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Stable Semisimple Modules, Stable t- Semisimple Modules and Strongly Stable t-Semisimple Modules

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Abstract:

Throughout this paper, three concepts are introduced namely stable semisimple modules, stable t-semisimple modules and strongly stable t-semisimple. Many features co-related with these concepts are presented. Also many connections between these concepts are given. Moreover several relationships between these classes of modules and other co-related classes and other related concepts are introduced.

Key words: Fully invariant submodules, Fully stable modules, Semisimple modules, Stable injective modules, Stable submodules, t-essential submodules, t-semisimple modules,

Introduction:

Let R be a ring with unity and M be a right $R-$module. It is known that" an $R-$module is semisimple if every submodule is a direct summand" (1). Asgari introduced and studied t-semisimple modules as a generalization of semisimple modules, where " an $R-$module M is called t-semisimple if for each submodule $(N \leq M)$, there exists a direct summand K $(K \leq^{\oplus} M)$ such that K is a t-essential in N $(K \leq_{tes} N)$" (2) . In fact " a submodule A of M is called t-essential in M $(A \leq_{tes} M)$ if whenever $B \leq M, A \cap B \subseteq Z_2(M)$, then $B \subseteq Z_2(M)$". (2)

$Z_2(M)$ is defined by" $Z\left(\frac{M}{Z(M)}\right) = \frac{Z_2(M)}{Z(M)}$"[5] is called the second torsion submodule of M. The concept of t-essential is a generalization of the concept of essential, where " a submodule A of M is essential in M $(A \leq_{ess} M)$ if whenever $B \leq M, A \cap B = (0)$, then $B = (0)$" (2) . "The two concepts are equivalent if M is nonsingular $(Z(M) = (0))$" (2). "M is called Z_2-torsion (singular) if $Z_2(M) = M(Z(M) = M)$ "Moreover $Z_2(M) = \{x \in M: xI = 0 \; for \; some \; I \leq_{tes} R\} = \{x \in M: ann_R(x) \leq_{tes} R\}$.

Asgari showed that, for a module M: semisimple ⇒ t-semisimple ⇒ t-extending, and each of the reverse inclusion may be not true." M is called t-extending if every submodule of M is t-essential in a direct summand". (2)

In other words: "M is t-extending if and only if every t-closed submodule A of M is a direct summand" (2), where " A is t-closed if A has no proper t-essential extension in M" (2).

Hadi I-M.A. and Shyaa F.D. in (3) extend the notion of t-semisimple in to strongly t-semisimple modules and studied them.

In (4), they introduced and studied these concept FI-semisimple modules, where "an $R-$module M is called FI-semisimple if every fully invariant submodule is a direct summand" (4)." M is called FI-t-semisimple module if for each fully invariant submodule A of M, there exists $B \leq^{\oplus} M$ such that $B \leq_{tes} A$" (4). "M is called strongly FI-t-semisimple if for each fully invariant submodule A of M, there exists a fully invariant submodule B of M with $B \leq_{tes} A$" (4).

"A submodule A of M is called fully invariant if for each endomorphism f (i.e. $f \in End(M)$), $f(A) \subseteq A$ " (1). "A is called stable if for each homomorphism $f: A \to M, f(A) \subseteq A$ (5). "M is called Duo (fully stable) if every submodule is fully invariant (stable)" (6) and (5). Obviously "every stable submodule is fully invariant but the converse is not true in general", see (5), (7). This motivate us to introduce and study these types of modules: stable semisimple, stable t-semisimple and strongly stable t-semisimple modules.

Section 2 is devoted for studying stable semisimple modules. The direct sum of stable semisimple modules is stable semisimple (see proposition 3). However a direct summand of stable semisimple inherits the property under certain condition (see proposition 4). Also, stable submodules inherit the property if the module is stable injective (see proposition 5).

In Section 3, the stable t-semisimple modules are introduced and studied which as a generalization of t-semisimple modules and also a generalization of FI-t-semisimple modules. The direct sum of stable t-semisimple modules M_1 and M_2 is stable t-semisimple and the converse hold if $M = M_1 \oplus M_2$ is stable injective and $annM_1 + annM_2 = R$(see Theorm 1). Beside this, many characterizations of stable t-semisimple module (with certain conditions) are presented.

