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Properties of the Adjoint Operator of a General Fuzzy Bounded Operator

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Abstract:

Our goal in the present paper is to recall the concept of general fuzzy normed space and its basic properties in order to define the adjoint operator of a general fuzzy bounded operator from a general fuzzy normed space V into another general fuzzy normed space U. After that basic properties of the adjoint operator were proved then the definition of fuzzy reflexive general fuzzy normed space was introduced in order to prove that every finite dimensional general fuzzy normed space is fuzzy reflexive.

Keywords: Adjoint operator, Fuzzy reflexive space, General fuzzy normed space GFB(V, ℝ), General fuzzy continuous operator, General Fuzzy bounded operator.

Introduction:

The fuzzy topological structure of a fuzzy normed space was studied by Sadeqi and Kia in 2009 (1). Kider introduced a fuzzy normed space in 2011 (2). Also he proved this fuzzy normed space has a completion in (3). Again Kider introduced a new fuzzy normed space in 2012 (4). The properties of fuzzy continuous mapping which is defined on a fuzzy normed space was studied by Nadaban in 2015 (5).

Kider and Kadhum in 2017 (6) introduced the fuzzy norm for a fuzzy bounded operator on a fuzzy normed space and proved its basic properties then other properties were proved by Kadhum in 2017 (7). Ali in 2018 (8) proved basic properties of complete fuzzy normed algebra. Kider and Ali in 2018 (9) introduced the notion of fuzzy absolute value and studied the properties of finite dimensional fuzzy normed space.

The notion of general fuzzy normed space was introduced by Kider and Gheeab in 2019 (10, 11) also they proved basic properties of this space and the general fuzzy normed space GFB(V, U). Kider and Kadhum in 2019 (12) introduced the notion fuzzy compact linear operator and proved its basic properties.

In this paper first, the definition of a general fuzzy normed space and also its basic properties is recalled so that to define the adjoint operator of a general fuzzy bounded operator from a general fuzzy normed space V into another general fuzzy normed space U. Then the important properties for the adjoint operator were proved. Finally the notion fuzzy reflexive general fuzzy normed space is defined in order to prove that every finite dimensional general fuzzy normed space is fuzzy reflexive.

Basic Properties of General Fuzzy Norm

Definition 1:(9)

A binary operation $\odot$:[0,1]×[0,1]→[0,1] satisfying (1) $a\odot b = b\odot a$,(2) $b\odot 1 = b$, (3) $a\odot[b\odot t] = [a\odot b]\odot t$
(4) if $b\leq a$ and $t\leq s$ then $b\odot t \leq a\odot s$.
for all a, b, s, t ∈[0,1] is called a continuous **triangular norm [or t-norm].**

Definition 2:(9)

Let ℝ be a vector a space of real numbers over filed ℝ and $\odot,\otimes$ be continuous t-norm. A fuzzy set $L_{\mathbb{R}}:\mathbb{R}\times[0,\infty)$ is called **fuzzy absolute value on** ℝ if it satisfies

(A1) $0\leq L_{\mathbb{R}}(n, a) < 1$ for all a>0.
(A2) $L_{\mathbb{R}}(n, a) = 1 \Leftrightarrow n = 0$ for all a>0.
(A3) $L_{\mathbb{R}}(n + m, a + b) \geq L_{\mathbb{R}}(n, a)\odot L_{\mathbb{R}}(m, b)$.
(A4) $L_{\mathbb{R}}(nm, ab) \geq L_{\mathbb{R}}(n, a) \otimes L_{\mathbb{R}}(m, b)$.
(A5) $L_{\mathbb{R}}(n, .):[0,\infty)\to[0,1]$ is continuous function.
(A6) $\lim_{a\to\infty} L_{\mathbb{R}}(n, a) = 1$.

For all m , n ∈ ℝ and for all a, b ∈[0,1]. Then $(\mathbb{R}, L_{\mathbb{R}}, \odot,\otimes)$ is called **a fuzzy absolute value space**.

Example 1:(9)

Define $L_{\mathbb{R}}(a, t) = \frac{t}{t+|a|}$ for all a ∈ ℝ then $L_{\mathbb{R}}$ is a fuzzy absolute value on ℝ where $t \odot s = t \cdot s$ and t

⊗ s = t · s for all t, s ∈ **[0, 1]** where t · s is the ordinary multiplication of t and s.

