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Best Multiplier Approximation of Unbounded Periodic Functions in $L_{p,\emptyset_n}(B), B = [0,2\pi]$ Using Discrete Linear Positive Operators

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Abstract:

The purpose of this paper is to find the best multiplier approximation of unbounded functions in $L_{p,\emptyset_n}$ –space by using some discrete linear positive operators. Also we will estimate the degree of the best multiplier approximation in term of modulus of continuity and the averaged modulus.

Key words: Multiplier convergence, Multiplier integral, Multiplier modulus.

Introduction:

Here we will review some researchers who have studied the approximation of unbounded functions and who have obtained important results in this field, for example In (2015), S. K. Jassim and (1) studied the approximation of unbounded functions in locally –global space $L_{p,\delta,\omega}(X)$.In(2015),S. K. Jassim and Alaa A. Auad (2) estimated the degree of the best one –sided approximation of unbounded functions by using some discrete operator in $L_{p,\omega}$-space (weighted space).Also the researchers have studied the approximation of bounded functions and obtained important results. For example, in (2016),Jafarov S. (3) studied the approximation of bounded functions by de la vallee-poussin sums in weighted Orlicz spaces.Again in (2016),Sadigova ,S. (4) studied the approximation of bounded functions by Shifting operators in Morrey Type Space. In this paper we will deal with unbounded functions by multiplier approximation and we shall use the Jackson polynomials and the Korovkin polynomials to find the best multiplier approximation of periodic unbounded functions in $L_{p,\emptyset_n}(B), B = [0,2\pi]$ in terms of the modulus of continuity and the Averaged modulus. First we introduce some definitions and some results that are used throughout this work.

Definition 1 (5):

A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergence if there is a Sequence $\{\emptyset_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} a_n \ \emptyset_n < \infty$, and we will say that $\{\emptyset_n\}$ is a multiplier for the convergence, for example.

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent series and the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ convergent sequence since $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent series then the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a multiplier convergence.

***Note* 1:**

If $\sum a_n$ is convergent series then it is multiplier convergent, this by taken $\{\emptyset_n\}_{n=0}^{\infty} = \{1\}_{n=0}^{\infty}$, but the reverse is not true

Definition 2 (5):

For any real valued function f if there is a sequence $\{\emptyset_n\}_{n=0}^{\infty}$ such that $\int_B f\emptyset_n(x) < \infty$, then we say that $\emptyset_n$ is a multiplier for the Integral

Definition 3:

Let $L_{P,\emptyset_n}(B), 1 \le p < \infty$ be the space of all real valued unbounded functions f such that $\int_B f\emptyset_n(x)\, dx < \infty$ with the following norm.

$$\|f\|_{L_{P,\emptyset_n}} = \sup\left\{\left(\int_B |f\emptyset_n(x)|^p\, dx\right)^{\frac{1}{p}} : x \in B\right\},$$

where $\emptyset_n$ is the multiplier for the integral, and $B = [0,2\pi]$.

For simplicity let us denote the norm $\|f\|_{L_{P,\emptyset_n}}$ by $\|f\|_{p,\emptyset_n}$

Definition 4:

For $f \in L_{P,\emptyset_n}(B)$ and $\delta > 0$ we will define the following concepts.

- $\omega\ (f,\delta)_{P,\emptyset_n} = sup_{|h|<\delta}\|f(x+h) - f(x)\|_{P,\emptyset_n}$, the multiplier modulus of continuity of function f
- $\tau^k(f,\delta)_{p,\emptyset_n} = \left\|\omega^k(f,.,\delta)\right\|_{p,\emptyset_n}$ $1 \le p < \infty$, $\delta > 0$, $k \in$N

Is the multiplier Averaged modulus of smoothness of f of order k, where k^{th} modulus of smoothness for f is defined by

$\omega^k(f,x,\delta)_{p,\emptyset_n} = sup_{|h|<\delta}\left\{\left\|\Delta_h^k(f,t)\right\|_{p,\emptyset_n} : t\,,t + kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right]\right\}$ Where $\Delta_h^k(f,x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}f\left(x - \frac{kh}{2} + ih\right), x \mp \frac{kh}{2} \in B$ the k^{th} symmetric difference of the function f .

Definition 5:

Let $f \in L_{P,\emptyset_n}(B)$ then the degree of best multiplier approximation of a function f with respect to trigonometric polynomial $g_n \in \Pi_n$ is given by

$E_n(f)_{P,\emptyset_n} = \inf\{\|f - g_n\|_{P,\emptyset_n}, g_n \in \Pi_n\}$

where Π_n be the set of all trigonometric polynomials.

