Semihollow-Lifting Modules and Projectivity

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Abstract:
Throughout this paper, T is a ring with identity and F is a unitary left module over T. This paper study the relation between semihollow-lifting modules and semiprojective covers, proposition 5 shows that If T is semihollow-lifting, then every semilocal T-module has semiprojective cover. Also, give a condition under which a quotient of a semihollow-lifting module having a semiprojective cover. proposition 2 shows that if K is a projective module. K is semihollow-lifting if and only if For every submodule A of K with $\frac{K}{A}$ is hollow, then $\frac{K}{A}$ has a semiprojective cover.

Keywords: Projective cover, Projective modules, Semihollow lifting modules.

Introduction:
Let T be a ring with identity and F a unitary left module over T. A submodule E from a T-module F is called small on F (E $\ll$ F) if whenever a submodule S of J with F = E + S implies S = F. 1. A submodule E of an T-module F is called semiasmall in F (E $\ll$ F) if E = 0 or E/V $\ll$ F/V for every nonzero submodule V of E. 2. Let F be an T-module and let V,N be submodule of F such that V$\subseteq$N$\subseteq$F. K is called semiessential submodule of N in F (N$\subseteq_{sc}$E in F) if $\frac{N}{V}$ $\ll$ $\frac{F}{V}$ 3. A non-zero T-module F is called a hollow module if every proper submodule of F is a small submodule of F 4. An T-module F is called semihollow-lifting if for every submodule $H$ of F with $\frac{F}{H}$ hollow, there exists a submodule K of F such that F = K$\oplus$K$^c$ and $K \subseteq_{sc} H$ in F 5. An T-module P is projective if and only if, for S, B are any two T-module and Z:S $\rightarrow$ B is epimorphism and for any homomorphism L:P $\rightarrow$ B, $\exists$ a homomorphism h:P $\rightarrow$ S such that Zoh = L.

A T-module U is called semiprojective cover of a T-module F if, U is projective and $\exists$ an epimorphism $\varphi$:U $\rightarrow$ F with ker$\varphi$ $\ll$ S U.

Lemma 1 Suppose that F is a T-module such that F has a semiprojective cover. If V is projective with an epimorphism f:V $\rightarrow$ F, then V has a decomposition $V = V_1 \oplus V_2$ such that $V_1 \subseteq$ kerf and $f|V_2: V_2 \rightarrow F$ is a semiprojective cover of F.

Proof: Since V is projective, there is a commuting diagram

\[ \begin{array}{ccc} V & \xrightarrow{f} & F \\ h \downarrow & & \downarrow \phi \\ Q & \xrightarrow{q} & 0 \\ \end{array} \]

with exact row and column, as q is a small epimorphism and $qf = f$, since Q is projective, thus h splits, i.e. there is a monomorphism g:Q $\rightarrow$ V such that hg = $I_Q$, then V = Img $\oplus$ Kerh. Now, put $V_1 = Img$ and $V_2 = kerh$. But $qh = f$, $fg$ thus $V_2 \subseteq$ kerf. Since $f(V_1) = f(V) = F$ then $V_1 \rightarrow F \rightarrow 0$ is exact, so is a projective cover from fg $= qhg = q$, it follows that ker($f|V_1$) = g(kerq), a small submodules of $g(Q) = V_1$; then $f|V_2: V_2 \rightarrow F$ is a semiprojective cover of F.
Proposition 2 Let $K$ be a projective module. Then the following statements are equivalent:
1. $K$ is semihollow-lifting.
2. For every submodule $A$ of $K$ with $\frac{K}{A}$ is hollow, then $\frac{K}{A}$ has a semiprojective cover.