Definition 3:(10), (11)

Let V be a vector space over the field ℝ and ⨀,⊗ be a continuous t-norms. A fuzzy set G_V:V×[0,∞) is called a **general fuzzy norm on V** if it satisfies the following conditions for all u, v∈V and for all $\alpha \in$ ℝ, s, t ∈ [0,∞):

(G1) 0≤ G_V(u, s)<1 for all s>0.

(G2) G_V(u, s)=1 ⟺ u= 0 for all s>0.

(G3) $G_V(\alpha$u, st) ≥ $L_{\mathbb{R}}(\alpha$, s) ⊗ G_V(u, t) for all $\alpha \neq 0$ ∈ ℝ.

(G4) G_V(u+ v, s +t) ≥ G_V(u, s) ⨀ G_V(v, t).

(G5) G_V(u, .) : [0,∞) → [0,1] is continuous function of t.

(G6) $\lim_{t\to\infty} G_V$(u ,t) = 1

Then (V, G_V,⨀,⊗) is called **a general fuzzy normed space.**

Example 2:(10)

Define $G_{|.|}(u,a)=\frac{a}{a+|u|}$ for all u∈ℝ .Then (ℝ ,$G_{|.|}$,⨀,⊗) is a general fuzzy normed space with s⨀ t = s.t and t⊗ s = t.s for all s ,t ∈ [0,1]. Then $G_{|.|}$ is called the **standard general fuzzy norm induced by the absolute value |.|**.

Example 3 :(10)

If (V, ‖.‖) is normed space and $G_{\|.\|}$: V × [0,∞) → [0,1] is defined by :

$G_{\|.\|}(u,a) = \frac{a}{a+\| u\|}$ then (V,$G_{\|.\|}$,⨀ ,⊗) is general fuzzy normed space where s⨀t = s ∧ t and t⊗ s= t. s for all t ,s ∈ [0,1]. Then $G_{\|.\|}$ is called the **standard general fuzzy norm induced by the norm ‖.‖**.

The next result is proved in (10)

Proposition 1:

Suppose that (V, ‖.‖) is a normed space define $G_V(u, s) = \frac{s}{s+\|u\|}$ for all u ∈ V and 0< s. Then (V, G_V,⨀,⊗) is general fuzzy normed space where a ⨀ b = a ⊗ b = a . b for all a, b ∈ [0,1] .

Proposition 2:(10)

In the general fuzzy normed space (V, G_V,⨀,⊗) if 0< t <s then G_V(u, t) < G_V(u, s).

This means that G_V(u, .) is a nondecreasing function.

Definition 4:(10)

If (V,G_V,⨀,⊗) is a general fuzzy normed space. Then GFB(u ,n ,s) = { m ∈V : G_V(u−m, s)>(1−n) } is called a **general fuzzy open ball** with center u∈V radius n and s>0 and GFB[u, n ,s] = {m ∈V : G_V(u−m ,s) ≥ (1−n)} is called a **general fuzzy closed ball** with center u∈V radius n and s>0**.**

Definition 5:(10)

Suppose that (V, G_V ,⨀,⊗) is a general fuzzy normed space and M⊆ V. Then M is called a **general fuzzy open** if for any u ∈ M it can find 0< n<1, s>0 with FB(u, n, t) ⊆ M . A subset W ⊆ V is called a **general fuzzy closed** set if W^C is a general fuzzy open.

Definition 6:(10)

Suppose that (V,G_V,⨀,⊗) is a general fuzzy normed space. A sequence (u_n) in V is said to be **general fuzzy approaches** to u if every 0 < ε < 1 and 0< s there is N ∈ ℕ such that $G_V(u_n - u , s) > (1 -\varepsilon)$ for every n≥ N. If (u_n) is general fuzzy approaches to the fuzzy limit u it is written by $\lim_{n\to\infty} u_n$= u or $u_n \to$ u. Also $\lim_{n\to\infty} G_V(u_n -$ u, s) = 1 if and only if (u_n) is general fuzzy approaches to u.

Definition 7(10)

Suppose that $(V, G_V,⨀,⊗)$ is a general fuzzy normed space. A sequence (v_n) in V is called a **general Cauchy sequence** if for each $0 < r < 1$, $t > 0$ there exists a positive number N ∈ ℕ such that $G_V[v_m - v_n, t] > (1 - r)$ for all m , n ≥ N.

Definition 8:(10)

Let (V,G_V,⨀,⊗) be a general fuzzy normed space and let M⊆ V. Then the **general closure** of M is denote by $\overline{M^G}$ or GCL(M) is smallest general fuzzy closed set contains M .