Definition 6 (6):

For $f \in L_{P,\emptyset_n}(B)$, $B = [0,2\pi]$,$1 \le p < \infty$ let define the multiplier Jackson's operator which is generated by the following kernel

$$\Phi_n(t) = \frac{3}{2n(2n^2+1)}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^4 = \frac{1}{2} + \frac{2n(n^2-1)}{2n^2+1}\cos t + \sum_{k=2}^{2n-2}\rho_{k,n}\cos kt$$ **Where**

$\rho_{k,n}$ Is a constant depends on (k, n) and $\rho_{1,n} = \frac{2n(n^2-1)}{2n^2+1}$,$1 \le k \le n$, $n \ge 1$ **as follows**

$$K_n^*(f,x) = \frac{3}{2n^2(2n^2+1)}\sum_{k=1}^{2n} f\emptyset_n(t_{k,n}^*)\left(\frac{\sin\frac{n}{2}(t_{k,n}^*-x)}{\sin\frac{1}{2}(t_{k,n}-x)}\right)^4$$

Note 2: For $t_{k,n}^* = \frac{k\pi}{n}$, $1 \le k \le n$, $n = 1,2,\cdots$ we have

$$K_n^*(f,x) = \frac{3}{2n^2(2n^2+1)}\sum_{k=1}^{2n} f\emptyset_n\left(\frac{k\pi}{n}\right)\left(\frac{\sin\frac{n}{2}(\frac{k\pi}{n}-x)}{\sin\frac{1}{2}(\frac{k\pi}{n}-x)}\right)^4$$

Definition 7 (6):

For every trigonometric polynomial T of order $< n-1$ we have

$$\frac{2}{n}\sum_{k=1}^{n}\mathrm{T}\left(\frac{2k\pi}{n}\right) = \frac{1}{\pi}\int_0^{2\pi}\mathrm{T}(t)dt \cdots\cdots\cdots (1)$$

Proposition 8:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi]$, $1 \le p < \infty$ we have

$$K_n^*(f,x) = \frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\Phi_n(t-x)dt$$

Proof:

$$K_n^*(f,x) = \frac{3}{2n^2(2n^2+1)}\sum_{k=1}^{2n} f\emptyset_n\left(\frac{k\pi}{n}\right)\left(\frac{\sin\frac{n}{2}(\frac{k\pi}{n}-x)}{\sin\frac{1}{2}(\frac{k\pi}{n}-x)}\right)^4 .$$

Then using (1) we get

$$K_n^*(f,x) = \frac{3}{2n^2(2n^2+1)}\sum_{k=1}^{2n} f\emptyset_n\left(\frac{2k\pi}{2n}\right)\left(\frac{\sin\frac{n}{2}(\frac{k\pi}{n}-x)}{\sin\frac{1}{2}(\frac{k\pi}{n}-x)}\right)^4 =$$
$$\frac{2n}{2}\frac{3}{2n^2(2n^2+1)}\frac{2}{2n}\sum_{k=1}^{2n} f\emptyset_n\left(\frac{2k\pi}{2n}\right)\left(\frac{\sin\frac{n}{2}(\frac{k\pi}{n}-x)}{\sin\frac{1}{2}(\frac{k\pi}{n}-x)}\right)^4$$
$$=\frac{3}{2n(2n^2+1)}\frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\left(\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right)^4 dt = \frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)dt\,\frac{3}{2n(2n^2+1)}\left(\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right)^4$$
$$= \frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\,\Phi_n(t-x)dt$$ **Where** $\Phi_n(t-x) = \frac{3}{2n(2n^2+1)}\left(\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right)^4$

$$K_n^*(f,x) = \frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\,\Phi_n(t-x)dt \quad \blacksquare$$

Definition 9 (6):

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi]$, $1 \le p < \infty$,let us define

the multiplier linear positive Korovkin's operator which is generated by the following kernel

$$\varphi_n(t) = \frac{1}{n}sin^2\left(\frac{\pi}{n}\right)\left(\frac{\cos\left(\frac{nt}{2}\right)}{\cos t - \cos\frac{\pi}{n}}\right)^2 = \frac{1}{2} + \cos\left(\frac{\pi}{n}\right)\cos t + \sum_{k=2}^{n-2}\rho_{k,n}\cos kt$$

where $\rho_{k,n}$ is a constant depends on (k,n) and $\rho_{1,n} = \cos\frac{\pi}{n}$ **as follows**

$$K_n^{**}(f,x) = \frac{2}{n^2}sin^2\left(\frac{\pi}{n}\right)\sum_{k=1}^{n} f\emptyset_n(t_{k,n}^{**})\left(\frac{\cos\left(\frac{n}{2}(t_{k,n}^{**}-x)\right)}{\cos(t_{k,n}^{**}-x) - \cos\frac{\pi}{n}}\right)^2$$

Note 3: For $t_{k,n}^{**} = \frac{2k\pi}{n}$, $1 \le k \le n$, $n = 1,2,\cdots$ we have

$$K_n^{**}(f,x) = \frac{2}{n^2}sin^2\left(\frac{\pi}{n}\right)\sum_{k=1}^{n} f\emptyset_n\left(\frac{2k\pi}{n}\right)\left(\frac{\cos\left(\frac{n}{2}(\frac{2k\pi}{n}-x)\right)}{\cos\left(\frac{2k\pi}{n}-x\right) - \cos\frac{\pi}{n}}\right)^2$$

Proposition 10:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi]$, $1 \le p < \infty$ we have

$$K_n^{**}(f,x) = \frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\varphi_n(t-x)dt$$

Proof:

