Open Access
2020, 17(3):869-877

Baghdad Science Journal

P-ISSN: 2078-8665
E-ISSN: 2411-7986

DOI: http://dx.doi.org/10.21123/bsj.2020.17.3.0869

Modified Weighted Pareto Distribution Type I (MWPDTI)

Mahmood A. Shamran

Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq.
E-mail: Mahmoodas_math@csw.uobaghdad.edu.iq
ORCID ID: https://orcid.org/0000-0001-8506-3876

Received 16/4/2019, Accepted 18/12/2019, Published 1/9/2020

Abstract:

In this paper, the Azzallini's method used to find a weighted distribution derived from the standard Pareto distribution of type I (SPDTI) by inserting the shape parameter (θ) resulting from the above method to cover the period (0, 1] which was neglected by the standard distribution. Thus, the proposed distribution is a modification to the Pareto distribution of the first type, where the probability of the random variable lies within the period $\left[\frac{k}{\theta}, \infty\right), k > 0, \theta \geq 1$. The properties of the modified weighted Pareto distribution of the type I (MWPDTI) as the probability density function ,cumulative distribution function, Reliability function , Moment and the hazard function are found. The behaviour of probability density function for MWPDTI distribution by representing the values of $x = \frac{k}{\theta}$ This means, the probability density function of this distribution treats the period (0,1] which is ignore in SPDTI.

Key words: Azzallini's method , Cumulative distribution function , Modified weighted Pareto distribution type I(MWPDTI), Reliability function , Standard Pareto distribution type I(SPDTI).

Introduction:

The weighted statistical distributions were established in (1934)(1) by Fisher, who pointed to develop and circulate standard statistical distributions to match biased data samples, which are often shown through medical science data and some aspects of biology and other fields of science, Khatree (1989)(1) studied the statistical properties of some weighted length distributions and compared them with the properties of their standard distributions .

The defect in the random sample selection of the population leads to bias in sample size and length, which leads to the failure of standard statistical distributions in the treatment and interpretation of these samples.

In (1978), Patil and Rao (2)presented some concepts on how to use weighted distributions to correct the probability of events in biased samples and apply them to environmental models and human populations. Azzallini (1985)(2) introduced a new technique to find weighted distribution from standard distribution summarized as follows :

$$f_w(x) = \frac{1}{P_r(x_2 < \theta x_1)} f(x_1) F(\theta x_1) \qquad (1)$$

Such that $f_w(x)$ is the probability density function for the weighted distribution, $f(x_1)$ and $F(\theta x_1)$ are the probability density function and cumulative distribution function of the standard distribution respectively.

Gupt and Kundu (2009)(3) used the Azzallini's method to find the shape parameter for exponential distribution, resulting in a new type of exponential distribution with two parameters.
In (2013) (4) Mervat. M. studied the weighted of the Weibull distribution and found its properties
Shakhatreh (2012) (5) studied weighted exponential distribution in two properties.

Improvement of the Azzallini's method was achieved in (2014) (6) by Badmus et al. which was used in the weighted Weibull model.

In (2015) (7) A. Mahdavi introduced two new classes of weighted distributions by using the Azzallini's method

Iden H. H. and Shayma G. S.(2017) (8) studied the methods to estimate the parameters for the Maxwell –Boltzmann distribution .

Filippo and Bozidar V. (2018) (9) studied the expansion of the Azilini's method on two dependent random variables of different distributions .

The standard Pareto distribution type I (SPDTI) has the probability density function (pdf) and cumulative distribution function(cdf) are given respectively by (10):

$$f(x;\alpha,k)=\frac{\alpha k^{\alpha}}{x^{\alpha+1}} \quad ,x\geq k>0,\alpha>0 \qquad (2)$$

$$F(x;\alpha,k)=1-\frac{k^{\alpha}}{x^{\alpha}} \quad , \; x\geq k>0,\alpha>0 \qquad (3)$$

The mean and the variance of (PDTI) are respectively given by:

$$\mu_x=E(x)=\frac{\alpha k}{(\alpha-1)} \; ,\alpha>1 \qquad (4)$$

$$V(x)=\frac{\alpha k^2}{(\alpha-1)^2(\alpha-2)} \; , \; \alpha>2 \qquad (5)$$

Main Result: Azzallini's technique will be used to find weighted distribution derived from Pareto distribution type I, called Modified Weighted Pareto distribution type I (MWPDTI), and study the statistical properties as mean ,variance, standard deviation , cumulative distribution function, Reliability function , hazard rate function, Skewness , Kurtosis and ordered statistics of the MWPDTI. The MWPDTI treated the period (0,1] which was neglected by the SPDTI, such that the value of the random variable is $\frac{k}{\theta}$, in this treatment the calculate probability values of the random variable by using two parameters k,θ . Thus, the MWPDTI includes the period $\left[\frac{k}{\theta},\infty\right)$ $,x\geq\frac{k}{\theta}>0,\theta\geq 1$.

Weighted Pareto Type I Distribution (WPDTI)

Lemma 1:

Let X_1,X_2 are a non negative independent random variables having Pareto type I probability density functions (pdf) $f(x_1),f(x_2)$ respectively with parameters α,k, then the $P_r(\theta x_1>x_2)=\frac{2\theta^{\alpha}-1}{2\theta^{\alpha}}$ where $\frac{2\theta^{\alpha}-1}{2\theta^{\alpha}}$ is weighted Pareto type I distribution.