Definition 9:(10)

Suppose that (V,G_V,⨀,⊗) is a general fuzzy normed space and let M⊆ V. Then M is said to be **general fuzzy dense** in V if $\overline{M^G}$ = V

Lemma 1:(10)

If (V,G_V,⨀,⊗) is a general fuzzy normed space and let M ⊆ V. Then m ∈ $\overline{M^G}$ if and only if it can find (m_n) in M such that $m_n \to m$.

Definition 10:(10)

Suppose that (V,G_V,⨀,⊗) is a general fuzzy normed space A sequence (v_n) in V is called a **general Cauchy sequence** if for each $0 < r < 1$, $t > 0$ it can find N ∈ ℕ with $G_V[v_j - v_k, t] > (1 - r)$ for all j, k≥N.

Definition 11:(10)

Let (V,G_V,⨀,⊗) be a general fuzzy normed space. A sequence (u_n) is said to be **general fuzzy bounded** if there exists 0 < q <1 such that $G_V(u_n$, s) >(1−q) for all s>0 and n∈ ℕ.

Definition 12:(10)

Let (V,G_V,⨀ ,⊗)and (U,G_U,⨀ ,⊗) be two general fuzzy normed spaces the operator S: V→ U is called **general fuzzy continuous at** v_0 ∈V for every s>0 and every 0 < γ < 1 there exist t and there exists δ such that for all v∈V with $G_V[v - v_0, s] > (1 - \delta)$ it have $G_U[S(v) - S(v_0), t] > (1 - \gamma)$ if S is fuzzy continuous at each point v∈V then S is said to be **general fuzzy continuous .**

Theorem 1:(10)

Suppose that $(V, G_V, \odot . \otimes)$ and $(U, G_U, \odot, \otimes)$ are general fuzzy normed spaces. Then $S: V \to U$ is a general fuzzy continuous at $u \in V$ if and only if $u_n \to u$ in V implies $S(u_n) \to S(u)$ in U.

Definition 13:(10)

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space. Then V is called **a general complete** if every general Cauchy sequence in V is general fuzzy approaches to a vector in V.

Definition 14 :(10)

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces. The operator $S: D(S) \to U$ is called **general fuzzy bounded** if it can find α, $0 < \alpha < 1$; with

$G_U(Sv, t) \geq (1 - \alpha)$ for each $v \in D(S)$ and $t > 0$ . . . . (1)

Notation 1: (11)

Suppose that $(V, G_V, \odot, \otimes)$and $(U, G_U, \odot, \otimes)$ are a general fuzzy normed spaces. Put GFB(V,U) = { $S: V \to U: G_U(Sv, t) \geq (1 - \alpha)$} with $0 < \alpha < 1$.

Theorem 2:(11)

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces. Put $\mathbf{G(T, t) = \inf_{v \in D(T)} G_U(Tv, t)}$ for all $T \in$ GFB(V, U), $t > 0$. Then $[GFB(V, U), G, \odot, \otimes]$ is general fuzzy normed space.

Notation 2:(11)

Rewrite 1 by :

$G_U(Sv, t) \geq G(S, t)$………………(2)

Theorem 3 :(11)

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are general fuzzy normed spaces with U is a general complete. Assume that $T: D(T) \to U$ be a linear operator and a general fuzzy bounded. Then T has an extension $S: \overline{D(T)} \to U$ with S is linear and general fuzzy bounded such that $G(T, t) = G(S, t)$ for all $t > 0$.

Theorem 4 :(11)

If $(V, G_V, \odot, \otimes)$and $(U, G_U, \odot, \otimes)$ are general fuzzy normed spaces and let $T: D(T) \to U$ be a linear operator where $D(T) \subseteq V$. Then T is general fuzzy continuous if and only if T is general fuzzy bounded.

Definition 15 :(11)

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and $(\mathbb{R}, L_{\mathbb{R}}, \odot, \otimes)$ is fuzzy absolute value space $(\mathbb{R}, L_{\mathbb{R}}, \odot, \otimes)$. Then a linear function $f: V \to \mathbb{R}$ is called **general fuzzy bounded** if there exists $\sigma \in (0, 1)$ with $L_{\mathbb{R}}[f(v), t] \geq (1 - \sigma)$ for any $v \in D(f)$, $t > 0$. Furthermore, the fuzzy norm of f is $L(f, t) = \inf L_{\mathbb{R}}(f(v), t)$

and $L_{\mathbb{R}}(f(v), t) \geq L(f, t)$.

