is said to have *the property (II)* if for every $\lambda, \mu \in \sigma_{ap}(T)$ and every bounded sequences of vectors x_n and y_n such that $\lambda \neq \mu$ and $\| (T-\lambda)x_n \| \to 0$, $\| (T-\lambda)y_n \| \to 0$, the sequence $\langle x_n, y_n \rangle$ converges to 0 as $n \to \infty$.

Theorem 3.2. [11] *If* T *has property (II), then* T *also has property* (β).

It is well known that dominant operator has Bishop's property (β) but couldn't find the proof, so we prove it.

Theorem 3.3. *Every dominant operator has Bishop's property* (β).

Proof. Let $\lambda, \mu \in \sigma_{ap}(T)$ $(\lambda \neq \mu)$ and sequences $\{x_n\}, \{y_n\}$ of bounded vectors in H satisfy $\| (T-\lambda)x_n \| \to 0$, $\| (T-\lambda)y_n \| \to 0$ (as $n \to 0$). Since T is dominant, then $\| (T-\lambda)^* y_n \| \to 0$ as $n \to \infty$. Hence

$$(\lambda-\mu)\langle x_n, y_n \rangle = \langle (\lambda - T)x_n, y_n \rangle + \langle x_n, (T-\mu)^* y_n \rangle \to 0$$

as $(n \to 0)$

This implies that $\langle x_n, y_n \rangle \to 0$. Then T has the property *(II)* Therefore T has property (β) by **Theorem (3.2).** □

Remark 3.4. *Every* $\theta-$*operator has Bishop's property* (β).

Now we give an example of $\theta-$operator which is not hypercyclic, We begin with the following result.

Corollary 3.5. [12] *If* $T \in B(H)$ *and* $\| T \| \leq 1$, *then* T *is not hypercyclic.*

Example 3.6. Let U be the unilateral shift operator defined on $\ell^2(\mathrm{N})$.

$U(x_1, x_2, x_{3,...}) = (0, x_1, x_2, x_3, \ldots)$

One can easily cheek that $((U^*U)(U+U^*))(x_1, x_2, x_3, \ldots) = (x_2, x_1 + x_3, x_1 + x_4, \ldots)$

$((U+U^*)(U^*U))(x_1, x_2, x_3, \ldots) = (x_2, x_1 + x_3, x_1 + x_4, \ldots)$

Which implies U is a $\theta-$operator. Since U is not hypercyclic by **Corollary (3.5)**. □

Now we give our Theorem.

Theorem 3.7. *If T is a $\theta-$operator on a separable Hilbert space H, then T^* is hypercyclic if and only if* $\sigma_T(x) \bigcap D \neq \phi$ and $\sigma_T(x) \bigcap (C \setminus \overline{D}) \neq \phi$ for every nonzero $x \in H$.

Proof. If T is a $\theta-$operator on H, then T has property (β) by **Remark (3.4)**. Thus T has property (C), and so T^* has property (δ). If the local spectra $\sigma_T(x) \bigcap D \neq \phi$ and $\sigma_T(x) \bigcap (C \setminus \overline{D}) \neq \phi$ for every nonzero $x \in H$, then T^* is hypercyclic by **Corollary (2.3).**

Conversely, suppose that T^* is hypercyclic. First we prove that every part of the spectrum of T meets both D and $C \setminus \overline{D}$, i.e., $\sigma(T|_M) \bigcap D \neq \phi$ and $\sigma(T|_M) \bigcap (C \setminus \overline{D}) \neq \phi$.

Let $S = T|_M$ for some $M \in Lat(T) \setminus \{0\}$. If x is a hypercyclic vector for T^*, then by the definition of hypercyclic vector $Orb(T^*, x) = \{x, T^*x, (T^*)^2 x, \ldots\}$ is dence in H.