By using (1) we have

$$K_n^{**}(f,x) = \frac{2}{n^2}sin^2\left(\frac{\pi}{n}\right)\sum_{k=1}^{n} f\emptyset_n\left(\frac{2k\pi}{n}\right)\left(\frac{\cos(\frac{n}{2}(\frac{2k\pi}{n}-x))}{\cos(\frac{2k\pi}{n}-x) - \cos\frac{\pi}{n}}\right)^2$$
$$= \frac{n}{2}\frac{2}{n^2}sin^2\left(\frac{\pi}{n}\right)\frac{2}{n}\sum_{k=1}^{n} f\emptyset_n\left(\frac{2k\pi}{n}\right)\left(\frac{\cos(\frac{n}{2}(\frac{2k\pi}{n}-x))}{\cos(\frac{2k\pi}{n}-x) - \cos\frac{\pi}{n}}\right)^2$$
$$= \frac{1}{n}sin^2\left(\frac{\pi}{n}\right)\frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\left(\frac{\cos(\frac{n}{2}(t-x))}{\cos(t-x) - \cos\frac{\pi}{n}}\right)^2$$

$$= \frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\, \frac{1}{n} sin^2\left(\frac{\pi}{n}\right)\left(\frac{\cos(\frac{n}{2}(t-x))}{\cos(t-x)-\cos\frac{\pi}{n}}\right)^2$$

Thus $K_n^{**}(f,x) = \frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\varphi_n(t-x)\mathrm{dt}$ **Where**

$$\varphi_n(t-x) = \frac{1}{n} sin^2\left(\frac{\pi}{n}\right)\left(\frac{\cos(\frac{n}{2}(t-x))}{\cos(t-x)-\cos\frac{\pi}{n}}\right)^2 \quad ■$$

Auxiliary Results:

In this section we recall some lemmas and proved some results which will be used to prove the main results.

Lemma 1:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi], 1 \le p < \infty$ There is a constant $A(p)$ depends on p such that $\|K_n^*(f,x)\|_{p,\emptyset_n} \le A(p)\|f\|_{p,\emptyset_n}$

Proof:

$$K_n^*(f,x) = \frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\,\Phi_n(t-x)\mathrm{dt}$$

$$\|K_n^*(f,x)\|_{p,\emptyset_n} = \mathrm{Sup}\left\{\left(\int_0^{2\pi}\left|\frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\,\Phi_n(t-x)\mathrm{dt}\right|^p dx\right)^{\frac{1}{p}}\right\}$$

$$\le \mathrm{Sup}\left\{\left(\int_0^{2\pi}\left| f\,\emptyset_n(t)\,dt\right|^p dt\,\frac{1}{\pi}\int_0^{2\pi}\Phi_n(t-x)\,dx\right)^{\frac{1}{p}}\right\} \text{(Jensen inequality)}$$

$$\le \mathrm{Sup}\left\{\left(\int_0^{2\pi}| f\,\emptyset_n(t)|^p\, dt\right)^{\frac{1}{p}}\right\}.A(p) = A(p)\|f\|_{p,\emptyset_n}$$

$\|K_n^*(f,x)\|_{p,\emptyset_n} \le A(p)\|f\|_{p,\emptyset_n}$ ■ Similarly we can prove the next result.

Lemma 2:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi], 1 \le p < \infty$ there is a constant $A(p)$ depends on p such that $\|K_n^{**}(f,x)\|_{p,\emptyset_n} \le A(p)\|f\|_{p,\emptyset_n}$

Lemma 3 (**7**):

For $f \in L_P(B), B = [0,2\pi], 1 \le p < \infty$ we have $\tau_1(f,\delta)_p \le \delta\|f'\|_p$.**Where** f' is the first derivative of the function f .

Lemma 4:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi], 1 \le p < \infty$ we have $\tau^1(f,\delta)_{p,\emptyset_n} \le \delta\|f'\|_{p,\emptyset_n}$

Proof:

For $f \in L_{P,\emptyset_n}(B)$, there is a sequence $\{\emptyset_n\}_{n=1}^{\infty}$ such that $\int_0^{2\pi} f\emptyset_n(x) < \infty$ exists, using lemma (3) we get $\tau_1(f\emptyset_n,\delta)_p \le \delta\|(f\emptyset_n)'\|_p$ .Thus

$\tau^1(f,\delta)_{p,\emptyset_n} \le \delta\|f'\|_{p,\emptyset_n}$ ■

Lemma 5:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi], 1 \le p < \infty$ we have $\tau^k(f,\delta)_{p,\emptyset_n} \le \tau^{k-1}(f'\delta)_{p,\emptyset_n}$

Proof:

Let use the identity in **(7)**

$\Delta_h^k f(t) = \int_0^h \Delta_h^{k-1}(f'(t+u))du \quad, h > 0$ Then

$$\tau^k(f,\delta)_{p,\emptyset_n} = \left\|\omega^k(f,.,\delta)\right\|_{p,\emptyset_n} = \sup\left\{\left(\int_0^{2\pi}\left|\omega^k(f,x,\delta)\right|^p dx\right)^{\frac{1}{p}}\right\}$$

$$= \sup\left\{\left(\int_0^{2\pi}\left|sup\left\{\left(\int_0^{2\pi}\left|\Delta_h^k(f(x)\right|^p dx\right)^{\frac{1}{p}}\right\}\emptyset_n(t)\right|^p dt\right)^{\frac{1}{p}}\right\}$$

$$= \sup\left\{\left(\int_0^{2\pi}\left|sup\left\{\left(\int_0^{2\pi}\left|\int_0^h \Delta_h^{k-1} f'(t+u)\,du\right|^p \emptyset_n(t)\right)^{\frac{1}{p}}\right\}dx\right|^p dt\right)^{\frac{1}{p}}\right\}$$

$$\le \sup\left\{\left(\int_0^{2\pi}\left|sup\left\{\left(\int_0^{2\pi}\left|\Delta_h^{k-1} f'(t+u)\right|^p dx\int_0^h du\right)^{\frac{1}{p}}\right\}\emptyset_n(t)\right|^p dt\right)^{\frac{1}{p}}\right\}$$

$$\le \sup\left\{\left(\int_0^{2\pi}\left|sup\left\{\left(\int_0^{2\pi}\left|\Delta_h^{k-1}(f'(t+u))\right|^p dx\right)^{\frac{1}{p}}\right\}\left(\int_0^h du\right)\emptyset_n(t)\right|^p dt\right)^{\frac{1}{p}}\right\}$$