Proof: $1 \Rightarrow 2$ Assume that $K$ is a semihollow-lifting module. Let $A$ be a submodule of $K$ with $\frac{K}{A}$ is hollow. Thus there is a submodule $A^*$ of $A$ such that $K = A^* \oplus A^{**}$ and $A \cap A^{**} \subseteq A^*$. But $K$ is projective, then by $6$, $A^{**}$ is projective. Let $\pi : K \to \frac{K}{A} \to 0$ be the natural epimorphism, thus, $\pi|A^{**} : A^{**} \to \frac{K}{A} \to 0$ is an epimorphism, to see this, let $x + A \in \frac{K}{A}$. It is clear that $\pi(x) = x + A$. But $x \in K$ and $K = A^* \oplus A^{**}$, this implies that $x = a^* + a^{**}$, where $a^* \in A^*$, $a^{**} \in A^{**}$. Now, $\pi(x) = \pi(a^*) + \pi(a^{**}) = \pi(a^*) = x + A$. Since $\ker(\pi|A^{**}) = A \cap A^{**}$ and $A^{**}$ is projective, then $\pi|A^{**} : A^{**} \to \frac{K}{A} \to 0$ is a semiprojective cover of $\frac{K}{A}$.

$2 \Rightarrow 1$ Let $A$ be a submodule of $K$ such that $\frac{K}{A}$ is hollow and let $\varphi : K \to \frac{K}{A}$ be the natural epimorphism. By (2), $\frac{K}{A}$ has a semiprojective cover. Thus by Lemma 1, there exists a decomposition $K = K_1 \oplus K_2$ such that $\varphi|K_2 : K_2 \to \frac{K}{A} \to 0$ is a semiprojective cover and $K_1 \subseteq \ker \varphi$, this implies that $K_1 \subseteq A$ and $\ker(\varphi|K_2) = A \cap K_2 \subseteq A$. Then $K$ is a semihollow-lifting module.

The following is an immediate corollary from prop. 2.

Corollary 3 The following statements are equivalent for any ring $T$.
1. $T$ is semihollow-lifting.
2. For every ideal $J$ of $T$ such that $\frac{T}{J}$ is hollow, then $\frac{T}{J}$ has a semiprojective cover.

A $T$-module $F$ is called semisimple if every submodule of $F$ is a direct summand of $F$.

Let $N$ and $L$ be submodules of a module $F$, $N$ is called semisupplement of $L$ in $F$ if $F = N + L$ and $N \cap L \subseteq L$. Now, give new definitions:

A $T$-module $F$ is called semimaximal submodule if and only if the quotient $F/D$ is a semisimple module.

A $T$-module $F$ is called semilocal if $F$ has a unique semimaximal submodule $N$ which contains all proper submodule of $F$. For example, every semisupplement of a semimaximal submodule in a module is a semilocal module.

Proposition 4 Assume that $F$ is a nonzero $T$-module, the following statements are equivalent:
1) $F$ is semihollow and $\text{Rad}(F) \neq F$
2) $F$ is semihollow and cyclic
3) $F$ is semilocal

Proof: $1 \Rightarrow 2$ Since $\text{Rad}(F) \subseteq F$ and $F/ \text{Rad}(F)$ semisimple then $F$ is cyclic.

$2 \Rightarrow 3 \Rightarrow 1$ Clear.

Proposition 5 Assume that $T$ is a ring. If $T$ is semihollow-lifting, then every semilocal $T$-module $F$ has semiprojective cover.