Corollary 1 :(11)

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and $(\mathbb{R}, L_{\mathbb{R}}, \odot, \otimes)$ is fuzzy absolute value space. Then a linear function $f: V \to \mathbb{R}$ with $D(f) \subseteq V$ is general fuzzy bounded if and only f is general fuzzy continuous.

Definition 16 :(11)

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space. Then GFB(V, $\mathbb{R}$) = { $f: V \to \mathbb{R}$: f is general fuzzy bounded linear } forms a general fuzzy normed space with general fuzzy norm defined by $L(f, t) = \inf L_{\mathbb{R}}(f(v), t)$ which is said to be the general fuzzy dual space of V.

Definition 17:(11)

Suppose that $(V, G_V, \odot, \otimes)$ is general fuzzy normed space. A sequence (v_n) in V is **general fuzzy weakly approaches** if it can find $v \in V$ with every $h \in$ GFB(V,$\mathbb{R}$) $\lim_{n \to \infty} h(v_n) = h(v)$. This is written $v_n \to^{w} v$ the element v is said to be the weak limit to (v_n) and (v_n) is said to be general fuzzy approaches weakly to v.

Theorem 5:(11)

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and (v_n) is in V.

1. If $v_n \to v$ then $v_n \to^{w} v$.
2. $v_n \to^{w} v$ implies $v_n \to v$ when dimension of V is finite.

Definition 18:(11)

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space. A sequence (h_n) with $h_n \in GFB(V, \mathbb{R})$ is called

1)**Strong general fuzzy approaches** in the general fuzzy norm on GFB(V, $\mathbb{R}$) that is $h \in GFB(V, \mathbb{R})$ with $G[h_n - h, t] \to 1$ for all $t > 0$ this written $h_n \to h$

2)**Weak general fuzzy approaches** in the fuzzy absolute value on $\mathbb{R}$ that is $h \in GFB(V, \mathbb{R})$ with $h_n(v) \to h(v)$ for every $v \in V$ written by $\lim_{n \to \infty} h_n(v) = h(v)$.

Theorem 6 :(11)

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces. Then GFB(V, U) is general complete when U is general complete.

Theorem 7:

The general fuzzy normed space $(\mathbb{R}, L_{\mathbb{R}}, \odot, \otimes)$ is general fuzzy complete

Proof:

Let (a_n) be a general Cauchy sequence in $\mathbb{R}$ then (a_n) has a monotonic subsequence (a_{n_j}) but (a_n) is a general fuzzy bounded hence (a_{n_j}) is a general fuzzy bounded thus (a_{n_j}) general approaches to a∈ $\mathbb{R}$. Then for every 0 < r <1 there exists N ∈ $\mathbb{N}$ such that

$L_{\mathbb{R}}(a_m - a_n \; \frac{t}{2}) \geq (1 - r)$ for all m, n $\geq N$ t > 0

$L_{\mathbb{R}}\,(a_{\mathrm{m}} - a_{\mathrm{N}}\ \frac{t}{2}) \geq (1-r)$ for all m ≥ K and t > 0

Choose K ≥ N suppose k≥ K then $k \geq N$

$L_{\mathbb{R}}\,(a_{n_{\mathrm{k}}} - a_{\mathrm{k}}\ \frac{t}{2}) \geq (1-r)$. Now

$$L_{\mathbb{R}}\,(a_k - a\ ,t) \geq L_{\mathbb{R}}\,(a_k - a_{n_k} + a_{n_k} - a, t)$$
$$\geq L_{\mathbb{R}}\left(a_k - a_{n_k}, \frac{t}{2}\right) \odot L_{\mathbb{R}}\,(a_k - a, \frac{t}{2})$$
$$\geq (1-r) \odot (1-r)$$

Choose $0 < \varepsilon < 1$ such that $(1-r) \odot (1-r) > (1-\varepsilon)$.

Then $L_{\mathbb{R}}\,(a_k - a\ ,t) > (1-\varepsilon)$.

Thus (a_n) is a general fuzzy approaches to a∈ ℝ.

Corollary 2 :

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and (ℝ, $L_{\mathbb{R}}$, ⊙,⊗) is fuzzy absolute value space. Then GFB(V, ℝ) is general complete if (ℝ, $L_{\mathbb{R}}$, ⊙,⊗) is general complete.

Theorem 8:

If a general fuzzy normed space $(V, G_V, \odot, \otimes)$ is finite dimensional then every linear operator T on V is a general fuzzy bounded.