Thus

$$\le \sup\left\{\left(\int_0^{2\pi}\left|sup\left\|\Delta_h^{k-1}(f'(t))\right\|\emptyset_n(t).h\right|^p dt\right)^{\frac{1}{p}}\right\}$$

$$\le \sup\left\{\left(\int_0^{2\pi}\left|\omega^{k-1}(f',x,\delta^*)\emptyset_n(t)\right|^p dt\right)^{\frac{1}{p}}\right\}.h$$

$$\le \left\|\omega^{k-1}(f',x,\delta^*)\right\|_{p,\emptyset_n}.h \le h$$

$\tau^{k-1}(f',\delta^*)_{p,\emptyset_n} \le \delta\tau^{k-1}(f',\delta^*)_{p,\emptyset_n}$ **Where** $\delta^* = \frac{k}{k-1}\delta$ **Thus**

$$\tau^k(f,\delta)_{p,\emptyset_n} \le \tau^{k-1}(f'\delta)_{p,\emptyset_n} \quad ■$$

Lemma 6:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi], 1 \le p < \infty$ we get $\tau^k(f,\delta)_{p,\emptyset_n} \le \delta^k\left\|f^{(k)}\right\|_{p,\emptyset_n}$. **Where** $f^{(k)}$ is the k^{th} derivative of the function f

Proof:

From Lemma (5) we have

$$\tau^k(f,\delta)_{p,\emptyset_n} \le \delta\tau^{k-1}(f',\delta)_{p,\emptyset_n} \le \delta\delta\tau^{k-2}(f'',\delta)_{p,\emptyset_n} \le \cdots \underbrace{\delta\delta\cdots\delta}_{k-1\ time}\,\tau^1\left(f^{(k-1)},\delta\right)_{p,\emptyset_n}$$

Then using lemma (4) we have

$\tau^k(f,\delta)_{p,\emptyset_n} \le \underbrace{\delta\delta\cdots\delta}_{k-1\ time}\ \tau^1\left(f^{(k-1)},\delta\right)_{p,\emptyset_n} \le \underbrace{\delta\delta\cdots\delta}_{k-1\ time}.\delta\left\|f^{((k-1)+1)}\right\|_{p,\emptyset_n} = \delta^k\left\|f^{(k)}\right\|_{p,\emptyset_n}$ **Thus**

$\tau^k(f,\delta)_{p,\emptyset_n} \le \delta^k\left\|f^{(k)}\right\|_{p,\emptyset_n}$ ■

Lemma 7 (6):

For any linear positive operator $L_n(f,x)$ of 2π −periodic and continuous functions f we have

$$|L_n(f,x)-f(x)| \le \left(L_n(1,x) + \pi\sqrt{L_n(1,x)}\right)\omega(f,\beta_n(x)) + |f(x)|.|L_n(1,x)-1|$$

Where $\beta_n(x) = \sqrt{L_n\left(sin^2(\frac{t-x}{2}),x\right)}$

Lemma 8 (7):

If f is a measurable bounded function on $[a,b]$ then

$$\omega^k(f,\delta)_p \le \tau^k(f,\delta)_p \le (b-a)^{\frac{1}{p}}\ \omega^k(f,\delta)_p$$

Lemma 9:

For $f \in L_{P,\emptyset_n}(B), B=[0,2\pi]$, $1 \le p < \infty$,$1 \le$ k ≤ n , n ≥ 1 we get

$\omega^k(f,\delta)_{p,\emptyset_n} \le \tau^k(f,\delta)_{p,\emptyset_n}$

Proof:

$$\omega^k(f,\delta)_{p,\emptyset_n} = sup\left\{\left\|\Delta_h^k(f)\right\|_{p,\emptyset_n}\right\} = \sup\left\{\begin{matrix} sup \\ |h|<\delta\end{matrix}\left\{(\int_0^{2\pi}\left|(\Delta_h^k(f(x))\emptyset_n\right|^p dx)^{\frac{1}{p}}\right\}\right\}$$

$$\le \sup\left\{\begin{matrix} sup \\ |h|<\delta\end{matrix}\left\{(\int_0^{2\pi-\frac{kh}{2}}\left((\omega^k(f(x),x+\frac{kh}{2},\delta)\emptyset_n\right)^p dx)^{\frac{1}{p}}\right\}\right\}$$

$$\le \sup\left\{\begin{matrix} sup \\ |h|<\delta\end{matrix}\left\{(\int_{0+\frac{kh}{2}}^{2\pi-\frac{kh}{2}}((\omega^k(f(x),x,\delta)\emptyset_n)^p\ dx)^{\frac{1}{p}}\right\}\right\}$$

$\omega^k(f,\delta)_{p,\emptyset_n} \le sup\left\{\left\|\omega^k(f,.,\delta)\right\|_{p,\emptyset_n}\right\} \le sup\tau^k(f,\delta)_{p,\emptyset_n} = \tau^k(f,\delta)_{p,\emptyset_n}$ **.Thus**

$$\omega^k(f,\delta)_{p,\emptyset_n} \le \tau^k(f,\delta)_{p,\emptyset_n}$$ ■

Main Results

We shall prove the direct inequality of $f \in L_{P,\emptyset_n}(B), B=[0,2\pi]$ by **Jackson** and **Korovkin** polynomials in terms of Average modulus and the modulus of continuity.