In Section 4, strongly stable t-semisimple is introduced and studied. This concept is a generalization of strongly t-semisimple, also a generalization of strongly FI-t-semisimple. Many connections between this concept and other concepts such as stable semisimple, Z_2 −torsion are given. Strongly stable t-semisimple modules and strongly FI-t-semisimple modules are coincide under certain conditions (see Remarks and Examples 3(6),(7)). The direct sum of two strongly stable t- semisimple modules M_1, M_2 with $annM_1 + annM_2 = R$ is strongly stable t-semisimple, and the converse hold if $M = M_1 \oplus M_2$ is stable injective. (Theorem 3). Also every stable direct summand of strongly stable t-semisimple module M is strongly stable-t- semisimple if M is stable-injective (see Proposition 4). Many other results are given in section 4.

Stable Semisimple:

In this section, the stable semisimple modules are introduced and studied.

Definition 1: An R-module M is called stable semisimple (briefly s- semisimple) if every stable submodule of M is a direct summand. A ring R_R is s- semisimple if every stable ideal of R is a direct summand of R.

Note that an R-module M is s- semisimple module if for each stable submodule N of M, there exists $K \leq^{\oplus} M$ such that $K \leq_{ess} N$.

Remarks and Examples 1:

1. Every semisimple module is s- semisimple, but the converse may be not correctly, for instance the $Z-$ module Z is s- semisimple since it has only two stable submodules namely $(0), Z$ and they are direct summands, and Z is not semisimple.
2. Since every stable submodule is fully invariant, then every FI- semisimple module is s- semisimple. However s- semisimple module may be not FI- semisimple; as: Z as $Z-$ module is s- semisimple and it is not FI- semisimple since every proper non zero submodule of Z is fully invariant but it is not direct summand.
3. "An R-module M is called stable extending ($S-$extending) if every stable submodule N of M is essential in a direct summand " (7).

   Note that every s- semisimple is s- extending, but the reverse inclusion may be not correct , like: let M be the $Z-$ module $Z_8 \oplus Z_2$. M is s- extending (7, Rem & Ex 3.1.3(3),p.75). let $N = <\overline{2}> \oplus Z_2$. Then N is a stable submodule of M but N is not a direct summand of M and so M is not s- semisimple.
4. Let M be a fully stable module. Then the following are equivalent:
   1) M is semisimple.
   2) M is FI-semisimple.
   3) M is s- semisimple.
5. Let M be a FI-quasi-injective (that is for each fully invariant submodule N of M and for each homomorphism $f: N \to M$, can be extended to a mapping $g: M \to M$) (7, Definition . 3.1.17).
   Then M is a FI- semisimple if and only if M is s- semisimple.

Proof:($\Rightarrow$) it see a(2).
($\Leftarrow$) let N be a fully invariant submodule of M. By (7, proposition . 3.1.19,p.85), N is stable and hence $N$is a direct summand of M.

6. Image of s-semisimple module is not necessarily stable semisimple, for example: the $Z-$ module Z is s- semisimple. Let $\prod: Z \to Z/4Z \simeq Z_4$ be the natural epimorphism. However Z_4 is not s- semisimple.
7. Let M, M' be two $R-$modules with M isomorphic to M', then M is s- semisimple if and only if M' is s- semisimple.

Lemma 1: Let M be an $R-$module and $N \leq U \leq M$. If $\frac{U}{N}$ is stable in $\frac{M}{N}$ and N is stable in M, then U is stable in M.

Proof: Let $f: U \to M$ be an $R-$homomorphism.
Define $g: \frac{U}{N} \to \frac{M}{N}$ by $g(u+N) = f(u) + N$ for each $+N \in \frac{U}{N}$. To show that g is well-defined.
Let $u_1 + N = u_2 + N$. Then $u_1 - u_2 \in N$, so that $f(u_1 - u_2) \in f(N)$. But $f|_N: N \to M$ implies that $f(N) \subseteq N$ (since N is stable in M). Thus $f(u_1 - u_2) \in N$ and this implies $f(u_1) + N = f(u_2) + N$;

that is $g(u_1)+N=g(u_2)+N$. But $\frac{U}{N}$ is stable in $\frac{M}{N}$, So $g(u+N)\in\frac{U}{N}$, hence $f(u)+N\in\frac{U}{N}$ which implies $f(u)\in U$, for each $u\in U$. Thus U is stable in M.

Proposition 1: Let M be a s- semisimple and N is a stable submodule of M. Then $\frac{M}{N}$ is s- semisimple.