Proof: Suppose that $T$ is semihollow-lifting. Let $F$ be a semilocal $T$-module, thus by prop. 4, $F = T$ for some $a \in F$. Define $\varphi : T \to T_a$, by $\varphi(t) = ta$, $\forall t \in T$. It is clear that $\varphi$ is an epimorphism. Then by the first isomorphism theorem $\frac{T}{\ker \varphi} \cong T_a$. It is clear that $\ker \varphi = \text{Ann}(a)$. Thus $\frac{T}{\text{Ann}(a)} \equiv T_a$. This implies that $\frac{T}{\text{Ann}(a)}$ is semilocal. Now, put $A = \text{Ann}(a)$ so $\frac{T}{A}$ is semihollow. But $T$ is semihollow-lifting, thus $\exists$ an ideal $A^*$ of $T$ such that $A^* \subseteq A$, $T = A^* \oplus A^{**}$ and $A \cap A^{**} \subseteq A^*$. Then $\pi : T \to \frac{T}{A}$ is the natural epimorphism, thus, $\pi|A^{**} : A^{**} \to \frac{T}{A} \to 0$ is an epimorphism. It is clear that $\ker(\pi|A^{**}) = A \cap A^{**}$, hence $\ker(\pi|A^{**}) \subseteq A^*$. Then $\pi|A^{**} : A^{**} \to \frac{T}{A}$ is a semiprojective cover of $\frac{T}{A}$. Hence $F$ has a semiprojective cover.

Let $F$ be an $T$-module. Let $K$ and $N$ be submodules of $F$, $K$ is a strong semisupplement of $N$ in $F$ if $K$ is a semisupplement of $N$ in $F$ and $K \cap N$ is a direct summand of $N$.

Theorem 6 Assume that $F$ is a $T$-module, then $F$ is semihollow-lifting if and only if for every submodule $V$ of $F$ with $\frac{F}{V}$ is hollow has a strong semisupplement in $F$.

Proof: Assume that $F$ is a semihollow-lifting module and $V$ is a submodule of $F$ such that $\frac{F}{V}$ is hollow. Hence $\exists$ a submodule $K$ of $V$ such that $K \subseteq V$ and $F = K \oplus K^*$, for some $K^* \subseteq F$. By modular law, $V = V \cap (K \oplus K^*) = K \oplus (V \cap K^*)$. It
is easy to show that $F = V + K'$. To show $V \cap K' \ll K'$. Let $(V \cap K') + D = K'$, where $D \subseteq K'$.
So $F = K + K' = (V \cap K') + D$. This implies that $F = V + D$ and $F = \frac{V + D}{K} = \frac{V}{K} \oplus \frac{D}{K}$. But $K \subseteq V$ in $F$, thus $F = D + K$. Since $F = (V \cap K') + D \subseteq K'$, then $D = K'$ and hence $V \cap K' \ll K'$. Thus $V$ has a strong semisupplement $K'$ in $F$.

Conversely, Take $V$ to be a submodule of $F$ such that $\frac{F}{F_0}$ is hollow. Thus by our assumption there is a strong semisupplement $K$ of $V$ in $F$, then $F = V + K \cap \subseteq K$ and $V = (V \cap K) \oplus L$, where $L \subseteq F$. Now, $F = V + K = (V \cap K) + L + K = L + K$. It is clear that $L \cap K = 0$, so $F = L \oplus K$. To show that $L \subseteq V$ in $F$. Let $\frac{L}{F} = \frac{D}{F}$, where $D \subseteq F$ containing $L$, thus $V + D = F$. Hence $F = (V \cap K) + L + D$. Since $V \cap K \ll K$, then $F = L + D$. But $L \subseteq F$, thus $F = D$. Then $F$ is semihollow-lifting.

A $T$-module $F$ is said to have the (finite) exchange property if for any (finite) index set $I$, whenever $F \oplus N = \bigoplus_{i \in I} A_i$, for modules $N$ and $A_i$, then $F \oplus N = F \bigoplus \left( \bigoplus_{i \in I} B_i \right)$ for submodules $B_i \subseteq A_i$.

**Lemma 7** Let $F_0$ be a direct summand of a module $F$ such that $F_0$ has a finite exchange property. If $M_0 \subseteq V \subseteq F$ and $V$ has a strong semisupplement in $F$, then $\frac{V}{F_0}$ has a strong semisupplement in $\frac{F}{F_0}$.