Theorem 1:

For $f \in L_{P,\emptyset_n}(B), B=[0,2\pi]$, $1 \le p < \infty$,$1 \le k \le n$ we get

$$\mathrm{E_n}(f)_{\mathrm{P},\emptyset_\mathrm{n}} \le \frac{1}{n^k}\left\|f^{(k)}\right\|_{p,\emptyset_n}$$

Proof:

Let Π_n be the set of all trigonometric polynomial , $f \in L_{p,\emptyset_n}(B)$, then

$\|f\emptyset_n(.)\|_p = \left[\int_B |f\emptyset_n(x)|^p dx\right]^{\frac{1}{p}}$ **, exists**

Then by (Sendov Theorem in (8)) there is a polynomial $T_n^* \in \Pi_n$ such that

$\|f\emptyset_n(\cdot) - T_n^*\emptyset_n(.)\|_p \le C_1(k)\omega^k(f\emptyset_n(x),\frac{1}{n})_p$

.Thus

$\|f\emptyset_n(\cdot) - T_n^*\emptyset_n(.)\|_p = \|(f-T_n^*)\emptyset_n\|_p = \|f-T_n^*\|_{p,\emptyset_n} \le C_1(k)\omega^k\left(f\emptyset_n(x),\frac{1}{n}\right)_p = C_1(k)\omega^k\left(f,\frac{1}{n}\right)_{p,\emptyset_n}$ **.Thus**

$\|f-T_n^*\|_{p,\emptyset_n} \le C_1(k)\omega^k\left(f,\frac{1}{n}\right)_{p,\emptyset_n}$ **Now** using lemma (9) and lemma (6) we get

$E_n(f)_{p,\emptyset_n} = \inf_{T_n\in\pi_n}\{\|f-T_n\|_{p,\emptyset_n}\} = \|f-T_n^*\|_{p,\emptyset_n} \le C(k)\omega^k\left(f,\frac{1}{n}\right)_{P,\emptyset_n} \le C(k)\tau^k\left(f,\frac{1}{n}\right)_{p,\emptyset_n} \le \frac{1}{n^k}\left\|f^k\right\|_{p,\emptyset_n}$ **Thus**

$E_n(f)_{P,\emptyset_n} \le \frac{1}{n^k}\left\|f^k\right\|_{p,\emptyset_n}$, where T_n^* is the best multiplier approximation of f ■

Theorem 2:

For $f \in L_{P,\emptyset_n}(B), B=[0,2\pi]$, $1 \le p < \infty$ we have

$\|K_n^*(f,.) - f(.)\|_{p,\emptyset_n} \le (1+\pi)\tau^1(f,\beta_n^*)_{p,\emptyset_n}$

,where $\beta_n^* = \frac{\sqrt{3}}{\sqrt{4n^2+2}}$

Proof:

Since $K_n^*(f,x)$ is a linear positive operator we can using lemma (7)

First we must prove the following

1-$K_n^*(1,x) = 1$

2- $\beta^*{}_n(x) = \sqrt{K_n^*\left(sin^2(\frac{t-x}{2}),x\right)} = \frac{\sqrt{3}}{\sqrt{4n^2+2}}$

Since $K_n^*(f,x) = \frac{1}{\pi}\int_0^{2\pi} f\emptyset_n(t)\ \Phi_n(t-x)\mathrm{dt}$

then

$$K_n^*(1,x) = \frac{1}{\pi}\int_{\mathbf{0}}^{\mathbf{2\pi}}\Phi_n(t-x)\mathrm{dt} = \frac{1}{\pi}\int_{\mathbf{0}}^{\mathbf{2\pi}}\Phi_n(t)\mathrm{dt} = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1}{2} + \frac{2n(n^2-1)}{2n^2+1}\cos t + \sum_{k=2}^{2n-2}\rho_{k,n}\cos kt\right)dt$$

$$K_n^*(1,x) = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}\ dt + \frac{1}{\pi}\int_0^{2\pi}\frac{2n(n^2-1)}{2n^2+1}\cos t\ dt + \frac{1}{\pi}\int_0^{2\pi}\ \sum_{k=2}^{2n-2}\rho_{k,n}\cos kt\ \ dt$$

$$K_n^*(1,x) = \frac{1}{\pi}\left[\frac{1}{2}t\right]_0^{2\pi} + \frac{1}{\pi}\ \frac{2n(n^2-1)}{2n^2+1}\ [\sin t]_0^{2\pi} + \frac{1}{k\pi}\sum_{k=2}^{2n-2}\rho_{k,n}[\sin kt]_0^{2\pi}$$