Proof: Let $\frac{U}{N}$ be a stable submodule of $\frac{M}{N}$ where U≤M and U contains N . By Lemma 1, U is a stable submodule of M. But M is s- semisimple, hence $N\leq^{\oplus}M$; that is $U\oplus V=M$ for some $V\leq M$. This implies $\frac{M}{N}=\frac{U}{N}\oplus\frac{V+N}{N}$ and so that $\frac{U}{N}\leq\frac{\oplus_M}{N}$ and $\frac{M}{N}$ is s- semisimple.

Corollary 1: Let $f:M\to M'$ be an epimorphism such that $Ker\,f$ is a stable submodule of M. If M is s- semisimple, then M' is s- semisimple.

Lemma 2: For any $R-$module M, $Z_2(M)$ is stable submodule of M.

Proof: Let $f:Z_2(M)\to M$ be any $R-$homomorphism. To prove that $(Z_2(M))\subseteq Z_2(M)$. Let $y\in f(Z_2(M))$, So $y=f(x)$ and $x\in Z_2(M)$, that is $ann(x)\leq_{tes}R$ Assume that $ann(f(x))\cap J\subseteq Z_2(R)$ where J is an ideal of R. Since $ann(x)\subseteq ann(f(x))$, then $ann(x)\cap J\subseteq ann(f(x))\cap J\subseteq Z_2(R)$. It follows that $J\subseteq Z_2(R)$, since $ann(x)\leq_{tes}R$. Thus $ann(f(x))\leq_{tes}R$ and $y\in Z_2(M)$.

Proposition 2: Let M be a s- semisimple. Then $\frac{M}{Z_2(M)}$ is s- semisimple and $M=Z_2(M)\oplus M'$ where M' is a nonsingular s-semisimple module.

Proof: By Lemma 2, $Z_2(M)$ is stable in M, and by Proposition. 1, $\frac{M}{Z_2(M)}$ is s- semisimple. On the other hand, since M is s- semisimple, $Z_2(M)\leq^{\oplus}M$; that is $M=Z_2(M)\oplus M'$ for some $M'\leq M$. But $M'\simeq\frac{M}{Z_2(M)}$, so M' is a nonsingular s- semisimple module.

Proposition 3: Let $M=M_1\oplus M_2$ where $M_1$and M_2 are $R-$modules. If $M_1$and M_2 are s-semisimple, then M is s- semisimple.

Proof: Let N be a stable submodule of M. Then by (8, Lemma: 2.11), $N=(N\cap M_1)\oplus(N\cap M_2)$ where $N\cap M_1$is stable in M_1 and $N\cap M_2$ is stable in M_2. Since $M_1$and M_2 are s- semisimple modules, then $(N\cap M_1)\leq^{\oplus}M_1$and $(N\cap M_2)\leq^{\oplus}M_2$. Therefore $N=(N\cap M_1)\oplus(N\cap M_2)\leq^{\oplus}M_1\oplus M_2=M$. Thus M is s- semisimple.

Example 1: By applying proposition 3, each of the $Z-$module $Z\oplus Z, Z\oplus Z_2, Q\oplus Z, Z\oplus Z_6$ is s-semisimple.

The following proposition shows that the property of s- semisimple inhirts to direct summands, under certain conditions. First the following Lemma is given.

Lemma 3: Let $M=M_1\oplus M_2$ be an $R-$module, with $M_1\leq M, M_2\leq M$ and $annM_1+annM_2=R$. Then M_1 and M_2 are stable submodules of M.

Proof: Since $annM_1+annM_2=R$, then $M_1=M_1annM_2$ and $M_2=M_2annM_1$. Assume $f:M_1\to M$ be an $R-$homomorphism. $f(M_1)=f(M_1annM_2)=f(M_1)annM_2$. But $f(M_1)\subseteq M=M_1\oplus M_2$, So $f(M_1)\subseteq(M_1\oplus M_2)annM_2=M_1annM_2=M_1$, thus $f(M_1)\subseteq M_1$ and M_1 is stable in M. Similarly M_2 is stable in M.

Proposition 4: Let $M=M_1\oplus M_2$ where $M_1, M_2\leq M$ and $annM_1+annM_2=R$. If M is s- semisimple $R-$module, then M_1 and M_2 are s- semisimple.

Proof: Since M_1 is stable in M (by Lemma 3) and M is s- semisimple by hypothesis. Then by proposition. 1, $\frac{M}{M_1}$ is s- semisimple , hence M_2 is s-semisimple since $M_2\simeq\frac{M}{M_1}$.