Proof:

$P_r(x_2<\theta x_1)$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\theta x_1} f(x_1)f(x_2)dx_2dx_1 \qquad (6)$$

Since $x_1 and\, x_2$ are independent Pareto type I random variables and $x\geq k,\theta\geq 1$ then

$$\int_{k}^{\infty}\int_{k}^{\theta x_1} f(x_1)f(x_2)dx_2dx_1$$

$$=\int_{k}^{\infty} f(x_1)\left[\int_{k}^{\theta x_1} f(x_2)\,dx_2\right]dx_1$$

$$=\int_{k}^{\infty}\frac{\alpha k^{\alpha}}{x_1^{\alpha+1}}\left[\int_{k}^{\theta x_1}\frac{\alpha k^{\alpha}}{x_2^{\alpha+1}}dx_2\right]dx_1$$

$$=\int_{k}^{\infty}\frac{\alpha k^{\alpha}}{x_1^{\alpha+1}}\left[1-\frac{k^{\alpha}}{\theta^{\alpha}x_1^{\alpha}}\right]dx_1$$

$$=\int_{k}^{\infty}\left(\frac{\alpha k^{\alpha}}{x_1^{\alpha+1}}-\frac{\alpha k^{2\alpha}}{\theta^{\alpha}x_1^{2\alpha+1}}\right)dx_1$$

$$=\int_{k}^{\infty}\frac{\alpha k^{\alpha}}{x_1^{\alpha+1}}dx_1-\int_{k}^{\infty}\frac{\alpha k^{2\alpha}}{\theta^{\alpha}x_1^{2\alpha+1}}dx_1$$

$$=1-\frac{1}{2\theta^{\alpha}}\int_{k}^{\infty}\frac{2\alpha k^{2\alpha}}{x_1^{2\alpha+1}}dx_1=1-\frac{1}{2\theta^{\alpha}}=\frac{2\theta^{\alpha}-1}{2\theta^{\alpha}}$$

$$P_r(x_2<\theta x_1)=\frac{2\theta^{\alpha}-1}{2\theta^{\alpha}} \; , \; \theta\geq 1 \qquad (7)$$

$$\therefore \; \frac{1}{P_r(x_2<\theta x_1)}=\frac{2\theta^{\alpha}}{2\theta^{\alpha}-1} \quad , \quad \theta\geq 1$$

By substituting eq. (7) in eq. (1) we get the weighted probability density function (wpdf) of the Pareto distribution type I (WPDTI) is $f_w(x)$ given by:

$$f_w(x;\alpha,k,\theta)=\frac{2\theta^{\alpha}}{2\theta^{\alpha}-1}\left(\frac{\alpha k^{\alpha}}{x^{\alpha+1}}\right)\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)$$

$$f_w(x;\alpha,k,\theta)=\frac{2\alpha\theta^{\alpha}k^{\alpha}x^{-\alpha-1}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)}{(2\theta^{\alpha}-1)} \; , \quad \theta\geq 1,x\geq k\geq 1,\alpha>0 \qquad (8)$$

Lemma 2:

The function $f_w(x;\alpha,k,\theta)=\frac{2\alpha\theta^{\alpha}k^{\alpha}x^{-\alpha-1}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)}{(2\theta^{\alpha}-1)}$ is a probability density function of the Pareto random variable X.

Proof:

$$\int_{k}^{\infty} f_w(x)dx=\int_{k}^{\infty}\left(\frac{2\alpha\theta^{\alpha}k^{\alpha}x^{-\alpha-1}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)}{(2\theta^{\alpha}-1)}\right)dx$$

$$Let\; u=\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right) \;\rightarrow\; du=\frac{ak^{\alpha}x^{-\alpha-1}}{\theta^{\alpha}}\,dx$$

$$=\frac{2\theta^{2\alpha}}{2\theta^{\alpha}-1}\int_{k}^{\infty}u\,du \quad =\frac{2\theta^{2\alpha}}{2\theta^{\alpha}-1}\int_{k}^{\infty}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)\left(\frac{ak^{\alpha}x^{-\alpha-1}}{\theta^{\alpha}}\right)dx$$

$$=\frac{\theta^{2\alpha}(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}})^2}{2\theta^{\alpha}-1}\Bigg|_{k}^{\infty}$$

$$=\lim_{b\to\infty}\frac{(\theta^{\alpha}x^{\alpha}-k^{\alpha})^2}{(2\theta^{\alpha}-1)x^{2\alpha}}\Bigg|_{k}^{b}=1$$

$\therefore\; f_w(x;\alpha,k,\theta)$ is a weighted probability density function (wpdf).

The eq.(8) is called weighted Pareto distribution Type I (WPDTI).

The purpose of this paper is to expand the standard Pareto distribution of type I by adding the period in which the random variable lies within the

period (0,1] which was neglected by the standard Pareto distribution of type I which started from the random variable values $x \geq k \geq 1$.
This treatment we call it MWPDTI.
Test the function $f_w(x)$ is a probability density function (pdf) over the suggested new interval $\left[\frac{k}{\theta}, \infty\right]$, $x \geq \frac{k}{\theta} > 0, \theta \geq 1$ as follow:

$$\int_{\frac{k}{\theta}}^{\infty} f_w(x)dx = \int_{\frac{k}{\theta}}^{\infty} \left(\frac{2\alpha\theta^{\alpha}k^{\alpha}x^{-\alpha-1}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)}{(2\theta^{\alpha}-1)}\right)dx$$

$$Let\ u = \left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right) \rightarrow du = \frac{ak^{\alpha}x^{-\alpha-1}}{\theta^{\alpha}} dx$$

$$= \frac{2\theta^{2\alpha}}{2\theta^{\alpha}-1}\int_{\frac{k}{\theta}}^{\infty} u\, du$$

$$= \frac{2\theta^{2\alpha}}{2\theta^{\alpha}-1}\left(\lim_{b\to\infty}\int_{\frac{k}{\theta}}^{b}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)\left(\frac{ak^{\alpha}x^{-\alpha-1}}{\theta^{\alpha}}\right)dx\right.$$

$$= \lim_{b\to\infty}\frac{\theta^{2\alpha}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)^2}{2\theta^{\alpha}-1}\Bigg|_{\frac{k}{\theta}}^{b}$$

$$= \lim_{b\to\infty}\frac{(\theta^{\alpha}x^{\alpha}-k^{\alpha})^2}{(2\theta^{\alpha}-1)x^{2\alpha}}\Bigg|_{\frac{k}{\theta}}^{b}$$