**Proof:** Let $K$ be a strong semisupplement of $V$ in $F$, then $F = V + K$. $V \cap K \ll K$ and $V = (V \cap K) \oplus L$, where $L \subseteq F$. So $F = V + K = (V \cap K) + L + K$ and hence $F = L \oplus K$. Since $F_0$ is a direct summand of $F$, thus $F = F_0 \oplus F_1$, for some $F_1 \subseteq F$. But $F_0$ has a finite exchange property, thus $F_0 \cap F_1 = F_0 \bigoplus L' \bigoplus K'$, where $L' \subseteq L$ and $K' \subseteq K$. Hence $F = F_0 + V + K'$. But $F_0 \subseteq V \subseteq F$, thus $F = V + K'$. Since $F = V + K$, then by minimality of $K$, $K = K'$. Now, put $L_0 = F_0 \oplus L'. But F = F_0 \bigoplus L' \bigoplus K'$, thus $F = F_0 \bigoplus L' \bigoplus K$ and $F = L_0 \oplus K$. Now, $\frac{F}{F_0} = \frac{L_0 \oplus K}{F_0} = \frac{L_0}{F_0} \bigoplus \frac{K}{F_0}$ and get, $\frac{F}{F_0} = F_0 \bigoplus \frac{L'}{F_0} \bigoplus \frac{K}{F_0}$.

**Proposition 8** Let $F_0$ be a direct summand of an $T$-module $F$ such that $F_0$ has the finite exchange property. If $F$ is a semihollow-lifting module, then $\frac{F}{F_0}$ is also semihollow-lifting.

**Proof:** Take a submodule $B$ of $F$, such that $F_0 \leq B$ and $\frac{F}{F_0}$ is a hollow module. From (third isomorphism theorem), $\frac{F}{B} \cong \frac{F}{F_0}$ and thus $\frac{F}{B}$ is a hollow module. But $F$ is semihollow-lifting, thus by Th. 6, $B$ has a strong semisupplement in $F$. Since $F_0$ is a direct summand of $F$ and has the finite exchange property, then by Lemma 7, $\frac{B}{F_0}$ has a strong semisupplement in $\frac{F}{F_0}$. Then by Th. 6, $\frac{F}{F_0}$ is a semihollow-lifting module.

**Proposition 9** Assume that $F$ is a semihollow-lifting module that has a semimaximal submodule. Then $F$ has a semilocal submodule which is a direct summand.

**Proof:** Let $F$ be a semihollow-lifting module and $S$ be a semimaximal submodule of $F$. Then, $\frac{F}{S}$ is a semisimple module and hence $\frac{F}{S}$ is a semihollow module. Then, $S$ has a strong semisupplement $H$ in $F$. Thus, $F = S + H$. $S \cap H \ll S$ and $S = (S \cap H) \oplus L$, where $L \subseteq F$. Hence $F = H \oplus L$. Then $H$ is a direct summand of $F$. But $H$ is a semisupplement of a semimaximal submodule, thus $H$ is semilocal.

The following proposition gives a decomposition of any projective hollow-lifting module.

**Proposition 10** Assume that $F$ be a $T$-module and let $X$ be an zero projective module, then there exists a semimaximal submodule in $X$.

**Proof:** Assume $\text{Rad} P = P$. To show that every finitely generated sub module $N \leq P$ is zero. Let $\{K_i\}_A$ be a family of finitely generated (cyclic) modules in $\sigma[F]$ and h: $\bigoplus A K_i \rightarrow P$ an epimorphism. Since $P$ projective, there exists $g:P \rightarrow \bigoplus E K_\lambda$ with $g|P = id$ with $N$, and also $g(N)$, finitely generated there is a finite subset $E \subseteq A$ with $g(N) \subseteq \bigoplus E K_\lambda$ with the canonical projection $\pi: \bigoplus E K_\lambda \rightarrow \bigoplus E K_\lambda$ obtain an endomorphism $f$:
Thus by Theorem 11, every sequence \( (f_1: c_1) \) is complete.

Theorem 12: Assume that \( F \) is a projective semihollow lifting module, then \( F \) has the finite exchange property.