"An $R-$module M is called stable injective (briefly s- injective) relative to an $R-$module X, if for each stable submodule $\mathcal{A}$ of X and each $R-$homomorphism $f:\mathcal{A}\to M$ can be extended to an $R-$homomorphism $g:X\to M$ " (7)." M is called s- injective if it is stable injective relative to any $R-$module X".(8).

Proposition 5: Let M be a s-injective $R-$module. If M is s- semisimple module, then every stable submodule of M is s-semisimple.

Proof: Let U be a stable submodule of M and let V be a stable submodule of N, then by (8,Lemma: 2.15). V is a stable submodule of M and hence $V\leq^{\oplus}M$; that is $V\oplus W=M$ for some $W\leq M$. Thus $U=(V\oplus W)\cap U=V\oplus(W\cap U)$ by modular Law. Therefore $V\leq^{\oplus}U$ and U is a s-semisimple module.

Stable t-Semisimple Modules:

In this section, the concept of stable t-semisimple modules are introduced and studied, which is a generalization of s- semisimple modules. Also it is a generalization of t- semisimple modules and FI-t-semisimple modules.

Definition 2: An R −module M is called stable t-semisimple (s-t-semisimple) if for each stable submodule N of M, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. A ring R is s- t-semisimple if R_R is a stable t-semisimple R −module.

Remarks and Examples 2:

1. clearly every s- semisimple module is s- t-semisimple, but the converse is not true in general, for example: the Z-module Z_4 is s- t-semisimple, since for each $N \leq Z_4$, N is stable and ($0 \leq_{tes} N$ because $0 + Z_2(N) = N \leq_{ess} N$), see (2, proposition1.1).
2. Every Singular (and hence Z_2-torsion) module is s- t-semisimple, since for each $N \leq M$, $(0) + Z_2(N) = (0) + N = N \leq_{ess} N$ and hence $(0) \leq_{tes} N$ by (2, proposition.1.1).
3. Every t-semisimple module is s- t-semisimple, but the converse may be not true, for example: Z as Z −module is not t-semisimple (since $\frac{Z}{Z_2(Z)} = \frac{Z}{(0)} \simeq Z$ is not semisimple), see (2, proposition 2.3).
   But Z is s- t-semisimple since It is s-semisimple. Also $M = Z \oplus Z_2$ as Z −module is s- semisimple by part 2., So it is s- t-semisimple, but M is not t-semisimple since $\frac{M}{Z_2(M)} \simeq Z$ is not semisimple see (2, Theorm.2.3).
   Note that under the class of fully stable modules the two notions (t-semisimple) and (s-t-semisimple)module are equivalent. Also they are equivalent under the class of comultiplication modules, since "every comultiplication modules is fully stable", see (9,lemma,1.2.12,p.39).
4. Every FI-t-semisimple is s- t-semisimple, but the convers may be false, as the following example shows: Z as Z −module is s- t-semisimple and it is not FI-t-semisimple by (4, Remarks and Examples.3.2(4)).
5. Let M be a FI-quasi-injective module. Then M is s- t-semisimple if and only if M is FI-t-semisimple.
   Proof: Since, since stable submodule and fully invariant submodule coincide in a FI-quasi-injective modules by (7,Propostion.3.1.19,p.85), the result is obtained.
6. Let M be a nonsingular module $(Z(M) = 0)$. Then M is s- t-semisimple if and only if M is s-semisimple.
   Proof: (⇒) Let N be a stable submodule of M. Since M is a s-t-semisimple, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. But M is nonsingular, implies N is nonsingular, so $K \leq_{ess} N$. On other hand $K \leq^{\oplus} M$ implies $K \oplus \mathrm{K}' = \mathrm{M}$ for some $K' \leq M$. Then $K \oplus (K' \cap N) = N$ by modular Law, hence $K \leq^{\oplus} N$ which implies K is closed in N. Thus $K = N$ and so $N \leq^{\oplus} M$.
   (⇐) it follows by Remarks &Examples. 2 (1).

Proposition 6: Let M be an s- injective module. If M is a s- t-semisimple module, then every stable submodule of M is s- t-semisimple.