We claim the projection $P_M x$ is hypercyclic for $S^* = P_M T^*|_M$. Since $M \in Lat(T) \setminus \{0\}$, then by **Corollary of Theorem 2, [3, P.39]**, $P_M T P_M = T P_M$. Consequently $P_M T^* P_M = P_M T^*$, and

$S^*(P_M x) = (P_M T^*|_M)(P_M x) = P_M T^*(P_M x)$

$= (P_M T^* P_M)(x) = P_M T^*(x) = P_M(T^* x)$

New a little bit calculation show that

$$\overline{Orb(S^*, P_M(x))} = \overline{\{P_M(x), S^*(P_M x), (S^*)^2(P_M x), \ldots\}}$$

$$= \overline{\{P_M(x), P_M(T^* x), S^*(P_M T^* x), \ldots\}}$$

$$= \overline{\{P_M(x), P_M T^*(x), P_M(T^*)^2(x), \ldots\}}$$

$$= P_M \overline{\{x, T^* x, (T^*)^2 x, \ldots\}} = \overline{P_M(H)} = M$$

i.e., the projection $P_M x$ is hypercyclic for $S^* = P_M T^*|_M$. Since S is a $\theta-$operator, then $r(S) = \| S \| = \| S^* \| > 1$ [If $\| S^* \| \leq 1$, then S^* is not hypercyclic this is impossible].

We prove $\sigma(S) \bigcap (C \setminus \overline{D}) \neq \phi$. Since $r(S) = \sup\{|\lambda| : \lambda \in \sigma(S)\} > 1$, this means that $\sigma(S)$ contains a complex number λ such that $|\lambda| > 1$ and since $C \setminus \overline{D} = \{\lambda : |\lambda| > 1\}$. Consequently $\sigma(S) \bigcap (C \setminus \overline{D}) \neq \phi$.

Now to show that $\sigma(S) \bigcap D \neq \phi$. If $\sigma(S) \subset (C \setminus \overline{D})$. i.e, $\sigma(S) \bigcap D = \phi$, then for all λ in $\sigma(S)$ is nonzero and hence $0 \in \rho(S)$, thus S is an invertiable and therefore S^{-1} is a $\theta-$operator.

Since $\sigma(S)$ contains a complex number λ such that $|\lambda| > 1$, then by **[3, P.171]**, $\sigma(S^{-1})$ contains a complex number λ^{-1} such that $|\lambda| \leq 1$. Thus $r(S^{-1}) = \inf\{|\lambda| : \lambda^{-1} \in \sigma(S)\} \leq 1$.

Consequently $\| S^{-1} \| \leq 1$. But S^* hypercyclic and invertiable, which implies that $(S^*)^{-1}$ is hypercyclic and thus $\| (S^*)^{-1} \| > 1$ by **Corollary (3.5)**.

Notice that $\| S^{-1} \| = \| (S^{-1})^* \| = \| (S^*)^{-1} \| > 1$, this is a contradiction since $\| S^{-1} \| \leq 1$, it follows that $\sigma(S) \bigcap D \neq \phi$.

Since T is a $\theta -$operator, then T has property (β) by **Remark (3.4)** and hence T has property (C). Thus by **Proposition (2.2)**, $\sigma_T(x) = \sigma(T|_{H_T(F)})$ whenever $F = \sigma_T(x)$ for every nonzero x and as in the previous paragraph, it follows that $\sigma_T(x) \bigcap D \neq \phi$ and $\sigma_T(x) \bigcap (C \setminus \overline{D}) \neq \phi$ for every nonzero $x \in H$. □

view of **Proposition (2.2)**, an equivalent way to state **Theorem (3.1)** is as follows.

Theorem 3.8. *If T is a $\theta -$operator on a separable Hilbert space H, then T^* is hypercyclic if and only if $\sigma(T|_M) \bigcap D \neq \phi$ and $\sigma(T|_M) \bigcap (C \setminus \overline{D}) \neq \phi$ for every hyperinvariant subspace M of T.*

4. Countably Hypercyclicity

It was shown in **[6]** that if T is a $\theta -$operator, then for fixed scalar, $\ker(T-\lambda)$ reduces T and $T|_{\ker(T-\lambda)}$ is normal. Recall that an operator $T \in B(H)$ is called pure if there is no reducing subspace M such that $T|_M$ is normal.