Proof:

Let dim V=n and $\{v_1, v_2, \dots, v_n\}$ is a basis for V then for each $v \in V$ there is $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in \mathbb{R}$ with $v = \sum_{j=1}^{n} \alpha_j v_j$. Now

$G_V[Tv, t] = G_V\left[\sum_{j=1}^{n} \alpha_j Tv_j\ , t\right] \geq$

$L_{\mathbb{R}}(\alpha_1, \frac{t}{n}) \otimes G_V(Tv_1, \frac{t}{n}) \odot \ldots \odot L_{\mathbb{R}}(\alpha_n, \frac{t}{n}) \otimes G_V(Tv_n, \frac{t}{n})$

Let $(1-p_j) = L_{\mathbb{R}}(\alpha_j, \frac{t}{n})$ and $(1-q_j) = G_V(Tv_j, \frac{t}{n})$ for some $0 < p_i < 1$, and for some $0 < q_i < 1$, for j=1, 2, …,n. Let $(1-p_1)\otimes(1-q_1)\odot\ldots\odot(1-p_n)\otimes(1-q_n) \geq (1-\sigma)$ for some $0 < \sigma < 1$. Thus

$G_V[Tv, t] \geq (1-p_1)\otimes(1-q_1)\odot\ldots\odot(1-p_n)\otimes(1-q_n) \geq (1-\sigma)$

Hence $G_V[Tv, t] \geq (1-\sigma)$. Thus T is a general fuzzy bounded operator.

The next result is well known in functional analysis

Theorem 9: (extension of a linear functional) :

Let V be vector space over ℝ and let p be a sublinear functional on V. Furthermore, let f be a linear functional which is defined a subspace W of V and satisfies f(u) ≤ p(u) for all u ∈ W. Then f has linear extension $\tilde{f}$:V → ℝ satisfying $\tilde{f}$(u) ≤ p(u) for all u ∈ V.

The proof of the next results is similar to the proof follows from Theorem 3 and hence is omitted.

Theorem 10:

Let (V, G_V ,⊙,⊗) be general fuzzy normed space and let W be a dense subspace of V. Let f be a fuzzy bounded linear functional on W then there exists a fuzzy bounded linear functional $\tilde{f}$:V → ℝ which is an extension of f and L(f, t) =L($\tilde{f}$, t) where L(f, t) =inf $L_{\mathbb{R}}$ (f ,t) and L($\tilde{f}$, t) = inf $L_{\mathbb{R}}$ ($\tilde{f}$,t)

Theorem 11:

Let (V,G_V, ⊙,⊗) be general fuzzy normed space and let $v_0 \neq 0 \in V$ suppose that W is dense subspace of V generated by v_0. Then there exists a fuzzy bounded linear functional $\tilde{f}$:V → ℝ such that $\tilde{f}(v_0)$ =$G_V(v_0$, t) where a⊗ b = a .b for all a, b ∈ [0,1]

Proof :

Define a linear functional f :W → ℝ by f(v) =f(αv_0) = $L_{\mathbb{R}}(\alpha\ ,t)\ .\ G_V(v_0\ ,t)$

for any v ∈ W and $\alpha \neq 0$. Then f is general fuzzy normed bounded since

$L_{\mathbb{R}}$(f(v),t) = $G_V(\alpha v_0, t)$ = $L_{\mathbb{R}}(\alpha\ ,t)\ .\ G_V(v_0\ ,t)$.

Now theorem 10 implies that f has a linear extension $\tilde{f}$: V → ℝ with

L(f, t) = L($\tilde{f}$,t) and $\tilde{f}(v_0) = f(v_0) = G_V(v_0\ ,t)$

The Adjoint operator of a General Fuzzy Bounded Operator:

Let (V,G_V, ⊙,⊗) and (U,G_U, ⊙,⊗) be two general fuzzy normed spaces consider a fuzzy bounded operator T:V →U. To define the adjoint operator T^a and T, start from fuzzy bounded linear functional g:U→ ℝ is defined for all v ∈ V setting u = T(v) to obtain a functional from U to ℝ. call it f with

f(v)=g T(v) [v ∈ V]………………(3)

f is linear since g and T are linear and f is general fuzzy bounded because

$L_{\mathbb{R}}$[f(v), t] =$L_{\mathbb{R}}$[g(T(v),t] ≥ L(g,t) ⊙ G(T,t)

Taking infinium to both sides obtaining the inequality

L(f,t) ≥ L(g,t) ⊙ G(T,t)

This shows that f ∈ GFB(V, ℝ) by assumption g ∈ GFB(U, ℝ) consequently the operator from GFB(U, ℝ) into GFB(V, ℝ) is called the adjoint operator of T and is denoted by T^a. Thus, T: V →U, T^a:GFB(U, ℝ) →GFB(V,ℝ).