Proof: Let U be a stable submodule of M and let V be a stable submodule of U. Since M is stable injective, V is stable in M by (8,Lemma: 2.15). It follows that there exists $K \leq^{\oplus} M$ and $K \leq_{tes} V$, since M is s- t-semisimple. Hence $M = K \oplus \mathrm{T}$ for some $T \leq M$ and so that $U = (K \oplus T) \cap U = K \oplus (\mathrm{T} \cap \mathrm{U})$, thus $K \leq^{\oplus} U$ and hence U is a stable t-semisimple.

Theorem: Let $M = M_1 \oplus M_2$ where $M_1, M_2 \leq M$. If M_1 and M_2 are s-t-semisimple, then M is s- t-semisimple, and the converse hold if M is s-injective and $annM_1 + annM_2 = R$.

Proof: Let N be a stable submodule of M. By (8,lemma 2.11), $N = (N \cap M_1) \bigoplus (N \cap M_2)$, and $N \cap M_1$is stable in M_1 and $N \cap M_2$ is stable in M_2. Since M_1 and M_2 are s- t-semisimple modules there exist $K_1 \leq^{\oplus} M_1, K_2 \leq^{\oplus} M$, $K_1 \leq_{tes} (N \cap M_1)$ and $K_2 \leq_{tes} (N \cap M_2)$. It follows that $K = K_1 \oplus K_2 \leq^{\oplus} M$ and $K_1 \oplus K_2 \leq_{tes} (N \cap M_1) \oplus (N \cap M_2) = N$. Thus M is s- t-semisimple.

Conversely , since $annM_1 + annM_2 = R$, then M_1 and M_2 are stable in M by lemma 3, but M is s- injective, hence by Proposition.3.3, M_1 and M_2 are stable t-semisimple.

Theorem 2: For an R −module M. Consider the following statements:

(1) M is s- t-semisimple.
(2) $\frac{M}{Z_2(M)}$ is s- semisimple.
(3) $M = Z_2(M) \oplus \mathrm{M}'$, where M' is nonsingular, s-semisimple.
(4) Every nonsingular stable submodule of $M$is a direct summand.
(5) Every stable submodule of M which contains $Z_2(M)$ is direct summand.

Then (3)⇔(5)⇒(2), (3)⇒(1)⇒(4) and (4)⇒(3) if M is s- injective and a complement of $Z_2(M)$ is stable.

Proof: (3)⇒(5) Let N be a stable submodule of M and $Z_2(M) \subseteq N$ since $M = Z_2(M) \oplus \mathrm{M}'$, then $N = \big(N \cap Z_2(M)\big) \oplus (N \cap M') = Z_2(M) \oplus (N \cap M')$ and $(N \cap M')$ is stable in M' by [8,Lemma2.11].But M' is s- semisimple, So that $(N \cap M') \leq^{\oplus} M'$; that is $(N \cap M') \oplus W = M'$ and so $Z_2(M) \oplus (N \cap M') \oplus \mathrm{W} = \mathrm{M}$ thus $N \oplus \mathrm{W} = \mathrm{M}$,that is $N \leq^{\oplus} M$.

(5)⇒(2) Let $\frac{N}{Z_2(M)}$ be a stable submodule of $\frac{M}{Z_2(M)}$. Since $Z_2(M)$ is stable submodule of, then by Lemma 1, N is a stable submodule of M. Beside this $Z_2(M) \subseteq N$, hence by condition (5), $N \leq^{\oplus} M$. Hence $N \oplus \mathrm{W} = \mathrm{M}$ for some $W \leq M$. It follows that $\frac{M}{Z_2(M)} = \frac{N}{Z_2(M)} \oplus \frac{W+Z_2(M)}{Z_2(M)}$, so $\frac{N}{Z_2(M)} \leq^{\oplus} \frac{M}{Z_2(M)}$ and $\frac{M}{Z_2(M)}$ is stable semisimple.

(5)⇒(3) $Z_2(M)$ is a stable submodule contain $Z_2(M)$. Hence by (5); $Z_2(M) \leq^{\oplus} M$. Thus $M = Z_2(M) \oplus \mathrm{M}'$, M' is nonsingular But $M' \simeq \frac{M}{Z_2(M)}$ and since part (5) implies (2) (i.e. $\frac{M}{Z_2(M)}$ is s- semisimple) therefore M' is s- semisimple.