Proposition 4.1. *If T is a pure $\theta -$operator, then T has no eigenvalues.*

Proof. If $\lambda \in \sigma_p(T)$, then $T|_{\ker(T-\lambda)}$ is normal, it is a contradiction to definition of pure. Therefore $\sigma_p(T) = \phi$.

Now we give an example of $\theta -$operator which is not countably hypercyclic.

Example 4.2. Let U be the unilateral shift operator defined on ℓ^2(N)

$U(x_1, x_2, x_{3,...}) = (0, x_1, x_2, x_3, \ldots)$

U is a $\theta -$operator by **Example (3.6)**. since $\| U \| = 1$, then $\| U^n \| \leq \| U \|^n = 1$ and hence $\sup \| U^n \| < \infty$. Thus can not exists a bounded set E with $Orb(U, E)$ dence by **part (a)** of **Proposition (2.6)**. Therefore U is not countably hypercyclic. □

Lemma 4.3.

a. *If T is a $\theta -$operator on a Hilbert space H, then for any open set $U \subseteq C$, we have $H_{T^*}(U)^\perp = H_T(C \setminus U)$.*

b. *If T is a pure $\theta -$operator for which $H_{T^*}(D)$ is finite dimensional, then $H_{T^*}(D) = \{0\}$.*

Proof.

a. Since T is $\theta -$operator, then T has property (β) by **Remark (3.4)**, and hence T^* has property (δ). Therefore by **[1, Proposition (2.5.14)]**, for any open set $U \subseteq C$, we have $H_{T^*}(U)^\perp = H_T(C \setminus U)$.

b. Suppose that $H_{T^*}(D)$ is a nonzero and finite dimensional. Since $H_{T^*}(D)$ is finite dimensional invariant subspace for T^*, it follows that T^* has eigenvectors with eigenvalues in D. Let λ be such an eigenvalue, then since $\ker(T^* - \lambda) \subseteq H_{T^*}(D)$, it follows that $\ker(T^* - \lambda)$ is finite dimensional. Thus by **[1, Lemma (3.1.2)]**, $(T - \overline{\lambda})$ has closed range. Since T is pure, then by **Proposition (4.1)**, $T - \overline{\lambda}$ is one to one with closed range, hence $\overline{\lambda} \in [\sigma(T) \setminus \sigma_{ap}(T)]$. However, $[\sigma(T) \setminus \sigma_{ap}(T)]$ is an open set and since $\lambda \in D \bigcap [\sigma(T) \setminus \sigma_{ap}(T)]$, it follows that $D \bigcap [\sigma(T) \setminus \sigma_{ap}(T)]$ is a non-empty open set. Hence for each $\mu \in D \bigcap [\sigma(T) \setminus \sigma_a p(T)]$ we have $\ker(T^* - \overline{\mu}) \neq \{0\}$ and $\ker(T^* - \overline{\mu}) \subseteq H_{T^*}(D)$. It follows that $H_{T^*}(D)$ is infinite dimensional, a contradiction. □

Theorem 4.2. *If T is a pure $\theta-$operator on a separable Hilbert space H, then T^* is countably hypercyclic if and only if for every hyperinvariant subspace M of T, $\sigma(T|_M) \bigcap (C \setminus \overline{D}) \neq \phi$ and $\sigma(T) \bigcap D \neq \phi$*