$$= \frac{\theta^{\alpha}}{2\theta^{\alpha}-1} \neq 1 .$$

$\therefore$ $f_w(x)$ is not a probability density function(pdf) over $[\frac{k}{\theta}, \infty]$ to make it a (pdf) Multiply eq. (8) by $\frac{2\theta^{\alpha}-1}{\theta^{2\alpha}}$ we get the modified weighted probability density function is $f_w^*(x;\alpha,k,\theta)$ given by:

$$f_w^*(x;\alpha,k,\theta) = \left(\frac{2\theta^{\alpha}-1}{\theta^{2\alpha}}\right) f_w(x)$$

$$f_w^*(x;\alpha,k,\theta) = \left(\frac{2\theta^{\alpha}-1}{\theta^{\alpha}}\right)\frac{2\alpha\theta^{\alpha}k^{\alpha}x^{-\alpha-1}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)}{(2\theta^{\alpha}-1)}$$

$$f_w^*(x;\alpha,k,\theta) = \frac{2\alpha k^{\alpha}x^{-\alpha-1}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)}{\theta^{\alpha}} \quad ,\alpha > 0, x \geq \frac{k}{\theta} > 0, \theta \geq 1, k > 0 \qquad (9)$$

Lemma 3:
The function $f_w^*(x;\alpha,k,\theta) = \frac{2\alpha k^{\alpha}x^{-\alpha-1}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)}{\theta^{\alpha}}$, $\alpha > 0, x \geq \frac{k}{\theta} > 0, \theta \geq 1, k > 0$ is modified weighted probability density function of the Pareto distribution type I.

Proof:
(1) $f_w^*(x;\alpha,k,\theta) \geq \mathbf{0}$

(2) $\int_{\frac{k}{\theta}}^{\infty} f_w^*(x)dx$

$$= \int_{\frac{k}{\theta}}^{\infty}\left(\frac{2\alpha k^{\alpha}x^{-\alpha-1}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)}{\theta^{\alpha}}\right)dx = 1$$

$$Let\ u = \left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right) \rightarrow du = \frac{ak^{\alpha}x^{-\alpha-1}}{\theta^{\alpha}} dx$$

$$= 2\int_{\frac{k}{\theta}}^{\infty} u\, du \quad = 2\int_{\frac{k}{\theta}}^{\infty}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)\left(\frac{ak^{\alpha}x^{-\alpha-1}}{\theta^{\alpha}}\right)dx$$

$$= \lim_{b\to\infty}\left(1-\frac{k^{\alpha}}{\theta^{\alpha}x^{\alpha}}\right)^2\Bigg|_{\frac{k}{\theta}}^{b} = \lim_{b\to\infty}\frac{(\theta^{\alpha}x^{\alpha}-k^{\alpha})^2}{\theta^{2\alpha}x^{2\alpha}}\Bigg|_{\frac{k}{\theta}}^{b}$$

$$= 1$$

As shown in Fig. 1.

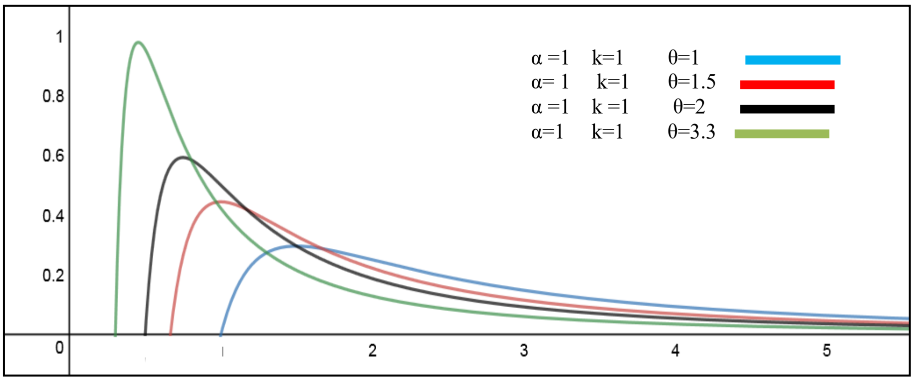

Figure 1. (MWPDTI) pdf $f_w^*(x;\alpha,k,\theta)$

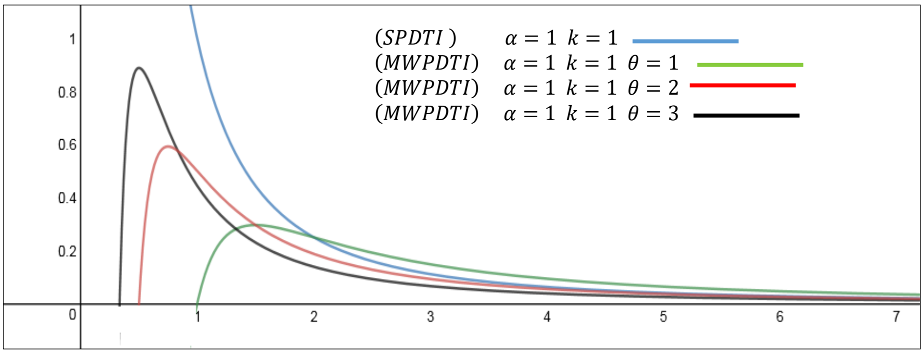

Figure 2. Comparison between standard Pareto and modified weighted Pareto

Showing that the graph of modified weighted Pareto is different from the Pareto distribution when entering the shape parameter and the curve of shape parameter (θ) is interest and process when $x \geq \frac{k}{\theta} > 0$. The MWPDTI take the value of $x \geq \frac{k}{\theta}$ and SPDTI take the value of $x \geq k \geq 1$. As shown in Fig.2.