(3)⇒(1) By hypothesis, $M = Z_2(M) \oplus \mathrm{M}'$,M' is nonsingular s- semisimple. Let N be a stable submodule of M, then $N = \big(N \cap Z_2(M)\big) \oplus (N \cap M')$ and $N \cap Z_2(M)$ is stable in $Z_2(M)$, $(N \cap M')$ is stable in M'. by (8,Lemma 2.11). Since M' is stable semisimple, $(N \cap M') \leq^{\oplus} M'$ which implies $(N \cap M') \leq^{\oplus} M$. On the other hand, $\frac{N}{N \cap M'} \simeq \frac{N+M'}{M'} \leq \frac{M}{M'}$ which is Z_2-torsion, hence $\frac{N}{N \cap M'}$ is Z_2-torsion and so that $(N \cap M') \leq_{tes} N$ by (2, proposition.1.1), (10, proposition 2.2). Thus $(N \cap M') \leq^{\oplus} M$ and $(N \cap M') \leq_{tes} N$; that is M is s- t-semisimple.

(1)⇒(4) Let N be a nonsingular stable submodule of M. By (1) there exists $K \leq^{\oplus} M$ and $K \leq_{tes} N$. Hence $K \leq_{ess} N$ since N is nonsingular. But K is closed in M since $K \leq^{\oplus} M$, so that $K = N$ thus $N \leq^{\oplus} M$.

(4)⇒(3) Let M' be a complement of $Z_2(M)$, so by hypothesis $M'$is stable in M. Also $M' \oplus Z_2(M) \leq_{ess} M$, hence by (2, proposition.1.1) $M' \leq_{tes} M'$ and so $\frac{M}{M'}$ is Z_2-torsion. Then M' is nonsingular. To show this, Let $x \in M'$, so $x \in M' \subseteq M$ and $ann(x) \leq_{ess} R$, so $ann(x) \leq_{tes} R$, it follows that $x \in Z_2(M) \cap M' = (0)$, thus $x = 0$ and M' is nonsingular. So that by (4) , $M' \leq^{\oplus} M$ this implies $M = L \oplus \mathrm{M}'$ for some $L \leq M$. Then $Z_2(M) = Z_2(L) \oplus Z_2(M') = Z_2(L) \oplus (0) = Z_2(L)$ but L is Z_2-torsion since $\frac{M}{M'} \simeq L$ and $\frac{M}{M'}$ is Z_2-torsion, hence $Z_2(L) = L$ and hence $Z_2(M) = L$. Thus $= Z_2(M) \oplus \mathrm{M}'$, M' is nonsingular. To prove M' is s- semisimple. Let N be a stable submodule of M'. Since M' is stable in M, so N is stable in M by (8,Lemma 2.15) But $M'$is nonsingular, so N is nonsingular and hence by (4), $N \leq^{\oplus} M$; that is $N \oplus \mathrm{W} = \mathrm{M}$ for some $W \leq M$. Then $M' = (N \oplus \mathrm{W}) \cap M' = N \oplus (\mathrm{W} \cap \mathrm{M}')$, and so $N \leq^{\oplus} M'$. therefore M' is s- semisimple.

Proposition 7: Let M be a stable t-semisimple such that every direct summand contains $Z_2(M)$. Then M is stable semisimple and hence M is s-extending.

Proof: Let N be a stable submodule of M. Since M is a s- t-semisimple module, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. But $K \leq^{\oplus} M$ implies K is closed in M and since $Z_2(M) \subseteq K$, then K is a t-closed in M. by (10, proposition. 2.2(c)) thus $K = N$; that is $N \leq^{\oplus} M$ and M is s- semisimple.

Recall that for any submodule N of M, N is contained in a t-closed submodule H of M , such that $N \leq_{tes} H$ by (10,Lemma 2.3). H is called a t-closure of N (10).

Proposition 8: Let M be an s- injective module such that a complement of $Z_2(M)$ is stable and a t-closure of stable submodule is stable. If M is s- t-semisimple, then M is t-stable extending.

Proof: By Theorem 2 ((1)⇒(5)), each stable submodule N of M with $Z_2(M) \subseteq N$, $N \leq^{\oplus} M$. Hence every t-closed stable submodule is direct summand, since every t-closed submodule contains $Z_2(M)$. On the other hand, by hypothesis a t-closure of stable submodule is stable, hence by (8, proposition. 2.5), M is t-stable extending.

Proposition 9: Let M be a s- injective module such that a complement of stable submodule is stable and a t-closure of stable submodule is stable. If M is s-t-semisimple, then $\frac{M}{C}$ is s- t-semisimple for each stable t-closed submodule C.