Proof. Suppose the spectral conditions are satisfied. We want to apply **Theorem (2.5)**. So, suppose that $H_{T^*}(D) = \{0\}$. Since T is $\theta-$operator, then by **part (a)** of **Lemma (4.1)**, $H_{T^*}(D)^{\perp} = H_T(C \setminus D)$, it follows that $H_T(C \setminus D) = H$. Thus by **Proposition (2.1)**, $\sigma(T) = \sigma(T|_{H_T(C \setminus D)}) \subseteq (C \setminus D)$ a contradiction. So, $H_{T^*}(D) \neq \{0\}$, now by **part (b)** of **Lemma (4.1)** $H_{T^*}(D)$ is infinite dimensional. Now, suppose that $H_{T^*}(C \setminus \overline{D})$ is not dense in H, i.e., $\overline{H_{T^*}(C \setminus \overline{D})} \neq H$, then $H_{T^*}(C \setminus \overline{D}) \neq H$. Thus $H_T(\overline{D})$ is a nonzero **[** If $H_T(\overline{D}) = 0$, then by **part (a) of Lemma (4.1)**, $H_{T^*}(C \setminus \overline{D})^{\perp} = H_T(\overline{D}) = 0$, and hence $H_{T^*}(C \setminus \overline{D})^{\perp\perp} = H$. So $\overline{H_{T^*}(C \setminus \overline{D})} = H$. contradicting our assumption**]**. Therefore $H_T(\overline{D})$ is a nonzero hyperinvariant subspace for T. Furthermore $\sigma(T|_{H_T(\overline{D})}) \subseteq \overline{D}$ contradicting our assumption. Thus it follows that $H_{T^*}(C \setminus \overline{D})$ is dense. So, by **Theorem (2.5)**, T^* is countably hypercyclic.

Conversely, suppose T^* is countably hypercyclic. Let E be a bounded set, that is bounded away from zero, with dense orbit by **part (a)** of **Remark (2.7)**. Let M be an invariant subspace for T and let P_M be the projection onto M. It is easy to prove $(T|_M)^* P_M = P_M T^*$ and $P_M(E)$ is bounded set.

Now

$$\overline{Orb((T|_M)^*, P_M(E))}$$

$$= \bigcup_{\substack{P_M(x) \in P_M(E) \\ x \in E}} \overline{\{P_M(x), (T|_M)^*(P_M(x)), ((T|_M)^*)^2(P_M(x)), \ldots\}}$$

$$= \bigcup_{\substack{P_M(x) \in P_M(E) \\ x \in E}} \overline{\{P_M(x), ((T|_M)^* P_M)(x), ((T|_M)^*(T|_M)^* P_M)(x), \ldots\}}$$

$$= \bigcup_{\substack{P_M(x) \in P_M(E) \\ x \in E}} \overline{\{P_M(x), (P_M T^*)(x), (T|_M)^*((P_M T^*)(x)), \ldots\}}$$

$$= \bigcup_{\substack{P_M(x) \in P_M(E) \\ x \in E}} \overline{\{P_M(x), P_M(T^* x), ((P_M T^*)T^*)(x), \ldots\}}$$

$$= \bigcup_{\substack{P_M(x) \in P_M(E) \\ x \in E}} \overline{\{P_M(x), P_M(T^* x), (P_M T^*)^2)(x), \ldots\}}$$

$$= P_M(\bigcup_{x \in E} \overline{\{x, T^* x, (T^*)^2 x, \ldots\}}) = \overline{P_M(H)} = M$$

Therefore $P_M(E)$ whose orbit under $(T|_M)^*$ is dense in M. Thus, we must have $\| T|_M \| = \| (T|_M)^* \| > 1$ [If $\| (T|_M)^* \| \leq 1$, then $\| ((T|_M)^*)^n \| \leq 1$, $n = 0,1,2,\ldots$ and hence

$\sup \| ((T|_M)^*)^n \| < \infty$. This is impossible by **part (a)** of **Proposition (2.6)]**.

Since T is $\theta-$operator, then $T|_M$ is $\theta-$operator and hence $T|_M$ is normoliad. Thus $r(T|_M) = \| T|_M \| > 1$. Since $r(T|_M) = \sup\{|\lambda| : \lambda \in \sigma(T|_M)\} > 1$, then there is $\lambda \in \sigma(T|_M)$ such that $|\lambda| > 1$, also since $C \setminus \overline{D} = \{\lambda : |\lambda| > 1\}$. So $\sigma(T|_M) \cap (C \setminus \overline{D}) \neq \phi$.