The Moments:

Lemma 4:

Let X be the MWPDTI random variable, then the r[th] moments of the (MWPDTI) is $E_w^*(x^r)$ given by:

$$E_w^*(x^r) = \frac{2\alpha^2 k^r}{\theta^r (r-2\alpha)(r-\alpha)} \quad , \quad \alpha > r \; , k > 0$$

Proof:

$$E_w^*(x^r) = \int_{\frac{k}{\theta}}^{\infty} x^r f_w^*(x) dx$$

$$E_w^*(x^r) = \int_{\frac{k}{\theta}}^{\infty} x^r \left(\frac{2\alpha k^\alpha x^{-\alpha-1}\left(1 - \frac{k^\alpha}{\theta^\alpha x^\alpha}\right)}{\theta^\alpha} \right) dx$$

$$= \frac{1}{\theta^\alpha} (\int_{\frac{k}{\theta}}^{\infty} \left(2\alpha k^\alpha x^{r-\alpha-1} \left(1 - \frac{k^\alpha}{\theta^\alpha x^\alpha}\right)\right) dx$$

$$= \frac{1}{\theta^\alpha}\left[\int_{\frac{k}{\theta}}^{\infty} 2\alpha k^\alpha x^{r-\alpha-1} dx - \frac{1}{\theta^\alpha}\int_{\frac{k}{\theta}}^{\infty} 2\alpha k^{2\alpha} x^{r-2\alpha-1} dx\right]$$

$$= \frac{1}{\theta^\alpha}\left[\lim_{b\to\infty}\left(\int_{\frac{k}{\theta}}^{b} 2\alpha k^\alpha x^{r-\alpha-1} dx\right)\right] - \frac{1}{\theta^{2\alpha}}\left[\lim_{b\to\infty}\left(\int_{\frac{k}{\theta}}^{b} 2\alpha k^{2\alpha} x^{r-2\alpha-1} dx\right)\right]$$

$$= \frac{2\alpha k^\alpha}{\theta^\alpha}\left[\lim_{b\to\infty}\left(\frac{x^{r-\alpha}}{r-\alpha}\right)\Bigg|_{\frac{k}{\theta}}^{b}\right] - \frac{2\alpha k^{2\alpha}}{\theta^{2\alpha}}\left[\lim_{b\to\infty}\left(\frac{x^{r-2\alpha}}{r-2\alpha}\right)\Bigg|_{\frac{k}{\theta}}^{b}\right]$$

$$= \frac{2\alpha k^\alpha}{\theta^\alpha (r-\alpha)}\left[\lim_{b\to\infty}\left(b^{r-\alpha} - \left(\frac{k}{\theta}\right)^{r-\alpha}\right)\right] - \frac{2\alpha k^{2\alpha}}{\theta^{2\alpha}(r-2\alpha)}\left[\lim_{b\to\infty}\left(b^{r-2\alpha} - \left(\frac{k}{\theta}\right)^{r-2\alpha}\right)\right]$$

$$= \frac{2\alpha k^\alpha}{\theta^\alpha (r-\alpha)}\left[\lim_{b\to\infty}\left(b^{r-\alpha} - \left(\frac{k}{\theta}\right)^{r-\alpha}\right)\right] - \frac{2\alpha k^{2\alpha}}{\theta^{2\alpha}(r-2\alpha)}\left[\lim_{b\to\infty}\left(b^{r-2\alpha} - \left(\frac{k}{\theta}\right)^{r-2\alpha}\right)\right]$$

$$E_w^*(x^r) = \frac{2\alpha^2 k^r}{\theta^r (r-2\alpha)(r-\alpha)} \quad , \quad \alpha > r \; , k > 0 \qquad (10)$$

The mean, variance and standard deviation of the (MWPDTI)

The mean of the (MWPDTI) is $E_w^*(x)$ given by:
When $r = 1$ in eq. (10) we get the mean $E_w^*(x)$

$$E_w^*(x) = \frac{2\alpha^2 k}{\theta(1-2\alpha)(1-\alpha)} \quad , \alpha > 1 \, , \theta \geq 1 , k > 0 \qquad (11)$$

In the same way calculate $E_w^*(x^2)$ when $r = 2$ in eq. (10) .

$$E_w^*(x^2) = \frac{2\alpha^2 k^2}{\theta^2(2-2\alpha)(2-\alpha)} \quad , \alpha > 2 \, , \theta \geq 1, k > 0 \qquad (12)$$

From the eq.s (11) and (12) the variance of the MWPDTI is $\mathrm{Var}_w^*(x)$ given by:

$$Var_w(x) = E_w^*(x^2) - [E_w^*(x)]^2$$

$$= \left[\frac{2\alpha^2 k^2}{(2-\alpha)(2-2\theta)\theta^2}\right] - \left[\frac{2\alpha^2 k}{\theta(1-\alpha)(1-2\theta)}\right]^2$$

$$Var_w^*(x) = \frac{(5\alpha-1)\alpha^2 k^2}{(\alpha-2)(\alpha-1)^2(2\alpha-1)\theta^2} \quad ,\alpha > 2\,,\theta \geq 1, k > 0 \qquad (13)$$

By eq.(13) The standard deviation of the modified weighted random variable X_w $is\ Sd_w^*(x)$ given by:

$$Sd_w^*(x) = \sqrt{\boldsymbol{Var_w^*(x)}}$$

$$= \sqrt{\frac{(5\alpha-1)\alpha^2 k^2}{(\alpha-2)(\alpha-1)^2(2\alpha-1)\theta^2}}$$

$$Sd_w^*(x) = \frac{\alpha k\sqrt{5\alpha-1}}{\sqrt{\alpha-2}(\alpha-1)(2\alpha-1)\theta} \qquad (14)$$

The cumulative distribution function of the MWPDTI

Lemma 5:

The cumulative distribution function (cdf) of the MWPDTI is $F_W^*(x)$ given by

$$F_W^*(x) = \frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha} x^{2\alpha}}, x \geq \frac{k}{\theta} > 0\,, \alpha > 0\,, \theta \geq 1$$