Proof: By Proposition 8, M is t-stable extending, so by (8, Proposition 2.5), every stable t-closed submodule C is a direct summand of M. Hence $M = C \oplus K$ for some $K \leq M$. It follows that K is a complement of C and hence K is a stable submodule of M. Thus by Proposition 6, K is s- t-semisimple. But $\frac{M}{C} \cong K$, so $\frac{M}{C}$ is stable t-semisimple.

Strongly Stable t-semisimple Modules:

Our concern in this section is extending the notions of s- t-semisimple modules into strongly stable t-semisimple. Also this concept is a generalization of the concept strongly t-semisimple which is introduced in (3) where " an R- module M is strongly t-semisimple if for each submodule N of M,there exists a fully invariant direct summand (hence stable direct summand) K of M such that $K \leq_{tes} N$" (3).

Definition 3: An R- module M is called strongly stable t-semisimple (shortly s-s- t-semisimple) if

for each stable submodule N of M, there exists a stable direct summand K of M with $K \leq_{tes} N$.

Remarks and Examples 3:

1) Every s-semisimple module M is s-s- t-semisimple but not conversely as can see by the example Z_{12} as Z-module is s-s t-semisimple, but not stable semisimple.
2) Every strongly t-semisimple is s-s- t-semisimple, but the converse may be not achieved , for example: Let $M = Z \oplus Z$ as Z-module. Since M has only two stable submodules which are M and (0), so M is s- semisimple and hence by (1) is s-t-semisimple. However M is not strongly t-semisimple since $\frac{M}{Z_2(M)} \simeq Z$ is not t-semisimple (9,Ex.4,p.26).
3) Every Z_2-torsion module is s- t-semisimple by (3, Rem &Ex.(3)), so It is s-s t-semisimple. Note that Z_4 as Z_4-module is s- t-semisimple but not Z_2-torsion.
4) Every s-s- t-semisimple implies s-t-semisimple.
5) "An R-module M is called strongly FI-t-semisimple if for each fully invariant submodule N of M, there exists a fully invariant direct summand K of M, with $K \leq_{tes} N$ " (4), Then every strongly FI-t-semisimple is s-s t-semisimple, but the converse is not achieved for example: the Z-module Z is s-s t-semisimple, but Z is not strongly FI-t-semisimple since if $N = nZ, n \in Z, n > 1$. then (0) is the only direct summand of Z such that $(0) \subseteq N$ but $(0) \not\leq_{tes} N$.
6) Let M be a FI-quasi-injective R-module. Then M is s-s -t-semisimple if and only if strongly FI-t-semisimple.
7) Let M be a fully stable R-module. Then the following statements are equivalent.
   (1) M is strongly t-semisimple.
   (2) M is s-s- t-semisimple.
   (3) M is strongly FI-t-semisimple.
   (4) M is FI-t-semisimple.
   (5) M is s-t-semisimple.
   (6) M is t-semisimple.

Proof: it is clear

Proposition 10: Let M be an R-module with a property a complement of any submodule is stable. Then M is s-s- t-semisimple if and only if M is stable t-semisimple.

Proof: (⇒) it is clear.

(⇐) Let N be a stable submodule of M. Since M is s-t-semisimple, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. Then $K \oplus U = M$ for some $U \leq M$. It is clear that K is a complement of U, hence K is stable by hypothesis. Thus M is s-s- t-semisimple.

Proposition 11: Let M be an s- injective R-module. If M is s-s- t-semisimple, then every stable direct summand of M is s-s- t-semisimple.

Proof: Let N be a stable direct summand of M, let W be a stable submodule of N. Since M is s-injective, W is stable in M by (8,Lemma 2.15). hence there exists a stable direct summand K of M with $K \leq_{tes} W$. Now, *since* $K \leq^{\oplus} M$ then $K \oplus K' = M$ *and so* $N = K \oplus (K' \cap N)$, *thus* $K \leq^{\oplus} M$. but by (9,Rem 1.1.36), K is stable in N. Thus K is a stable direct summand of N with $K \leq_{tes} W$, so that N is s-s- t-semisimple.

Corollary 2: Let M be an s- injective. If M is s-s- t-semisimple, then every nonsingular stable submodule of M is s-s- t-semisimple.