$K_n^*(1,x) = \frac{1}{\pi}\left[\frac{1}{2}\ 2\pi\right] + 0 + 0 = 1$ **Now**

$$(\beta^*{}_n(x))^2 = K_n^*\left(sin^2(\frac{t-x}{2}),x\right) = \frac{1}{\pi}\int_0^{2\pi} sin^2(\frac{t-x}{2})\ \Phi_n(t-x)\mathrm{dt}$$

$$(\beta^*{}_n(x))^2 = \frac{1}{\pi}\int_0^{2\pi} sin^2(\frac{t}{2})\ \Phi_n(t)\mathrm{dt}$$

$= \frac{1}{2\pi}\int_0^{2\pi}(1-\cos t)\Phi_n(t)\mathrm{dt}$ **By** $\left[sin^2(kt) = \frac{1}{2}(1-\cos 2kt)\right]$

$= \frac{1}{2}\left[\frac{1}{\pi}\int_0^{2\pi}\Phi_n(t)\mathrm{dt} - \frac{1}{\pi}\int_0^{2\pi}\cos t\,\Phi_n(t)\mathrm{dt}\right]$

$= \frac{1}{2}\left[1 - \frac{1}{\pi}\int_0^{2\pi}\cos t\left[\frac{1}{2} + \frac{2(n^2-1)}{2n^2+1}\cos t + \sum_{k=2}^{2n-2}\rho_{k,n}\cos kt\right]dt\right]$

$= \frac{1}{2}\left[1 - \frac{1}{2\pi}\int_0^{2\pi}\cos t\,dt - \frac{1}{\pi}\frac{2(n^2-1)}{2n^2+1}\int_0^{2\pi}cos^2 t\,dt - \frac{1}{\pi}\sum_{k=2}^{2n-2}\rho_{k,n}\int_0^{2\pi}\cos kt\ \ dt\right]$

$= \frac{1}{2}\left[1 - 0 - \frac{1}{\pi}\frac{2(\mathrm{n}^2-1)}{2\mathrm{n}^2+1}.\pi - 0\right] = \frac{1}{2}\left[1 - \frac{2(\mathrm{n}^2-1)}{2\mathrm{n}^2+1}\right] = \frac{3}{4\mathrm{n}^2+2}$ **Thus**

$\beta^*{}_n(x) = \frac{\sqrt{3}}{\sqrt{4n^2+2}}$ **.Therefor**

$\|K_n^*(f,.) - f(.)\|_{p,\emptyset_n} = sup\left\{\left(\int_0^{2\pi}\left|\left((K_n^*(f,x) - f(x)\right)\emptyset_n\right|^p dx\right)^{\frac{1}{p}}\right\}$

$\leq sup\left\{\left(\int_0^{2\pi}\left|\left(K_n^*(1,x) + \pi\sqrt{K_n^*(1,x)}\right)\omega^1\left(f,\beta^*{}_n(x)\right)\emptyset_n + |f(x)|\,|K_n^*(1,x)-1|\emptyset_n\right|^p dx\right)^{\frac{1}{p}}\right\}$

$= sup\left\{\left(\int_0^{2\pi}\left|\left((1+\pi)\omega^1\left(f,\beta^*{}_n(x)\right)\right)\emptyset_n\right|^p dx\right)^{\frac{1}{p}}\right\}$

$= (1+\pi)sup\left\{\left(\int_0^{2\pi}\left|\left(\omega^1\left(f,x,\beta^*{}_n(x)\right)\right)\emptyset_n\right|^p dx\right)^{\frac{1}{p}}\right\}$

$\leq (1+\pi)\|\omega^1(f,.,\beta_n^*)\|_{p,\emptyset_n}$

$= (1+\pi)\tau^1(f,\beta_n^*)_{p,\emptyset_n}$ **where** $\beta_n^* = \frac{\sqrt{3}}{\sqrt{4n^2+2}}$

Thus

$\|K_n^*(f,.) - f(.)\|_{p,\emptyset_n} \leq (1+\pi)\tau^1(f,\beta_n^*)_{p,\emptyset_n}$ ■

Corollary 3:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi]\,, 1 \leq p < \infty$ we have

$\|K_n^*(f,.) - f(.)\|_{p,\emptyset_n} \leq \frac{(1+\pi)\sqrt{3}}{\sqrt{4n^2+2}}\|f'\|_{p,\emptyset_n}$

Proof:

Using theorem (2) and lemma (4) we have

$\|K_n^*(f,.) - f(.)\|_{p,\emptyset_n} \leq (1+\pi)\tau^1(f,\beta_n^*)_{p,\emptyset_n} \leq (1+\pi)\beta_n^*\|f'\|_{p,\emptyset_n} = \frac{(1+\pi)\sqrt{3}}{\sqrt{4n^2+2}}\|f'\|_{p,\emptyset_n}$ **Thus**

$\|K_n^*(f,.) - f(.)\|_{p,\emptyset_n} \leq \frac{(1+\pi)\sqrt{3}}{\sqrt{4n^2+2}}\|f'\|_{p,\emptyset_n}$ ■

Corollary 4:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi]\,, 1 \leq p < \infty$ we have $\mathrm{K}_\mathrm{n}^*(f,\mathrm{x}) \xrightarrow{\mathrm{u.\ c.}} f(\mathrm{x})$

Proof:

By theorem (2) we have

$\|K_n^*(f,.) - f(.)\|_{p,\emptyset_n} \leq (1+\pi)\tau^1(f,\beta_n^*)_{p,\emptyset_n}$ **Then**