Proof:

$$F_W^*(x) = \int_{\frac{k}{\theta}}^{x} f_W^*(t)\, dt$$

$$F_W^*(x) = \int_{\frac{k}{\theta}}^{x} \left(\frac{2\alpha k^\alpha t^{-\alpha-1}\left(1 - \frac{k^\alpha}{\theta^\alpha t^\alpha}\right)}{\theta^\alpha}\right) dt$$

$$Let\ u = \left(1 - \frac{k^\alpha}{\theta^\alpha t^\alpha}\right) \rightarrow du = \frac{\alpha k^\alpha t^{-\alpha-1}}{\theta^\alpha}\, dt$$

$$= 2\int_{\frac{k}{\theta}}^{x} u\, du = 2\int_{\frac{k}{\theta}}^{x} \left(1 - \frac{k^\alpha}{\theta^\alpha t^\alpha}\right)\left(\frac{\alpha k^\alpha t^{-\alpha-1}}{\theta^\alpha}\right) dt$$

$$= 2\frac{\left(1 - \frac{k^\alpha}{\theta^\alpha t^\alpha}\right)^2}{2}\Bigg|_{\frac{k}{\theta}}^{x}$$

$$= \left(1 - \frac{k^\alpha}{\theta^\alpha t^\alpha}\right)^2\Bigg|_{\frac{k}{\theta}}^{x}$$

$$= \frac{\theta^{2\alpha}x^{2\alpha} - 2\theta^\alpha k^\alpha x^\alpha + k^{2\alpha}}{\theta^{2\alpha}x^{2\alpha}}$$

$$F_W^*(x) = \frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha}x^{2\alpha}}, x \geq \frac{k}{\theta} > 0\,, \alpha > 0\,, \theta \geq 1 \qquad (15)$$

The accuracy of cumulative probability distribution depends on the values parameter θ produced from the Azzallini's method. If the values of θ are increasing then the accurate of curve of cumulative distribution are increasing. As shown inFig.3.

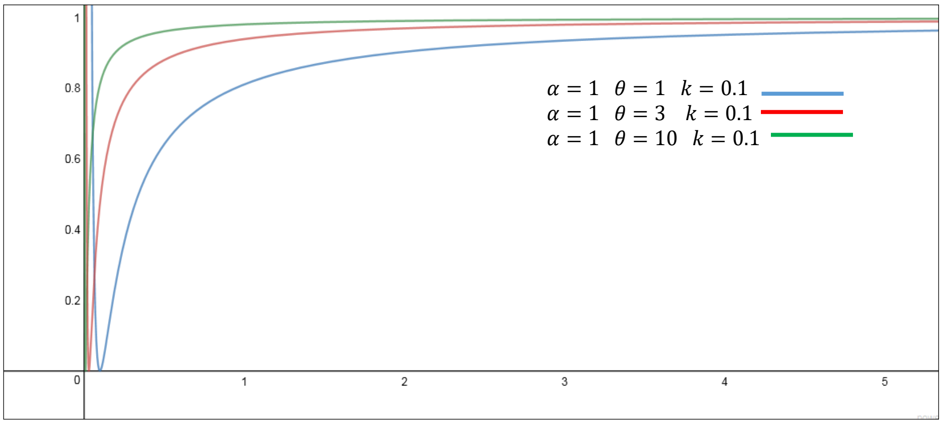

Figure 3. The cdf function $F_w^*(x)$ of the (MWPDTI)

The Reliability function of the MWPDTI

The reliability function of the MWPDTI is $\overline{F_W^*}(x)$ given by using eq.(15) :

$$\overline{F_W^*}(x) = 1 - F_W^*(x)$$

$$\overline{F_W^*}(x) = \left[1 - \frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha}x^{2\alpha}}\right] \ , \theta > 1, \alpha > 0, x \geq \frac{k}{\theta} > 0 \qquad (16)$$

Hazard rate function of the MWPTI

The hazard function of the MWPTI is $h_w^*(x)$ given by eq.(9) and eq.(16) ;

$$h_w^*(x) = \frac{f_W^*(x)}{\overline{F_W^*}(x)}$$

$$= \frac{\frac{2\alpha k^\alpha x^{-\alpha-1}\left(1 - \frac{k^\alpha}{\theta^\alpha x^\alpha}\right)}{\theta^\alpha}}{\left[1 - \frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha}x^{2\alpha}}\right]}$$

$$h_w^*(x) =. \frac{2\alpha\theta^\alpha x^{\alpha-1}}{2\theta^\alpha x^\alpha - k^\alpha} - \frac{2\alpha k^\alpha}{x(2\theta^\alpha x^\alpha - k^\alpha)} \ , \ x \geq \frac{k}{\theta} > 0, \alpha > 0, \theta \geq 1 \qquad (17)$$

The hazard function relies on the values of θ. If $\theta = 1$ the value of hazard function close to 1, and increasing θ then the curve of hazard function are decreasing . As shown in fig(4).

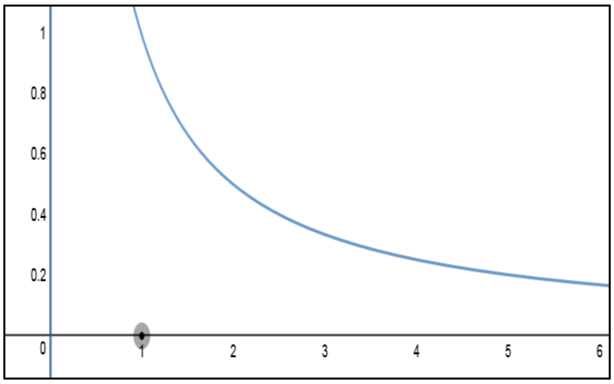

Figure 4. MWPDTI Hazard rate function $h_w^*(x)$

Mode of the MWPDTI

Lemma 6:

Let X be a nonnegative random variable having the MWPDTI probability density function ,then the mode of X is

$$x = \left[\frac{k^\alpha(2\alpha+1)}{\theta^\alpha(\alpha+1)}\right]^{\frac{1}{\alpha}}, \alpha > 0, \theta \geq 1.$$

Proof:

$$\frac{d}{dx} f_w^*(x) = 0$$

$$= \frac{d}{dx}\left(\frac{2\alpha k^\alpha x^{-\alpha-1}\left(1-\frac{k^\alpha}{\theta^\alpha x^\alpha}\right)}{\theta^\alpha}\right) = 0$$

$$\left(\frac{2(-\alpha-1)\alpha k^\alpha x^{-\alpha-2}\left(1-\frac{k^\alpha}{\theta^\alpha x^\alpha}\right)}{\theta^\alpha} + \frac{2\alpha^2 k^{2\alpha} x^{-2\alpha-2}}{\theta^{2\alpha}}\right) = 0$$

$$-\frac{2\alpha k^\alpha((\alpha+1)\theta^\alpha x^\alpha - (2\alpha+1)k^\alpha)}{\theta^{2\alpha} x^{2(\alpha+1)}} = 0$$

$$\frac{-2\alpha k^\alpha \theta^\alpha x^\alpha(\alpha+1) + 2\alpha k^{2\alpha}(2\alpha+1)}{\theta^{2\alpha} x^{2(\alpha+1)}} = 0$$

$$2\alpha k^{2\alpha}(2\alpha+1) - 2\alpha\theta^\alpha k^\alpha x^\alpha(\alpha+1) = 0$$

$$2\alpha k^{2\alpha}(2\alpha+1) = 2\alpha\theta^\alpha k^\alpha x^\alpha(\alpha+1)$$

$$x^\alpha = \frac{2\alpha k^{2\alpha}(2\alpha+1)}{2\alpha k^\alpha \theta^\alpha(\alpha+1)}$$

$$x^\alpha = \frac{k^\alpha(2\alpha+1)}{\theta^\alpha(\alpha+1)}$$

$$x = \left[\frac{k^\alpha(2\alpha+1)}{\theta^\alpha(\alpha+1)}\right]^{\frac{1}{\alpha}}, \alpha > 0, \theta \geq 1 \qquad (18)$$

The median of the MWPDTI

Lemma 7:

Let X be a non negative random variable having the MWPDTI probability density function, then the median of X is $m_w^*(x)$ given by:

$$m_w^*(x) = \left[\frac{\sqrt{2}k^\alpha}{\theta^\alpha(\sqrt{2}-1)}\right]^{\frac{1}{\alpha}}, \quad \theta \geq 1, \alpha > 0$$

Proof:

$$m_w^*(x) = \int_{\frac{k}{\theta}}^{m} f_w^*(x)dx = \frac{1}{2}$$

$$\int_{\frac{k}{\theta}}^{m}\left(\frac{2\alpha k^\alpha x^{-\alpha-1}\left(1-\frac{k^\alpha}{\theta^\alpha x^\alpha}\right)}{\theta^\alpha}\right)dx = \frac{1}{2}$$

$$\left(1-\frac{k^\alpha}{\theta^\alpha x^\alpha}\right)^2\Bigg|_{\frac{k}{\theta}}^{m} = \frac{1}{2}$$

$$\frac{\theta^{2\alpha}m^{2\alpha} - 2\theta^\alpha k^\alpha m^\alpha + k^{2\alpha}}{\theta^{2\alpha}m^{2\alpha}} = \frac{1}{2}$$

$$\frac{(\theta^\alpha m^\alpha - k^\alpha)^2}{\theta^{2\alpha}m^{2\alpha}} = \frac{1}{2}$$

$$2(\theta^\alpha m^\alpha - k^\alpha)^2 = \theta^{2\alpha}m^{2\alpha}$$

$$\sqrt{2(\theta^\alpha m^\alpha - k^\alpha)^2} = \sqrt{\theta^{2\alpha}m^{2\alpha}}$$

$$m^\alpha = \frac{\sqrt{2}k^\alpha}{\theta^\alpha(\sqrt{2}-1)}$$

$$m_w^*(x) = \left[\frac{\sqrt{2}k^\alpha}{\theta^\alpha(\sqrt{2}-1)}\right]^{\frac{1}{\alpha}}, \theta \geq 1, \alpha > 0 \qquad (19)$$

The Moment generating function

Lemma 8:

The moment generating function of the MWPDTI is $M_{wx}^*(x)$ given by;

$$M_{wx}^*(t) = \lim_{b\to\infty}\left[-\frac{2\alpha(-1)^\alpha k^\alpha t^\alpha(\Gamma(-\alpha,-tb)\theta^\alpha - \Gamma(-2\alpha,-tb)(-1)^\alpha k^\alpha t^\alpha}{\theta^{2\alpha}}\right]\Bigg|_{\frac{k}{\theta}}^{b}$$

Proof:

$$M_{wx}^*(t) = E(e^{tx})$$

$$= \int_{\frac{k}{\theta}}^{\infty} e^{tx}\left(\frac{2\alpha k^\alpha x^{-\alpha-1}\left(1-\frac{k^\alpha}{\theta^\alpha x^\alpha}\right)}{\theta^\alpha}\right)dx$$

$$= \frac{1}{\theta^\alpha}\left[\int_{\frac{k}{\theta}}^{\infty} 2\alpha k^\alpha x^{-\alpha-1}e^{tx}dx - \frac{1}{\theta^\alpha}\int_{\frac{k}{\theta}}^{\infty} 2\alpha k^{2\alpha}x^{-2\alpha-1}e^{tx}dx\right]$$

$$= \frac{2\alpha k^{\alpha}}{\theta^{\alpha}} \int_{\frac{k}{\theta}}^{\infty} x^{-\alpha-1} e^{tx} dx - \frac{2\alpha k^{2\alpha}}{\theta^{2\alpha}} \int_{\frac{k}{\theta}}^{\infty} x^{-2\alpha-1} e^{tx} dx$$