Definition 19:

Let (V,G_V, ⊙,⊗) and (U,G_U, ⊙,⊗) be two general fuzzy normed spaces and T: V →U be a fuzzy bounded linear operator then the adjoint operator

T^a: GFB(U, ℝ)→ GFB(V,ℝ) is defined by

f(v) = T^ag(v) = g [T(v)] for all g ∈ GFB(U, ℝ). ………..(4)

Our first goal is to prove that the adjoint operator has the same general fuzzy norm as the operator itself

Theorem 12:

The adjoint operator T^a in definition 19 is linear and G(T^a, t) = G(T, t) for all t > 0

Proof :

The operator T^a is linear since it's domain GFB(V, ℝ) is a vector space and obtaining $T^a(\alpha\, g_1 + \beta\, g_2)$ = $\alpha\, T^a\,(g_1) + \beta\, T^a\,(g_2)$

⟺ $T^a(\alpha\, g_1 + \beta\, g_2](v)$= [$\alpha\, T^a\,(g_1) + \beta\, T^a\,(g_2)$](v)

Now

$T^{a}(\alpha\, g_1 + \beta\, g_2](v)$ = $[\alpha\, g_1 + \beta\, g_2]\ (Tv)$
$= \alpha\, g_1\,(Tv) + \beta\, g_2(Tv)$
$=\alpha\ (T^{a}g_1)(v) + \beta\ (T^{a}\ g_2\ )$(v)
$= [\alpha\ (T^{a}g_1) + \beta\ (T^{a}\ g_2\ ]$(v)

Now to prove G(T^{a},t)= G(T, t) from (2) with f = $T^{a}g$ it follows that

G($T^{a}g, t) = L(f,t)$ ≥ L(g,t) ⊙ G(T,t)

Taking the infimum over all g ∈ GFB(U, ℝ) of fuzzy absolute value one obtaining

G($T^{a}g, t) \geq G(T,t)$ ……………(5)

To prove G($T^{a}g, t) \leq G(T,t)$ but Theorem 2.38 implies that for every $v_0 \in V$ there is g_0 ∈ GFB(U, ℝ) such that $g_0(Tv_0)$ = G_U(Tv_0, t) here g_0(Tv_0) = [$T^{a}g_0](v_0)$ by definition of the adjiont operator T^{a} written f_0 = $T^{a}g_0$. Thus obtaining

G_U(Tv_0, t) = g_0(Tv_0) ≥ $G_V(f_0, t)$ = G_V $(T^{a}g_0\ , t) \geq G_V\ (T^{a}, t)$

Thus for every $v_0 \in V$

$G_V(T^{a}g_0\ , t) \geq G_V\ (T^{a}$, t) but always $G_U(Tv_0\ , t) \geq$ G(T, t).

Hence (1−r) = G_U(T, t) is the largest constant (1− r) such that

$G_U(Tv_0\ , t) \geq (1-r)$ hold for all v_0 ∈V. Hence G (T^{a}, t) cannot be larger than G(T, t) this is it must have G(T^{a}, t) ≤ G(T, t)

From this and (3) it follows that G (T^{a}, t) = G (T, t).

Proposition 3:

Let (V,G_V, ⊙, ⊗) and (U,G_U, ⊙, ⊗) be two general fuzzy normed spaces. If S,T ∈ GFB(V, U) .Then (i)$(S+T)^{a} = S^{a} + T^{a}$ (ii)$(\alpha\, T)^{a} = \alpha\ T^{a}$ for any $\alpha \neq 0 \in \mathbb{R}$

Proof :

(i)$[(S+T)^{a}(g)](v) = g[S+T](v)$
= g[S(v) + T (v)]= gS(v) + gT (v)
= $[S^{a}(g)](v)$ + $[T^{a}($ g)] (v)
= $[S^{a}(g)$ + $T^{a}($ g)] (v)

Hence $(S+T)^{a} = S^{a} + T^{a}$

(ii)[$(\alpha\, T)^{a}(g)](v)$ = g[αT(v)]
= α[gT(v)] = α [T^{a}(g)] (v)

Hence [$(\alpha\, T)^{a}$](g) = α [T^{a}(g)]

that is $(\alpha\, T)^{a} = \alpha\, T^{a}$

Proposition 4 :