$\lim_{n\to\infty}\|K_n^*(f,.) - f(.)\|_{p,\emptyset_n} \leq \lim_{n\to\infty}(1+\pi)\tau^1(f,\beta_n^*)_{p,\emptyset_n}$

$= (1+\pi)\lim_{n\to\infty}\tau^1(f,\beta_n^*)_{p,\emptyset_n} = (1+\pi)\tau^1(f,\lim_{n\to\infty}\beta_n^*)_{p,\emptyset_n}$

$= (1+\pi)\tau^1(f,0)_{p,\emptyset_n}$ **Thus**

$\|K_n^*(f,.) - f(.)\|_{p,\emptyset_n} \to 0$ **As** n $\to \infty$ **Then** $\mathrm{K}_\mathrm{n}^*(f,\mathrm{x}) \xrightarrow{\mathrm{u.\ c.}} f(\mathrm{x})$ ■

Theorem 5:

For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi]\,, 1 \leq p < \infty$ we have

$\|K_n^{**}(f,.) - f(.)\|_{p,\emptyset_n} \leq (1+\pi)\tau^1(f,\beta_n^{**})_{p,\emptyset_n}$

where $\beta_n^{**} = \frac{\sqrt{1-\cos\frac{\pi}{n}}}{\sqrt{2}}$

Proof:

Since $K_n^{**}(f,x)$ is linear positive operator we can using lemma (7)

First we must prove the following

1-$K_n^{**}(1,x) = 1$

2-$\beta_n^{**}(x) = \sqrt{K_n^{**}\left(sin^2\left(\frac{t-x}{2}\right),x\right)} = \frac{\sqrt{1-\cos\frac{\pi}{n}}}{\sqrt{2}}$

Since $K_n^{**}(f,x) = \frac{1}{\pi}\int_0^{2\pi}f\emptyset_n(t)\varphi_n(t-x)\mathrm{dt}$

Thus

$$K_n^{**}(1,x) = \frac{1}{\pi}\int_0^{2\pi}\varphi_n(t-x)\mathrm{dt} = \frac{1}{\pi}\int_0^{2\pi}\varphi_n(t)\mathrm{dt}$$

$$K_n^{**}(1,x) = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1}{2} + \cos\left(\frac{\pi}{n}\right)\cos t + \sum_{k=1}^{n-2}\rho_{k,n}\cos kt\right)dt$$

$K_n^{**}(1,x) = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}\,dt + \frac{1}{\pi}\int_0^{2\pi}\cos\frac{\pi}{n}\cos t\ dt + \frac{1}{\pi}\int_0^{2\pi}\sum_{k=1}^{n-2}\rho_{k,n}\cos kt\ dt$

$K_n^{**}(1,x) = \frac{1}{\pi}\left[\frac{1}{2}t\right]_0^{2\pi} + \frac{1}{\pi}\cos\frac{\pi}{n}[\sin t]_0^{2\pi} + \frac{1}{k\pi}\sum_{k=2}^{2n-2}\rho_{k,n}[\sin kt]_0^{2\pi}$

$K_n^{**}(1,x) = \frac{1}{\pi}\left[\frac{1}{2}2\pi\right] + 0 + 0 = 1$ **Now**

$(\beta_n^{**})^2(x) = K_n^{**}\left(sin^2(\frac{t-x}{2}), x\right) =$
$\frac{1}{\pi}\int_0^{2\pi} sin^2(\frac{t-x}{2})\ \varphi_n\ (t-x)\text{dt}$
$(\beta_n^{**})^2(x) = \frac{1}{\pi}\int_0^{2\pi} sin^2(\frac{t}{2})\ \varphi_n(\text{t})\ \text{dt}$
$= \frac{1}{2\pi}\int_0^{2\pi}(1-\cos t)\varphi_n(\text{t})\ \text{dt}$ **By** $\left[sin^2(kt) = \frac{1}{2}(1-\cos 2kt)\right]$
$= \frac{1}{2}\left[\frac{1}{\pi}\int_0^{2\pi}\varphi_n(\text{t})\ \text{dt} - \frac{1}{\pi}\int_0^{2\pi}\cos t\,\varphi_n(\text{t})\ \text{dt}\right]$
$= \frac{1}{2}\left[1 - \frac{1}{\pi}\int_0^{2\pi}\cos t\left[\frac{1}{2} + \cos\frac{\pi}{n}\cos t + \sum_{k=2}^{n-2}\rho_{k,n}\cos kt\right]\right]$
$= \frac{1}{2}\left[1 - \frac{1}{2\pi}\int_0^{2\pi}\cos t - \frac{1}{\pi}\cos\frac{\pi}{n}\int_0^{2\pi}cos^2 t\ dt - \frac{1}{\pi}\sum_{k=2}^{n-2}\rho_{k,n}\int_0^{2\pi}\cos kt\ \ dt\right]$
$= \frac{1}{2}\left[1-\ 0\ -\ \frac{1}{\pi}\ \cos\frac{\pi}{n}\ .\pi\ -0\right] = \frac{1}{2}\left[1-\cos\frac{\pi}{n}\right]$
.Thus

$\beta_n^{**}\ (x) = \sqrt{K_n^{**}\left(sin^2(\frac{t-x}{2}), x\right)} = \frac{\sqrt{1-\cos\frac{\pi}{n}}}{\sqrt{2}}$