$$= -\left[\frac{2(-1)^{\alpha} k^{\alpha} t^{\alpha}(\Gamma(-2\alpha,-tx)}{\theta^{\alpha}} + \frac{2\alpha(-1)^{2\alpha} k^{2\alpha} t^{2\alpha}(\Gamma(-2\alpha,-tx)}{\theta^{2\alpha}}\right]\Bigg|_{\frac{k}{\theta}}^{\infty}$$

$$M_{wx}^{*}(t) = \lim_{b\to\infty}\left[-\frac{2\alpha(-1)^{\alpha} k^{\alpha} t^{\alpha}(\Gamma(-\alpha,-tb)\theta^{\alpha} - \Gamma(-2\alpha,-tb)(-1)^{\alpha} k^{\alpha} t^{\alpha}}{\theta^{2\alpha}}\right]\Bigg|_{\frac{k}{\theta}}^{b} \qquad (20)$$

Such that Γ is incomplete gamma function.

The Characteristics Function

Lemma 9:

The characteristic function of the MWPDTI is $Q_{wx}^{*}(it)$ given by:

$$Q_{wx}^{*}(it) = \lim_{b\to\infty}\left(\left[-\frac{2\alpha k^{\alpha} t^{\alpha}(\theta^{\alpha}(-1)^{\frac{3\alpha}{2}}\Gamma(-\alpha,-itb) - k^{\alpha} t^{\alpha}(-1)^{3\alpha}\Gamma(-2\alpha,-itb)}{\theta^{2\alpha}}\right]\Bigg|_{\frac{k}{\theta}}^{b}\right), \alpha > 1, t \neq 0$$

Proof:

$Q_{wx}^{*}(it) = E(e^{itx})$

$$= \int_{\frac{k}{\theta}}^{\infty} e^{itx}\left(\frac{2\alpha k^{\alpha} x^{-\alpha-1}\left(1 - \frac{k^{\alpha}}{\theta^{\alpha} x^{\alpha}}\right)}{\theta^{\alpha}}\right) dx$$

$$= \frac{1}{\theta^{\alpha}}\left[\int_{\frac{k}{\theta}}^{\infty} 2\alpha k^{\alpha} x^{-\alpha-1} e^{itx} dx - \frac{1}{\theta^{\alpha}}\int_{\frac{k}{\theta}}^{\infty} 2\alpha k^{2\alpha} x^{-2\alpha-1} e^{itx} dx\right]$$

$$= \frac{2\alpha k^{\alpha}}{\theta^{\alpha}} \int_{\frac{k}{\theta}}^{\infty} x^{-\alpha-1} e^{itx} dx - \frac{2\alpha k^{2\alpha}}{\theta^{2\alpha}} \int_{\frac{k}{\theta}}^{\infty} x^{-2\alpha-1} e^{itx} dx$$

$$= \lim_{b\to\infty}\left(\frac{2(-1)^{3\alpha}\alpha k^{2\alpha} t^{2\alpha}(\Gamma(-2\alpha,-itb)}{\theta^{\alpha}} - \frac{2\alpha(-1)^{\frac{3\alpha}{2}} k^{\alpha} t^{\alpha}(\Gamma(-\alpha,-itb)}{\theta^{2\alpha}}\Bigg|_{\frac{k}{\theta}}^{b}\right)$$

$$Q_{wx}^{*}(it) = \lim_{b\to\infty}\left(-\frac{2\alpha k^{\alpha} t^{\alpha}(\theta^{\alpha}(-1)^{\frac{3\alpha}{2}}\Gamma(-\alpha,-itb) - k^{\alpha} t^{\alpha}(-1)^{3\alpha}\Gamma(-2\alpha,-itb)}{\theta^{2\alpha}}\Bigg|_{\frac{k}{\theta}}^{b}\right), \alpha > 1, t \neq 0 \qquad (21)$$

Such that Γ is incomplete gamma function.

Skewness

The skewness of the MWPDTI is $C.S_{w}^{*}$ given by:

$$C.S_{w}^{*} = E\left[\left(\frac{x-E(x)}{\sigma(x)}\right)^{3}\right]$$

By using eq. (11) and (14) we have

$$C.S_{w}^{*} = E\left[\left(\frac{\left(x - \frac{2\alpha^{2}k}{\theta(1-2\alpha)(1-\alpha)}\right)}{\frac{\alpha k\sqrt{5\alpha-1}}{\sqrt{\alpha-2}(\alpha-1)(2\alpha-1)\theta}}\right)^{3}\right] \qquad (22)$$

Kurtosis

Lemma10:

The kurtosis of the MWPDTI is $kurtosis_{w}^{*}[X]$ given by:

$$kurtosis_{w}^{*}[X] = \left(\frac{E\left[\left(x - \frac{2\alpha^{2}k}{\theta(1-2\alpha)(1-\alpha)}\right)^{4}\right]}{E\left[\left(x - \frac{2\alpha^{2}k}{\theta(1-2\alpha)(1-\alpha)}\right)^{2}\right]^{2}} - 3\right)$$

Proof:

$$kurtosis_{w}^{*}[X] = \frac{k_4}{k_2^{2}}$$

$$= \frac{E\left[(x-E(x))^{4}\right] - 3E\left[(x-E(x))^{2}\right]}{E\left[(x-E(x))^{2}\right]^{2}}$$

$$= \left(\frac{E\left[(x-E(x))^{4}\right]}{E\left[(x-E(x))^{2}\right]^{2}} - 3\right)$$

By eq.(11) we get

$$kurtosis_w^*[X] = \left(\frac{E\left[\left(x - \frac{2\alpha^2 k}{\theta(1-2\alpha)(1-\alpha)}\right)^4\right]}{E\left[\left(x - \frac{2\alpha^2 k}{\theta(1-2\alpha)(1-\alpha)}\right)^2\right]^2} - 3 \right) \quad (23)$$