Let (V,G_V, ⊙, ⊗), (U,G_U, ⊙, ⊗) and (W,G_W, ⊙, ⊗) be three general fuzzy normed spaces. Suppose that T ∈GFB(V,U) and S ∈ GFB(U,W) then $(ST)^{a} = T^{a}S^{a}$

Proof :

If T:V→U and S:U→W then ST:V→W also $(ST)^{a}$: GFB(W,ℝ)→ GFB(V,ℝ**)** by above definitions of the adjoint operator. On the other hand T^{a}:GFB(U, ℝ**)** →GFB(V,ℝ) and S^{a}:GFB(W, ℝ**)** →GFB(U,ℝ) thus $T^{a}S^{a}$: GFB(W,ℝ)→ GFB(V,ℝ). Also by simple calculation one can see that $(ST)^{a}$[h] = $T^{a}S^{a}$[h] for all h ∈ GFB(W,ℝ).

Hence $(ST)^{a} = T^{a}S^{a}$.

Proposition 5:

Let (V,G_V, ⊙, ⊗) and (U,G_U, ⊙, ⊗) be two general fuzzy normed spaces if T ∈ GFB(V, U) and T^{-1} ∈ GFB (U, V) . Then $(T^{a})^{-1}$ exists $(T^{a})^{-1}$ ∈ GFB[GFB(V, ℝ), GFB (U, ℝ)] and $(T^{a})^{-1}$ =($T^{-1})^{a}$

Proof :

If T:V→U then T^{-1}: U→V also by the definition of the adjoint operator,

T^{a}:GFB(U, ℝ**)** →GFB(V,ℝ). Since T^{a} is one to one and onto $(T^{a})^{-1}$ exists

and $(T^{a})^{-1}$: GFB(V, ℝ**)** →GFB(U,ℝ). Now one the other hand by the definition of the adjoint operator ($T^{-1})^{a}$: GFB(V, ℝ**)** →GFB(U,ℝ). Also, by simple calculation one can see that $(T^{a})^{-1}$[h] =($T^{-1})^{a}$[h] for all h ∈ GFB(V, ℝ**).**

Hence $(T^{a})^{-1}$ =($T^{-1})^{a}$.

Corollary 3:

Let (V,G_V, ⊙, ⊗) be a general fuzzy normed space then for any v∈V it has $G_V(v,t)$ =$inf_{f\in GFB(V,\mathbb{R}),\, f\neq 0} L_{\mathbb{R}}$(f(v),t). Hence if v_0 is such that f(v_0) $= 0$ for all f∈ GFB (V,ℝ) then v_0 = 0.

Proof :

From Theorem 11 writing v for v_0,

$inf_{f\in GFB(V,\mathbb{R}),\, f\neq 0} L_{\mathbb{R}}$(f(v), t) ≤ G_V(v, t) and from

$L_{\mathbb{R}}$(f(v), t) ≥ L (f , t) ≥ G_V(v, t).

So $inf_{f\in GFB(V,\mathbb{R}),\, f\neq 0} L_{\mathbb{R}}$(f(v),t) ≥ $G_V(v,t)$.

Hence $inf_{f\in GFB(V,\mathbb{R}),\, f\neq 0} L_{\mathbb{R}}$(f(v),t) = $G_V(v,t)$

Reflexive spaces :

Consider a general fuzzy normed space (V,G_V, ⊙, ⊗) it's dual GFB(V, ℝ) and moreover the dual space of GFB(V, ℝ) is denoted by GFB[GFB(V, ℝ), ℝ] and is called the second dual space of V. The definition of a functional g_v on GFB(V, ℝ) based on choosing a fixed-point v ∈ V and setting

g_v(f) = f(v), f ∈ GFB(V, ℝ) is variable ………………..(6)

Proposition 6:

Let (V,G_V, ⊙, ⊗) be general fuzzy normed space . Then for any fixed v ∈ V the functional g_v is defined by (5) is a general fuzzy bounded linear function on GFB (V, ℝ) that is g_V ∈ GFB[GFB(V, ℝ), ℝ] and has fuzzy norm

L(g_V,t) =G_V(v,t) ………………(7)

Proof :