Now, if $\sigma(S) \cap D = \phi$, i.e., $\sigma(S) \subset (C \setminus D)$, then for all λ in $\sigma(T)$ is nonzero and hence $0 \in \rho(T)$, thus T is an invertiable and therefore T^{-1} is $\theta-$operator. Since $\sigma(T)$ contains a complex number λ such that $|\lambda| > 1$, then by **[3, P.171]**, $\sigma(T^{-1})$ contains a complex number λ^{-1} such that $|\lambda| \leq 1$. Thus $r(T^{-1}) = \inf\{|\lambda| : \lambda^{-1} \in \sigma(T)\} \leq 1$. Consequently $\| (T^*)^{-1} \| = \| T^{-1} \| \leq 1$, hence $\| T^* x \| \geq \| x \|$ for all $x \in H$, contradicting **part (b)** of **Proposition (2.6)** □

Proposition 4.3. *If T is a $\theta-$operator, then T^* has a bounded set with dense orbit if and only if for every hyperinvariant subspace M of T, $\sigma(T|_M) \cap (C \setminus \overline{D}) \neq \phi$.*

Proof. Suppose that every hyperinvariant subspace M of T, $\sigma(T|_M) \cap (C \setminus \overline{D}) \neq \phi$, we want to show $H_{T^*}(C \setminus \overline{D})$ is dense in H. So, suppose that $H_{T^*}(C \setminus \overline{D})$ is not dense in H, i.e., $\overline{H_{T^*}(C \setminus \overline{D})} \neq H$, then $H_{T^*}(C \setminus \overline{D}) \neq H$ and hence $H_T(\overline{D})$ is a nonzero hyperinvariant subspace for T. Furthermore $\sigma(T|_{H_T(\overline{D})}) \subseteq \overline{D}$ contradicting our assumption. Thus $H_{T^*}(C \setminus \overline{D})$ is dense in H. It follows that if $Z = H_{T^*}(C \setminus \overline{D})$, then condition(2) of the Countably Hypercyclic Criterion is satisfied , see **[7, Theorem 3.2]**. However, condition (2) of the Countably Hypercyclic Criterion easily implies that the unit ball has dense orbit, then by **part (b)** of **Remark (2.7)** has a bounded set with dense orbit. The converse is similar to the proof of **Theorem (4.2)** □

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فوق الدائرية وفوق الدائرية المعدودة لمرافق المؤثر من النمط $\theta-$

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الخلاصة

ليكن H فضاء هلبرت قابلا للفصل وغير منتهي البعد على حقل الأعداد العقدية وليكن $B(H)$ ، حيث $B(H)$ هو جبر بناخ لكافة المؤثرات الخطية المقيدة على H في هذا البحث نبرهن انه

اذا كان T في $B(H)$ هو مؤثر من النمط $\theta-$ فان

1. T^* هومؤثر فوق الدائرية اذا وفقط اذا كان $\sigma(T|_M)\bigcap D \neq \phi$ و $\sigma(T|_M)\bigcap(C\setminus\overline{D}) \neq \phi$ لكل فضاء جزئي عالي الثبوتية M لـ H.
2. اذا كان T هو صرف، فان T^* هومؤثر فوق الدائرية والقابل للعد اذا كان $\sigma(T|_M)\bigcap(C\setminus\overline{D}) \neq \phi$ و $\sigma(T)\bigcap D \neq \phi$ لكل فضاء جزئي عالي الثبوتية M لـ H.
3. T^* يمتلك مجموعة مقيدة ذات مدار كثيف اذا وفقط اذا كان لكل فضاء جزئي عالي الثبوتية M لـ H $\sigma(T|_M)\bigcap(C\setminus\overline{D}) \neq \phi$.