Proof: Take the projective semihollow lifting module \( F \). By prop. 10, \( F \) has a semisimple submodule \( F \), and \( F \) is a semisimple submodule of \( F \). Then by prop. 9, \( F \) has a semisimple submodule \( F \), and \( F \) is a semisimple submodule of \( F \). Then by Theorem 12, \( F \) has the finite exchange property.

Proposition 13: Let \( F \) be a projective semihollow lifting module. Then \( F \) has a semisimple submodule \( F \). Let \( F \) be a semisimple submodule of \( F \). Then \( F \) is a semisimple submodule of \( F \).

Proof: Assume that \( S = End(F) \) is a semilocal ring and \( S \) is semilocal. Then either \( 1 \) is a unit in \( S \), or \( S \) is a semilocal ring. Then by Theorem 12, \( F \) has the finite exchange property.

Theorem 14: A module \( F \) is said to be an SIE module if \( F \) is an endomorphism ring \( End(F) \) is semilocal and \( B \) has the finite exchange property.

Proof: If \( F \) is a left \( T \)-module and \( f : F \to F \) is an endomorphism such that the sequence of submodules \( f \), \( f^2 \), \( f^3 \), \( \ldots \), \( f^n \), \( \ldots \), stable, \( f \), \( f^2 \), \( f^3 \), \( \ldots \), \( f^n \), \( \ldots \), stable, \( f \), \( f^2 \), \( f^3 \), \( \ldots \), \( f^n \), \( \ldots \), stable, \( f \), \( f^2 \), \( f^3 \), \( \ldots \), \( f^n \), \( \ldots \), stable, the proof is complete.

Corollary 14: Let \( F \) be an indecomposable ring. Then F is semiprimary.

Proof: 1. Take \( T \) is semiprimary. Since \( T \) is semiprimary, thus by prop. 14, we have \( F = \mathbb{B} \oplus \mathbb{B} \). Then \( B \) has the finite exchange property. Also, give a condition under which \( F \) is semiprimary.

1. Let \( K \) be a projective module. \( K \) is semiprimary. Since \( T \) is semiprimary, thus by prop. 14, we have \( F = \mathbb{B} \oplus \mathbb{B} \). Then \( B \) has the finite exchange property. Also, give a condition under which \( F \) is semiprimary.

2. Suppose \( T \) satisfies the following chain condition for every \( F \), there exists a direct summand \( Y \) of \( F \) such that \( Y \) is semi-simple. Then \( F \) is semiprimary.

3. \( T \) is semiprimary lifting. Then \( F \) is semiprimary.

4. \( T \) is semiprimary lifting. Then \( F \) is semiprimary.
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Mukdad Qaess Hussain, Anfal Hasan Dheyab and Rana Aziz Yousif contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript.

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مقاسات شبه الرفع المجوفة والاسقاطية

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الخلاصة:
تكن T حلقة ذات عنصر محايد وليكن F مقاسا ايسر معرف على T. هذا البحث درس العلاقة بين المقاسات شبه الرفع المجوفة وغطاء المقاسات شبه الاسقاطية. بينت القضية 5 إذا كان T شبه رفع مجوف فان كل مقاس شبه محلي يمتلك غطاء شبه اسقاطي واعتى الشرط الذي يكون فيه المقاسات الكسري للمقاسات شبه الرفع المجوفة يمتلك غطاء شبه اسقاطي. القضية 2 تبين انه إذا كان K مقاس اسقاطي يمتلك غطاء شبه اسقاطي. K/A محوف فان K/A محوف فان A في K بحيث K/FAN يكون شبه رفع مجوف إذا وفقط إذا كل مقاس جزئي في K/FAN يكون شبه رفع مجوف.

الكلمات المفتاحية: غطاء المقاسات الاسقاطية، المقاسات الاسقاطية، مقاسات شبه الرفع المجوفة.