First to prove that g_V is linear it must show that $g_V(\alpha f_1 + \beta f_2) = \alpha\, g_V\,(f_1) + \beta\, g_V(f_2)$. Now
$g_V(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(v) = \alpha f_1(v) + \beta f_2(v) = \alpha\, g_V(f_1) + \beta\, g_V(f_2)$
For every $f_1, f_2 \in GFB(V, \mathbb{R})$ and $\beta \in \mathbb{R}$ by using corollary 2 and equation (5),
$L(g_V, t) = inf_{\substack{f \in GFB(V,\mathbb{R}) \\ f \neq 0}} L_{\mathbb{R}}(g_V(f), t) = inf_{\substack{f \in GFB(V,\mathbb{R}) \\ f \neq 0}} L_{\mathbb{R}}(f(v), t) = G_V(v, t)$

Definition 20:
Let $(V, G_V, \odot, \otimes)$ be general fuzzy normed space then for every $v \in V$ there corresponds a unique general fuzzy bounded linear functional $g_V \in$ [GFB[GFB(V, ℝ), ℝ] given by (5) this defines a mapping C: V→ [GFB[GFB(V, ℝ), ℝ] defined by
C[v] = g_V ……………(8)
C is called the **canonical mapping** of V into [GFB[GFB(V, ℝ), ℝ]. Now to show that C is linear and injective and preserves the general fuzzy norm.

Proposition 7:
Let $(V, G_V, \odot, \otimes)$ be general fuzzy normed space then canonical mapping C given by (7) is a general fuzzy isomorphism of V onto the general fuzzy normed space R(C) the range of C

Proof :
The linearity of C is clear since $g_v - g_u = g_{v-u}$ .Hence by (6) obtaining
$L(g_v - g_u, t) = L(g_{v-u}, t) = G_V(v - u, t)$. This show that C is fuzzy isometric and it preserves the general fuzzy norm. also, it is known that fuzzy isometric implies injectivity. Hence C is bijective regarded as mapping onto its range.

Definition 21:
A general fuzzy normed space $(V, G_V, \odot, \otimes)$ is said to be embeddable in a general fuzzy normed space $(U, G_U, \odot, \otimes)$ if V is a general fuzzy isomorphic with subspace W of U

Definition 22:
A general fuzzy normed space $(V, G_V, \odot, \otimes)$ is said to be fuzzy reflexive if R(C) = [GFB[GFB(V, ℝ), ℝ] where C:V→ [GFB[GFB(V, ℝ), ℝ] is canonical mapping given by(7)

Theorem 13:
If a general fuzzy normed space $(V, G_V, \odot, \otimes)$ is fuzzy reflexive then it is general complete

Proof :
Since GFB[GFB(V, ℝ), ℝ] is the fuzzy dual of GFB(V, ℝ) it is general complete by Theorem 7 fuzzy reflexivity of V means that R(C) = [GFB[GFB(V, ℝ), ℝ]
Now general completeness of V follows from that [GFB[GFB(V, ℝ), ℝ] by Proposition 7.

Theorem 14:
Every finite dimensional general fuzzy normed space is fuzzy reflexive.

Proof:
Indeed, if dim V= n then every linear functional on V is general fuzzy bounded by Theorem 8 so that GFB (V, ℝ) and the algebraic reflexivity of V holds, having R(C) = GFB[GFB(V, ℝ), ℝ].

Conclusion:

Our aim in this work is to recall the concept of general fuzzy normed space and its basic properties in order to define the adjoint operator of a general fuzzy bounded operator from a general fuzzy normed space V into another general fuzzy normed space U. Then basic properties of the adjoint operator were proved after that the definition of fuzzy reflexive general fuzzy normed space was introduced in order to prove that every finite dimensional general fuzzy normed space is fuzzy reflexive.

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- Ethical Clearance: The project was approved by the local ethical committee in University of Technology.

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خواص لمؤثر المرافق لمؤثر مقيد ضبابيا عام

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الخلاصة:

هدفنا في هذا البحث إعادة استخدام مفاهيم الفضاء الضبابي العام وخواصه الأساسية لتعريف المؤثر المرافق لمؤثر مقيد ضبابيا عام معرف من فضاء ضبابي عام V إلى فضاء ضبابي عام U. بعد ذلك الخواص الأساسية للمؤثر المرافق تم برهانها ثم تعريف فضاء القياس الضبابي العام الانعكاسي لبرهان كل فضاء القياس الضبابي المنتهي البعد يكون انعكاسي ضبابيا.

الكلمات المفتاحية : المؤثر المرافق، الفضاء الانعكاسي الضبابي، فضاء القياس الضبابي العام $GFB(V, \mathbb{R})$، المؤثر المستمر ضبابيا العام، المؤثر المقيد ضبابيا العام.