Therefor
$\|K_n^{**}(f,.) - f(.)\|_{p,\emptyset_n} =$
$sup\left\{\left(\int_0^{2\pi}\left|\left(\left(K_n^{**}(f,x) - f(x)\right)\right)\emptyset_n\right|^p dx\right)^{\frac{1}{p}}\right\}$
$\leq$
$sup\left\{\left(\int_0^{2\pi}\left|\left(K_n^{**}(1,x) + \pi\sqrt{K_n^{**}(1,x)}\right)\omega^1(f,\beta_n^{**}\ (x))\emptyset_n + |f(x)|\ |K_n^{**}(1,x)-1|\emptyset_n\right|^p dx\right)^{\frac{1}{p}}\right\}$
$=$
$(1+\pi)sup\left\{\left(\int_0^{2\pi}\left|\left(\omega^1(f,x,\beta_n^{**}(x))\right)\emptyset_n\right|^p dx\right)^{\frac{1}{p}}\right\}$
$\leq (1+\pi)\|\omega^1(f,.,\beta_n^{**})\|_{p,\emptyset_n}$
$= (1+\pi)\tau^1(f,\beta_n^{**})_{p,\emptyset_n}$ **.Thus**
$\|K_n^{**}(f,.) - f(.)\|_{p,\emptyset_n} \leq (1+\pi)\tau^1(f,\beta_n^{**})_{p,\emptyset_n}$,
with $\beta_n^{**} = \frac{\sqrt{1-\cos\frac{\pi}{n}}}{\sqrt{2}}$ ■

Corollary 6:
For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi]\ , 1 \leq p < \infty$ we have
$\|K_n^{**}(f,.) - f(.)\|_{p,\emptyset_n} \leq \frac{(1+\pi)}{\sqrt{2}}\sqrt{1-\cos\frac{\pi}{n}}\|f'\|_{p,\emptyset_n}$

Proof:
Using theorem (5) and lemma (4) we have
$\|K_n^{**}(f,.) - f(.)\|_{p,\emptyset_n} \leq (1+\pi)\tau^1(f,\beta_n^{**})_{p,\emptyset_n} \leq (1+\pi)\beta_n^{**}\|f'\|_{p,\emptyset_n}$
$= (1+\pi)\frac{\sqrt{1-\cos\frac{\pi}{n}}}{\sqrt{2}}\ \|f'\|_{p,\emptyset_n}$ **Thus**
$\|K_n^{**}(f,.) - f(.)\|_{p,\emptyset_n} \leq \frac{(1+\pi)}{\sqrt{2}}\sqrt{1-\cos\frac{\pi}{n}}\|f'\|_{p,\emptyset_n}$ ■

Corollary 7:
For $f \in L_{P,\emptyset_n}(B), B = [0,2\pi]\ , 1 \leq p < \infty$ we have
$\text{K}_\text{n}^{**}(f,\text{x}) \xrightarrow{\text{u. c.}} f(\text{x})$

Proof:
Using corollary (6) we get
$\|K_n^{**}(f,.) - f(.)\|_{p,\emptyset_n} \leq \frac{(1+\pi)}{\sqrt{2}}\sqrt{1-\cos\frac{\pi}{n}}\|f'\|_{p,\emptyset_n}$ **Then**
$\lim_{n\to\infty}\|K_n^{**}(f,.) - f(.)\|_{p,\emptyset_n} \leq$
$\lim_{n\to\infty}\frac{(1+\pi)}{\sqrt{2}}\sqrt{1-\cos\frac{\pi}{n}}\|f'\|_{p,\emptyset_n} =$
$\frac{(1+\pi)}{\sqrt{2}}\ \|f'\|_{p,\emptyset_n}\lim_{n\to\infty}\sqrt{1-\cos\frac{\pi}{n}} =$
$\frac{(1+\pi)}{\sqrt{2}}\ \|f'\|_{p,\emptyset_n}\sqrt{1-\lim_{n\to\infty}\cos\frac{\pi}{n}} =$
$\frac{(1+\pi)}{\sqrt{2}}\ \|f'\|_{p,\emptyset_n}\sqrt{1-1}$ **Thus**
$\|K_n^{**}(f,.) - f(.)\|_{p,\emptyset_n}$ $\to$ **0As**n $\to$ **0** **Then**
$\text{K}_\text{n}^{**}(f,\text{x}) \xrightarrow{\text{u. c.}} f(\text{x})$ ■

Conclusion:

By using the multiplier Jackson's operator$K_n^{*}(f,x)$and by the multiplier linear positive Korovkin's operator $K_n^{**}(f,x)$ we obtained the best Multiplier approximation of $f \in L_{p,\emptyset_n}(B), B = [0,2\pi]$ and we estimated the degree of the best multiplier approximation in terms of modulus of continuity and the averaged modulus.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
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أفضل تقريب مضروب للدوال الدورية الغير مقيدة باستخدام المؤثرات الخطية المتقطعة الموجبة

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الخلاصة:

الغرض من هذا البحث ايجاد افضل تقريب لمضروب الدوال الغير مقيدة في الفضاءات االدورية باستخدام الموءثرات الخطية المتقطعة وكذلك تقدير درجة افضل تقريب مضاعف من ناحية نموذج الاستمرارية ونموذج المتوسط.

الكلمات المفتاحية: التقارب المضروب، التكامل المضروب، النموذج المضروب.