Proof: Let N be a nonsingular stable submodule of M. Since M is s-s- t-semisimple, then M is stable t-semisimple by Rem &Ex 3(4). And by Theorem 3.5(1⇒4), $N \leq^{\oplus} M$. ***thus*** N ***is*** s-s- t-semisimple by Proposition 4.

Corollary 3: For an s-injective R-module M which satisfies (a Complement of $Z_2(M)$ is stable). If M is s-s- t-semisimple, then every stable submodule N of M which contains $Z_2(M)$ is s-s- t-semisimple.

Proof: since M is s-s- t-semisimple module, then by Theorem 3.5(1⇒5), $N \leq^{\oplus} M$. it follows that N is s-s- t-semisimple by Proposition 4.

Corollary 4: Let M be a s- injective such that a complement of $Z_2(M)$ is stable. If M is s-s- t-semisimple, then $\frac{M}{Z_2(M)}$ is stable t-semisimple, hence s-s- t-semisimple.

Proof: By Theorem 2 (1⇒2) and Rem &Ex. 3 (1), the result is obtained.

Theorem 3: Let $M = M_1 \oplus M_2$ with M_1 and M_2 are submodules of M and $annM_1 + annM_2 = R$. If M_1 and M_2 are s-s-t-semisimple modules, then M is s-s-t-semisimple. The converse hold if M is stable injective.

Proof: (⇒) Let N be a stable submodule of M. Then $N = (N \cap M_1) \oplus (N \cap M_2)$, $N \cap M_1$ is stable in M_1 and N_2 is stable in M_2 by (8,Lemma 2.11). put $N \cap M_1 = N_1, N \cap M_2 = N_2$. Since M_1 and M_2 are s-s- t-semisimple, there exist stable direct summands K_1, K_2 in M_1 and M_2 respectively where that $K_1 \leq_{tes} N_1$ and $K_2 \leq_{tes} N_2$. But $K_1 \leq^{\oplus} M_1$ and $K_2 \leq^{\oplus} M_2$ implies $K_1 \oplus K_2 \leq^{\oplus} M$. ***Also*** $K_1 \leq_{tes} N_1$ and $K_2 \leq_{tes} N_2$ implies $K_1 \oplus K_2 \leq_{tes} N_1 \oplus N_2 = N$ by (9, Proposition 1.1.21). To show that $K_1 \oplus K_2$ is stable in M, it is enough to show that $K_1 \oplus K_2$ is fully invariant. Let $f \in$

$End(M) \cong \begin{pmatrix} End(M_1) & Hom(M_2, M_1) \\ Hom(M_1, M_2) & End(M_2) \end{pmatrix}$. But by (9, Lemma 1.2.8), $Hom(M_1, M_2) = 0$ and $(M_2, M_1) = 0$, hence $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ for some $f_1 \in End(M_1)$, $f_2 \in End(M_2)$. It follows that $f(K_1 \oplus K_2) = f(K_1) \oplus f(K_2) \subseteq K_1 \oplus K_2$; that is $K = K_1 \oplus K_2$is afully invariant of M. Hence $K_1 \oplus K_2$ is stable by (7,Lemma 1.2.6).

Conversely ,Since $M = M_1 \oplus M_2$ and $annM_1 + annM_2 = R$, then M_1 and M_2 are stable in M by Lemma 3. It follows that M_1 and M_2 are s-s- t-semisimple by Proposition 4.

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- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
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المقاسات شبه البسيطة المستقرة، المقاسات شبه البسيطة المستقرة من النمط t والمقاسات شبه المستقرة من النمط t - القوية

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الخلاصة:

في هذا البحث ، قُدمتْ ثلاث مفاهيم والتي أُطلقَ عليها : المقاسات شبه البسيطة المستقرة ، المقاسات شبه البسيطة المستقرة من النمط t - والمقاسات شبه البسيطة المستقرة من النمط t - القوية. العديد من النتائج حول هذه الالمفاهيم قد قُدمت كذلك العديد من العلاقات بين هذه المفاهيم قد أُعطيت. بالإضافة الى هذا عدة علاقات بين هذه الاصناف من هذه الموديلات واصناف واخرى من المقاسات قد قُدمت.

الكلمات المفتاحية: المقاسات شبه البسيطة – المقاسات الجزئية المستقرة المقاسات شبه البسيطة من النمط t – المقاسات الجزئية الواسعة من النمط t – المقاسات الانغمارية المستقرة – المقاسات الجزئية الثابتة تماماً – المقاسات التامة الاستقرارية .