The order statistics of the MWPDTI
Lemma 11:

If $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denotes the order statistics of a random sample $x_1, x_2, \dots, x_n$ from a continuous population with (cdf) $F_X(x)$ and (pdf) $f_X(x)$, then the (pdf) of the r^{th} order statistics $X_{(r)}$ is $g_w^*(x)$ in the form

$$g_w^*(x) = \left(\frac{n\left(2\alpha k^\alpha x^{-\alpha-1}\left(1 - \frac{k^\alpha}{\theta^\alpha x^\alpha}\right)\right)}{\theta^\alpha} \right) \left(\frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha} x^{2\alpha}} \right)^{r-1} \left(1 - \left(\frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha} x^{2\alpha}} \right)\right)^{n-r} \quad for\ r = 1,2,\dots,n$$

Proof:

$f_{X_{(r)}}(x) = \binom{n}{r} f_X(x)\left(F_X(x)\right)^{r-1}(1 - F_X(x))^{n-r} \quad for \quad r = 1,2,3,\dots,n$

By eq. (9) , eq.(10) and eq.(16) we get

$g_w^*(x) = \frac{n!}{(r-1)!(n-r)!} f_w^*(x)\left(F_{wX}^*(x)\right)^{r-1}(1 - F_{wX}^*(x))^{n-r} \quad for \quad r = 1,2,3,\dots,n$

$$= \frac{n!}{(r-1)!\,(n-r)!} \left(\frac{2\alpha k^\alpha x^{-\alpha-1}\left(1 - \frac{k^\alpha}{\theta^\alpha x^\alpha}\right)}{\theta^\alpha} \right) \left(\frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha} x^{2\alpha}} \right)^{r-1} \left(1 - \frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha} x^{2\alpha}} \right)^{n-r} \quad for\ r = 1,2,\dots,n$$

$$g_w^*(x) = \left(\frac{n\left(2\alpha k^\alpha x^{-\alpha-1}\left(1 - \frac{k^\alpha}{\theta^\alpha x^\alpha}\right)\right)}{\theta^\alpha} \right) \left(\frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha} x^{2\alpha}} \right)^{r-1} \left(1 - \left(\frac{(\theta^\alpha x^\alpha - k^\alpha)^2}{\theta^{2\alpha} x^{2\alpha}} \right)\right)^{n-r} \quad for\ r = 1,2,\dots,n \quad (24)$$

Conclusion:

The modified weighted distribution of pareto distribution depends on the values of shape parameter θ obtained from the method of this research which effect directly on the probability density function, survival function and hazard function. In this paper interduce new probability density function contains the shape parameter θ which given a new weighted distribution for the Pareto distribution type I.

And study the statistical properties for the suggested modified weighted probability density function and other properties which appear by using Azzalini's method to weighted pareto distribution.

The suggested distribution deal with the interval $(0, \frac{k}{\theta}]$ which neglects by standard Pareto distribution type I.

The modified weighted standard Pareto distribution type I can be used in a state of the standard Pareto distribution type I, in values for the random variable x less than 1 to near the value of x is 0, otherwise the standard Pareto distribution type I in which the values of x startup the value 1 to ∞.

Author's declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

References:

1. Aamir S G, Hamedani G. Weighted Distributions: A Brief Review, Perspective and Characterizations. ijsp . 2017 May; 6(3).
2. Azzalini A . A class of distributions which includes the normal ones.. sjos. 1985,12:171-178.
3. Rameshwar D, Debasis G. A new class of weighted exponential distributions. Taylor&Francis. JS. 2009 December;43(6):621-634.
4. Mervat M R. A class of weighted weibull distributions and its properties.JMS. 2013; 6(1): 35-45.
5. Shakhatreh MK. A two-Parameter of weighted exponential distributions. j.spl. 2012 February;82(2): 252-261.
6. Badmus N I, Bamiduro A, Ogunobi T G, Lehmann S. Type II weighted weibull distribution. IJPS.2014; 9(4): 71-78.
7. Kharazmi O, Mahdavi A. Generalized weighted exponential distribution. CSSC J. 2015 July;44(6):1557-1569.

8. Iden H. H, Shayma G S. Bayes and Non-Bayes estimation methods for the parameter of Maxwell-Boltizmann distribution. Baghdad Sci. J. August(2017);14(4)
9. Filippo D, Bozidar V P. An extension of Azzalini's method. rXiv:1803.01240[math.ST],2018 March.
10. Arnold B C. Pareto distributions, ojs. Fairland, Maryland. 1983.

التوزيع المعدل الموزون لتوزيع باريتو القياسي من النوع الاول

محمود عريبي شمران

قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق.

الخلاصة:

في هذا البحث استخدمنا طريقة Azzallini لإيجاد توزيع موزون مشتق من توزيع باريتو القياسي النوع الاول(SPDTI) عن طريق إدخال معلمة الشكل (θ) الناتجة عن الطريقة المذكورة أعلاه لتغطية الفترة (0 ، 1] التي أهملها التوزيع القياسي.وبالتالي ، فإن التوزيع المقترح هو تعديل لتوزيع باريتو من النوع الأول ، حيث تكون القيم الاحتمالية للمتغير العشوائي خلال الفترة $\left[\frac{k}{\theta},\infty\right)$, $k>0,\theta\geq 1$تم ايجاد الخصائص الاحصائية لتوزيع باريتو المعدل الموزون من النوع الاول (MWPDTI) كدالة كثافة الاحتمالة ودالة التوزيع ودالة البقاء والعزوم والدوال الاحصائية الاخرى .

الكلمات المفتاحية :_طريقة ازيليني, دالة التوريع التجميعية، توزيع باريتو المعدل الموزون من النوع I, دالة البقاء, توزيع باريتو القياسي من النوع I